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Brief paper

Integral quadratic constraint approach vs. multiplier approach ☆

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Abstract

Integral quadratic constraints (IQC) arise in many optimal and/or robust control problems. The IQC approach can be viewed as a generalization of the classical multiplier approach in the absolute stability theory. In this paper, we study the relationship between the two approaches for robust stability analysis. Using a slightly modified multiplier approach, we show that the existence of an IQC is equivalent to the existence of a multiplier in most known cases. It is hoped that this result provides some new insight into both approaches and makes them more useful in robust control applications.

Keywords: Integral quadratic constrains; Multipliers; Robust stability; Nonlinear systems

1. Introduction

Integral quadratic constraints (IQCs) often arise in robustness analysis of linear and nonlinear dynamical systems. They are used as a convenient tool for describing parametric uncertainties, time delays, unmodeled dynamics and nonlinearity of the system, as well as design objectives such as LQG costs or H_∞ performances.

The terminology of IQC was formally introduced by Yakubovich (1967, 1971) for robust stability analysis of systems subject to complicated perturbations. The underlying idea, however, had been around since the seminal work by Popov (1962) on absolute stability. Popov's idea of using a quadratic constraint to "overbound" sectorial nonlinearity led to a frequency domain condition for absolute stability in terms of a *multiplier* function. The absolute stability theory developed in the 1960–1970s offers a rich class of multipliers for robustness analysis with various nonlinear functions. Strong connections between multipliers and the network realization theory are well established. Further, a Lyapunov function is associated with each multiplier. See, e.g., Brockett and Willems (1965), Narendra and Taylor (1973), Desoer and Vidyasagar (1975), Safonov (1980), Willems (1971) and Vidyasagar (1992) for details. Many of the classical papers on absolute stability can be found in an edited book by Aggarwal and Vidyasagar (1977). The multiplier approach also finds useful applications in searching for parameter-dependent Lyapunov functions for analysis and synthesis of uncertain systems; see, e.g., Dasgupta, Chockalingam, Anderson, and Fu (1994).

Generalized from the multiplier approach, the IQC approach is able to treat a larger class of uncertainty and nonlinearity. Many IQCs are collected in a paper by Megretski and Rantzer (1997). The examples where IQCs apply include real and complex uncertainties, fast and slow time-varying parameters, time delays, nonlinearity, H_∞ optimization constraints, etc. The so-called Kalman–Yakubovich–Popov (KYP) Lemma (see, Anderson, 1967; Willems, 1971) plays a vital role in the analysis of IQCs. Recent development in the IQC approach incorporates the theory of linear matrix inequality (LMI) to derive

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more advanced robust stability and robust control results; see, e.g., Boyd, El Ghaoui, Feron, and Balakrishnan (1994), Feron, Apkarian, and Gahinet (1995), Haddad and Bernstein (1991), How and Hall (1995), Fu, Barabanov, and Li (1995) and Fu and Barabanov (1997). The IQC approach has also been used to study parameter-dependent Lyapunov functions; see Gahinet, Apkarian, Chilali, and Feron (1995) and Fu and Dasgupta (2000). The advantage of the LMI approach is that much more complicated uncertainties can be handled using convex optimization, and hence, it differs sharply from the traditional absolute stability theory where the main goal was to obtain simple graphical tests.

The purpose of this paper is to study the following converse problem: To what extent does the IQC approach generalize the multiplier approach? In other words, we would like to know under what conditions the existence of an IQC implies the existence of a multiplier. This problem is motivated by the fact that the multiplier approach is simpler and more intuitive. So we want to know when we can apply the simpler approach.

To this end, we modify the classical multiplier approach slightly by allowing a somewhat more general class of multipliers. More specifically, the classical multiplier approach uses multipliers of the following form:

$$M(s) = M_1^*(s)M_2(s),$$

where both $M_1(s)$ and $M_2(s)$ are stable square matrix functions with stable inverses. Our modification simply allows multipliers which have "tall" $M_1(s)$ and $M_2(s)$. Such a modification does not alter the validity of the multiplier approach.

With this modification of the multipliers, we prove a surprising result: The existence of an IQC is equivalent to the existence of a multiplier under a technical condition called *convexity condition*. To explain this condition, we note that an IQC describes the relationship between the input z and the output w of an uncertainty block Δ as a quadratic functional inequality:

$$\Gamma(z, w) \leqslant 0.$$

The convexity condition requires the mapping of Δ to be $L_2[0, \infty) \to L_2[0, \infty)$ and the functional to be convex in w. The convexity condition is a natural description of most uncertainties in applications. In words, the condition simply means that a suitably transformed output of the uncertainty block is bounded by a suitably transformed input in some sense. A simple example of such an uncertainty block is a passive nonlinearity which is described by

$$\int_0^{\infty} (w^2(t) - z^2(t))\, \mathrm{d}t \leqslant 0$$

which is obviously convex in w. If the convexity condition is violated, then the output can be "large" in some sense for a "small" input.

It is natural that the convexity condition is satisfied for almost all known IQCs. We thus conclude that the IQC approach can be viewed as a reformulation of the multiplier approach to a large extent. We hope that this new connection between the IQC approach and the multiplier approach may make both approaches more useful in many applications.

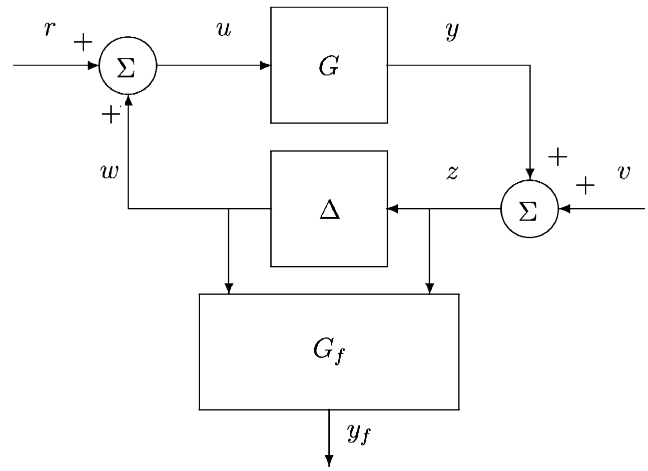

Fig. 1. Interconnected feedback system.

This paper is organized as follows: Section 2 introduces the IQC approach. Section 3 reviews the classical multiplier approach. Section 4 contains the main result of the paper. Section 5 gives a discussion on the main result. Section 6 concludes the paper.

2. The IQC approach

Consider the interconnected system in Fig. 1 which is also described by the following equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \\ z &= y + v, \\ u &= r + w, \\ w &= \Delta(z), \end{aligned} \tag{1}$$

where $\Delta(\cdot) \in \mathbf{\Delta}$ which is a set of linear or nonlinear dynamic operators to be specified later. Denote

$$G(s) = C(sI - A)^{-1}B + D \tag{2}$$

and assume A to be asymptotically stable in the sequel.

The feedback block $\Delta(\cdot)$ is assumed to satisfy an IQC which is constructed via a *filter* given by

$$G_{\mathrm{f}}(s) = C_{\mathrm{f}}(sI - A_{\mathrm{f}})^{-1}B_{\mathrm{f}} + D_{\mathrm{f}}, \tag{3}$$

where A_{f} is asymptotically stable. It is also assumed that $\mathbf{\Delta}$ is a connected set containing the zero operator.

The IQC used in this paper is then described by the following inequality:

$$\int_{-\infty}^{+\infty} [z^*(\mathrm{j}\omega)\ w^*(\mathrm{j}\omega)]\Phi(\mathrm{j}\omega) \begin{bmatrix} z(\mathrm{j}\omega) \\ w(\mathrm{j}\omega) \end{bmatrix} \mathrm{d}\omega \geqslant 0, \quad \forall \Delta \in \mathbf{\Delta}, \tag{4}$$

where $z(\mathrm{j}\omega), w(\mathrm{j}\omega)$ are Fourier transforms of $z(t), w(t)$, respectively, and

$$\Phi(s) = G_{\mathrm{f}}^*(s)\tilde{\Phi}G_{\mathrm{f}}(s) \tag{5}$$

with some constant symmetric matrix $\tilde{\Phi}$.

We now introduce a notion of stability, absolute total stability, for robust stability analysis with IQC. This stability notion is stronger than asymptotic stability and L_2 BIBO stability.

Definition 1. System (1) is called *totally stable* (or simply called stable) if there exists some constant ρ such that for all $r, v \in L_2[0,\infty)$ and the initial state $x(0)$, the response signals $w(t)$ and $x(t)$ (and hence all other signals) are well-defined at all $t \geqslant 0$, and the following holds:

$$\int_0^\infty (x'(t)x(t) + w'(t)w(t))\,\mathrm{d}t \leqslant \rho\left(x'(0)x(0) + \int_0^\infty (r'(t)r(t) + v'(t)v(t))\,\mathrm{d}t\right). \tag{6}$$

Further, a family of systems of the form (1) is called *absolutely totally stable* (or simply called absolutely stable) if there exists a common $\rho > 0$ such that (6) holds for every member system.

The following result serves the foundation of the IQC approach (see Megretski & Rantzer, 1997).

Theorem 1 (*The IQC Theorem*). *Given a set of operators* $\mathbf{\Delta}$ *for the feedback block of system* (1), *the system is absolutely stable if there exists some* $\Phi(s)$ *of the form* (5) *and a constant* $\varepsilon > 0$ *such that both* (4) *and the following condition are satisfied*:

$$[G^*(\mathrm{j}\omega)\ \ I]\Phi(\mathrm{j}\omega)\begin{bmatrix} G(\mathrm{j}\omega) \\ I \end{bmatrix} + \varepsilon I \leqslant 0, \quad \forall|\omega| < \infty. \tag{7}$$

3. The multiplier approach

Let us briefly review the classical multiplier approach to absolute stability analysis. The following result can be found in Desoer and Vidyasagar (1975), Vidyasagar (1992) and Zames and Falb (1968). We use U to denote the set of all asymptotically stable square transfer matrices with an asymptotically stable inverse.

Lemma 1. *Consider the system in Fig. 1 with* $\mathbf{\Delta}$ *being a set of* $L_2[0,\infty) \to L_2[0,\infty)$ *operators. Suppose there exist a multiplier* $M(s)$ *of the following form*:

$$M(s) = M_1^*(s)M_2(s),\ M_1(s), M_2(s) \in U \tag{8}$$

and a constant $\varepsilon > 0$ *such that the following two passivity conditions are satisfied*:

$$\int_{-\infty}^{\infty} \mathrm{Re}[z^*(\mathrm{j}\omega)M(\mathrm{j}\omega)w(\mathrm{j}\omega)]\,\mathrm{d}\omega \geqslant 0, \quad \forall z \in L_2[0,\infty), \quad \varDelta \in \mathbf{\Delta} \tag{9}$$

$$M^*(\mathrm{j}\omega)G(\mathrm{j}\omega) + G^*(\mathrm{j}\omega)M(\mathrm{j}\omega) \leqslant -\varepsilon I, \quad \forall\omega \in (-\infty,\infty). \tag{10}$$

Then, the system in Fig. 1 is absolutely stable.

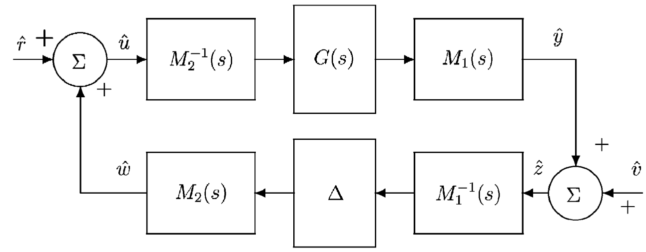

Fig. 2. Transformed feedback system.

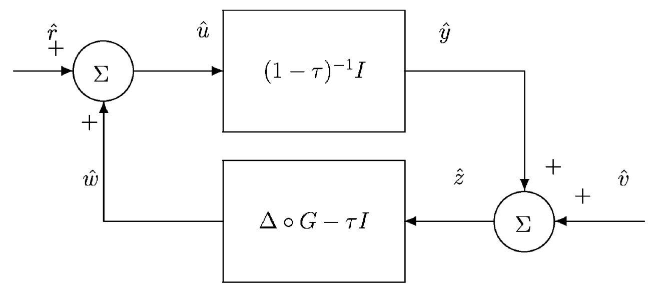

Fig. 3. Transformed feedback system.

Remark 1. The physical interpretation of the lemma above is clearly given in Fig. 2. It is obvious to see that Figs. 1 and 2 are identical, provided that $\hat{y} = M_1 y$, $\hat{z} = M_1 z$, $\hat{v} = M_1 v$, $\hat{r} = M_2 r$, $\hat{u} = M_2 u$ and $\hat{w} = M_2 w$ are taken. The conditions in (9) and (10) simply mean that the lower block of Fig. 2 is passive and the negated upper block is strictly passive.

Now let us consider a modified version of Lemma 1. This modification is obtained by converting Fig. 1 into Fig. 3, where $\tau \in [0, 1)$ is an arbitrary parameter. It can be easily verified that the signals in Fig. 3 are related by

$$(\varDelta \circ G - \tau I)(\hat{y} + \hat{v}) + \hat{r} = \hat{u} = (1-\tau)\hat{y}.$$

Simplifying the above gives

$$(I - \varDelta \circ G)(\hat{y} + \hat{v}) = \hat{r} + (1-\tau)\hat{v}. \tag{11}$$

Operating the both sides by G gives

$$(I - G \circ \varDelta)(G(\hat{y}) + G(\hat{v})) = G(\hat{r}) + G((1-\tau)\hat{v}). \tag{12}$$

In comparison, the signals in Fig. 1 are related by

$$(I - \Delta \circ G)(u) = r + \Delta(v), \tag{13}$$

$$(I - G \circ \Delta)(y) = G(r) + G \circ \Delta(v). \tag{14}$$

For any $r, v \in L_2[0, \infty)$, if we take

$$\hat{r} = r, \quad \hat{v} = \frac{1}{1-\tau}\Delta(v), \tag{15}$$

then $\hat{r}, \hat{v} \in L_2[0, \infty)$ and

$$u = \hat{y} + \hat{v}, \quad y = G(\hat{y}) + G(\hat{v}). \tag{16}$$

Hence, the absolute stability of Fig. 3 implies that of Fig. 1.

Applying Lemma 1 to Fig. 3, we obtain the following result.

Lemma 2. *Consider the system in Fig. 1 with $\mathbf{\Delta}$ being a set of $L_2[0, \infty) \to L_2[0, \infty)$ operators. Suppose there exists a multiplier $M(s)$ of the form* (8) *and a constant $\varepsilon > 0$ such that the following conditions are satisfied*:

$$M(j\omega) + M^*(j\omega) \leqslant -\varepsilon I, \tag{17}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} (\mathrm{Re}[u^*(j\omega)M(j\omega)(w(j\omega) - u(j\omega))] \\ &\quad - \varepsilon u^*(j\omega)u(j\omega))\, d\omega \geqslant 0, \\ &\forall u \in L_2[0, \infty), \quad \Delta \in \mathbf{\Delta}, \quad w = \Delta \circ G(u). \end{aligned} \tag{18}$$

Then, the system in Fig. 1 is absolutely stable.

Proof. Suppose (17) and (18) hold. With $u = \hat{z}$ and $0 < \tau < 1$ and

$$\hat{w} = (\Delta \circ G - \tau I)(\hat{z}),$$

we can rewrite (18) as

$$\int_{-\infty}^{\infty} \mathrm{Re}[\hat{z}^*(j\omega)M(j\omega)\hat{w}(j\omega)]\, d\omega \geqslant J$$

for all $\hat{z} \in L_2[0, \infty)$, $\Delta \in \mathbf{\Delta}$, where

$$J = \int_{-\infty}^{\infty} \mathrm{Re}[\hat{z}^*(j\omega)((1-\tau)M(j\omega) + \varepsilon)\hat{z}(j\omega)]\, d\omega.$$

It is obvious that $J \geqslant 0$ when τ is sufficiently close to 1. Also, (17) implies

$$M^*(j\omega)\left(\frac{1}{1-\tau}I\right) + \left(\frac{1}{1-\tau}I\right)M(j\omega) \leqslant -\frac{\varepsilon}{1-\tau}I.$$

Applying Lemma 1 to the system in Fig. 3, we conclude that this system is absolutely stable. Hence, so is the system in Fig. 1. □

Remark 2. Note that (17) is implied by (18) because $\mathbf{\Delta}$ contains the zero operator. But we state it to make it explicit.

4. IQC vs. multiplier

We are now ready to establish a relationship between IQCs and multipliers. To this end, we consider a more general class of multipliers than (8). More specifically, the class of multipliers, we allow, has the following form:

$$M(s) = M_1^*(s)M_2(s), \tag{19}$$

where $M_1(s)$ and $M_2(s)$ are $r \times m$ stable transfer matrices with $r \geqslant m$. That is, we allow $M_1(s)$ and $M_2(s)$ to be "tall" to take the advantage of larger dimensions. Note that $M(s)$ is square.

Also, let us express

$$\Phi(s) = \begin{bmatrix} Q(s) & F(s) \\ F^*(s) & R(s) \end{bmatrix}. \tag{20}$$

The key technical condition we require is the *convexity condition* discussed in the Introduction. When the uncertainty block is described by an IQC as expressed in (4), (5) and (20), we claim that the convexity condition is the same as requiring $R(j\omega) \leqslant 0$ for all ω. This is formalized below.

Lemma 3. *An IQC in* (4), (5) *and* (20) *is convex in w if $R(j\omega) \leqslant 0, \ \forall \omega$.*

Proof. Recall from (5), $\Phi(j\omega)$ is Hermitian. In particular, $R(j\omega)$ is Hermitian. Using (20),we can rewrite (4) as

$$\begin{aligned} &\int_{-\infty}^{\infty} \{z^*(j\omega)Q(j\omega)z(j\omega) + \mathrm{Re}[z^*(j\omega)F(j\omega)w(j\omega)] \\ &\quad + w^*(j\omega)R(j\omega)w(j\omega)\}\, d\omega \geqslant 0. \end{aligned}$$

Note that the integrand is a quadratic function in w. It follows that the inequality above is convex in w if $R(j\omega) \leqslant 0$ for all ω. □

We emphasize that most IQCs experienced in applications satisfy the convexity condition; see Section 5 for a discussion.

The main result of this paper is given below.

Theorem 2. *Consider the system in Fig. 1 with the assumption that $\mathbf{\Delta}$ is a set of $L_2[0, \infty) \to L_2[0, \infty)$ operators. Suppose there exist some multiplier $M(s)$ of the form* (19) *and some constant $\varepsilon > 0$ such that* (9) *and* (10) *are satisfied. Then,* (4) *and* (7) *hold with the following $\Phi(s)$*:

$$\Phi(s) = \begin{bmatrix} \frac{\varepsilon}{2\|G\|_\infty^2} & M(s) \\ M^*(s) & 0 \end{bmatrix} \tag{21}$$

which can realized in the form of (5) *with*

$$G_f(s) = \begin{bmatrix} I & 0 \\ M_1(s) & 0 \\ 0 & M_2(s) \end{bmatrix},$$

$$\tilde{\Phi} = \begin{bmatrix} \frac{\varepsilon}{2\|G\|_\infty^2} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}, \tag{22}$$

where $||G||_\infty$ *is the* H_∞ *norm of* $G(s)$. *Consequently, the system in Fig. 1 is absolutely stable.*

Conversely, suppose (4) *and* (7) *hold for some* $\varepsilon > 0$ *and some* $\Phi(s)$ *of form* (5) *with* $R(\mathrm{j}\omega) \leqslant 0$ *for all* $\omega \in (-\infty, \infty)$. *Then,* (17) *and* (18) *hold for*

$$M(s) = 2(G^*(s)F(s) + R(s)) \tag{23}$$

which can be realized in the form of (19) *with*

$$M_1(s) = 2G_\mathrm{f}(s)\begin{bmatrix} G(s) \\ I \end{bmatrix}, \quad M_2(s) = \tilde{\Phi}G_\mathrm{f}(s)\begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{24}$$

Remark 3. The result above shows that the existence of an IQC is equivalent to an multiplier if the IQC is restricted to have negative semidefinite $R(\mathrm{j}\omega)$. The advantage of the multiplier is that it is much smaller in dimension and hence in general easier to search for.

Proof of Theorem 2. Suppose there exist $M(s)$ of the form (19) and some constant $\varepsilon > 0$ such that (9) and (10) are satisfied. Using the $\Phi(s)$ in (21), it is trivially verified that (4) and (7) correspond to (9) and (10), respectively. Also, it is easy to check that this $\Phi(s) = G_\mathrm{f}^*(s)\tilde{\Phi}G_\mathrm{f}(s)$ for the $\tilde{\Phi}$ and $G_\mathrm{f}(s)$ defined in (22). In particular, $G_\mathrm{f}(s)$ is asymptotically stable because both $M_1(s)$ and $M_2(s)$ are. The assertion about absolute stability follows from the IQC Theorem.

Conversely, suppose (4) and (7) hold for some $\Phi(s)$ in the form of (5) with $R(\mathrm{j}\omega) \leqslant 0$ for all ω. Using (20), we rewrite (4) and (7) as follows:

$$\int_{-\infty}^{\infty} (z^*Qz + z^*Fw + w^*F^*z + w^*Rw)\,\mathrm{d}\omega \geqslant 0, \tag{25}$$

$$G^*QG + G^*F + F^*G + R \leqslant -\varepsilon I. \tag{26}$$

Take any $u \in L_2[0,\infty)$. It follows that $z = G(u) \in L_2[0,\infty)$ and $w = \Delta \circ G(u) \in L_2[0,\infty)$. Then, (25) and (26) become

$$\int_{-\infty}^{\infty} (u^*G^*QGu + u^*G^*Fw + w^*F^*Gu + w^*Rw)\,\mathrm{d}\omega \geqslant 0, \tag{27}$$

$$\int_{-\infty}^{\infty} u^*(G^*QG + G^*F + F^*G + R + \varepsilon I)u\,\mathrm{d}\omega \leqslant 0. \tag{28}$$

The difference between the two integrals above yields

$$\begin{aligned}\int_{-\infty}^{\infty} &(u^*G^*F(w-u) + (w-u)^*F^*Gu \\ &+ w^*Rw - u^*Ru - \varepsilon u^*u)\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} (u^*(G^*F+R)(w-u) + (w-u)^*R(w-u) \\ &\quad + (w-u)^*(F^*G+R)u - \varepsilon u^*u)\,\mathrm{d}\omega \\ &\geqslant 0.\end{aligned}$$

Since $R \leqslant 0$, the above implies

$$\begin{aligned}\int_{-\infty}^{\infty} &(u^*(G^*F+R)(w-u) + (w-u)^* \\ &\times (F^*G+R)u - \varepsilon u^*u)\,\mathrm{d}\omega \geqslant 0,\end{aligned}$$

which is the same as (18) with $M(s)$ given by (23). It is a trivial matter to verify that $M(s) = M_1^*(s)M_2(s)$ for $M_i(s)$ in (24). □

Remark 4. Observe that the first part of Theorem 2, which generalizes Lemma 1 to allow a multiplier $M(s)$ with "tall" $M_i(s)$, is trivially proved using the IQC Theorem, although the use of such a multiplier seems to be difficult to justify using Fig. 2 because $M_i(s)$ are not invertible.

Next, we introduce two corollaries. The first one generalizes Lemma 2 by allowing "tall" multipliers. This is needed to ensure that the multiplier in (23) and (24) is adequate for guaranteeing the absolute stability of the system in Fig. 1. The second corollary specializes Theorem 2 to linear Δ blocks.

Corollary 1. *The result in Lemma* 2 *holds even when* $M_1(s)$ *and* $M_2(s)$ *are "tall," i.e., they are in the form of* (19).

Proof. Let $M_1(s)$ and $M_2(s)$ be in the form of (19) and the conditions in (17) and (18) be satisfied. In view of our earlier discussion about the relationship between Figs. 1 and 3, it is sufficient to show that the related system in Fig. 3 is absolutely stable for some $\tau \in (0,1)$. To this end, we denote

$$\bar{G} = \frac{1}{1-\tau}I, \quad \bar{\Delta} = \Delta \circ G - \tau I,$$

which are the feedforward and feedback blocks in Fig. 3. We can derive the following from (17) and (18):

$$M^*(s)\bar{G}(s) + \bar{G}^*(s)M(s) \leqslant -\frac{\varepsilon}{1-\tau}I,$$

$$\int_{-\infty}^{\infty} (\mathrm{Re}[u^*(\mathrm{j}\omega)M(\mathrm{j}\omega)\bar{w}(\mathrm{j}\omega)] - \Gamma(\omega))\,\mathrm{d}\omega \geqslant 0$$

for all $u \in L_2[0,\infty)$, $\bar{w} = \bar{\Delta}(u)$ and $\Delta \in \mathbf{\Delta}$, where

$$\Gamma(\omega) = \mathrm{Re}[(1-\tau)u^*(\mathrm{j}\omega)M(\mathrm{j}\omega)u(\mathrm{j}\omega)] + \varepsilon u^*(\mathrm{j}\omega)u(\mathrm{j}\omega).$$

When τ is sufficiently close to 1, $\Gamma(\omega) \geqslant 0$ for all ω and we have

$$\int_{-\infty}^{\infty} \mathrm{Re}[u^*(\mathrm{j}\omega)M(\mathrm{j}\omega)\bar{w}(\mathrm{j}\omega)]\,\mathrm{d}\omega \geqslant 0$$

for all $u \in L_2[0,\infty)$, $\bar{w} = \bar{\Delta}(u)$ and $\Delta \in \mathbf{\Delta}$. Now applying the first part of Theorem 2, we conclude that the system in Fig. 3 is absolutely stable. □

Corollary 2 (Fu & Barabanov, 1997). *Suppose* $\mathbf{\Delta}$ *is a set of causal and asymptotically stable LTI operators containing the zero operator. Then, the following two conditions, both guaranteeing the absolute stability of the system in Fig. 1, have the implication that* (i) ⇒ (ii).

(i) *There exists* $\Phi(s)$ *of the form* (5) *and some* $\varepsilon > 0$ *such that* (4) *and* (7) *hold and that* $R(\mathrm{j}\omega) \leqslant 0$ *for all* $\omega \in (-\infty, \infty)$;

(ii) *There exists a multiplier* $M(s)$ *of the form* (19) *such that*

$$M(\mathrm{j}\omega)[I - \Delta(\mathrm{j}\omega)G(\mathrm{j}\omega)][I - \Delta(\mathrm{j}\omega)G(\mathrm{j}\omega)]^* \times M^*(\mathrm{j}\omega) + \varepsilon I \leqslant 0. \quad (29)$$

Remark 5. The problem studied in the corollary above is commonly known as the structured singular value problem when $\mathbf{\Delta}$ is specially structured. It is known (Fu & Barabanov, 1997; Meinsma, Shrivastava, & Fu, 1996) that the multiplier approach gives a less conservative test for robustness analysis than the so-called *D–G* scaling method given in Fan, Tits, and Doyle (1991). In fact, the *D–G* scaling method amounts to a special multiplier; see details in Fu and Barabanov (1997) and Meinsma et al. (1996).

5. Discussion

As we see from Theorem 2, the technical condition for the existence of a multiplier is that $R(\mathrm{j}\omega) \leqslant 0$. Indeed, most IQCs used in applications satisfy this condition. Examples include norm bounded uncertainties and passive operators, although many more can be found in the literature (see, e.g., Megretski & Rantzer, 1997). To demonstrate the convexity condition, we consider the examples below.

Example 1 (*Popov criterion*). The well-known Popov criterion (Popov, 1962) considers a single-input single-output system as in Fig. 1 with $G(s) = C(sI - A)^{-1}B$ (without the D term) and $\mathbf{\Delta}$ being a set of nonlinear functions satisfying $0 \leqslant z(-\Delta(z)) \leqslant cz^2$ for some unknown constant $c > 0$. The Popov criterion asserts that such a system is absolutely stable if $(1 + ks)G(s)$ is SPR for some constant $k \geqslant 0$. The function $(1 + ks)$ is called a multiplier.

The Popov criterion can be verified by modifying $G(s)$ and Δ to

$$\bar{G}(s) = G(s)(1 + s), \quad \bar{\Delta} = \Delta \circ \left(1 + \frac{\mathrm{d}}{\mathrm{d}t}\right)^{-1}.$$

For the modified system, the function $\Phi(s)$, renamed as $\bar{\Phi}(s)$, for the associated IQC is given by

$$\bar{\Phi}(s) = \begin{bmatrix} 0 & -\frac{1+ks}{1+s}^* \\ -\frac{1+ks}{1+s} & 0 \end{bmatrix}.$$

See details in Megretski and Rantzer (1997). Returning to the original system, the corresponding $\Phi(s)$ is given by

$$\Phi(s) = \begin{bmatrix} 0 & -(1 + ks)^* \\ (1 + ks) & 0 \end{bmatrix}. \quad (30)$$

Example 2 (*Limit cycles of a digital quantizer; Xie, Fu, and Li, 1998*). Consider a digital quantizer described by

$$w(n) = -\mathrm{sat}(z(n)) = \begin{cases} 1, & z(n) < -1, \\ -z(n), & |z(n)| \leqslant 1, \\ -1, & z(n) > 1. \end{cases} \quad (31)$$

It follows that $z(n)w(n) \leqslant 0$ for all n. We may model this as a simple passive device. However, this description is too conservative in general. To overcome this difficulty, we let $H(z)$ be any stable function with L_1 norm less than or equal to 1, i.e.,

$$\sum_0^\infty |h(n)| \leqslant 1, \quad (32)$$

where $h(n)$ is the impulse response corresponding to $H(z)$. In addition, it is required that $1 + H(z)$ is invertible. Then, the IQC is given by

$$\Phi(z) = \begin{bmatrix} 0 & -(1 + H) \\ -(1 + H^*) & -(2 + H + H^*) \end{bmatrix}.$$

Example 3 (*Constant uncertain parameters*). Consider the case

$$w = \Delta z = \text{block diag}\{q_1 I_{k_1}, \ldots, q_p I_{k_p}\}z, \\ q_i \in [-1, 1], \quad i = 1, \ldots, p, \quad (33)$$

where q_i are all constant uncertain parameters. Let us take any

$$D(s) = \text{block diag}\{D_1(s), \ldots, D_p(s)\}, \\ V(s) = \text{block diag}\{V_1(s), \ldots, V_p(s)\}, \quad (34)$$

where $D_i(s)$ and $V_i(s)$ are square matrices of dimension k_i, and

$$D(\mathrm{j}\omega) = D^*(\mathrm{j}\omega) > 0, \quad V(\mathrm{j}\omega) = -V^*(\mathrm{j}\omega), \\ \forall \omega \in (-\infty, \infty). \quad (35)$$

We can build an IQC with the following $\Phi(s)$:

$$\Phi(s) = \begin{bmatrix} D(s) & V(s) \\ V^*(s) & -D(s) \end{bmatrix} \quad (36)$$

provided that $D(s)$ and $V(s)$ are such that the $\Phi(s)$ above can be expressed as in (5).

We see in all the examples above, the term $R(\mathrm{j}\omega)$ (or $R(\mathrm{e}^{\mathrm{j}\omega})$ in Example 2) is all non-positive.

6. Conclusions

In this paper, we have studied the relationship between the IQC approach and the multiplier approach. The main result is that these two approaches are equivalent under a fairly mild convexity condition. It should be pointed out that the purpose of this paper is not to undermine the significance of the IQC approach. Rather, we hope that the work of this paper provides some new insight into these two approaches and can motivate more research in this area.

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