Abstract—This paper proves that the problem of pole placement via static output feedback for linear time-invariant systems is NP-hard.

I. INTRODUCTION
This paper is motivated by the following long-standing static output feedback stabilization (SOFSP) problem: Given a linear time-invariant system, determine if it is stabilizable via static output feedback. This is arguably one of the most fundamental yet challenging control problems; see a recent survey [1]. There have been a number of recent attempts to analyze the computational complexity of this problem. In Blondel and Tsitsiklis [2], it is shown that the problem of finding a static output feedback stabilizer from a given bounded set (a hypercube) is NP-complete. In Fu and Luo [3], it is shown that a matrix inequality problem closely related to the SOFS problem is NP-hard. This matrix inequality problem, involving two linear matrix inequalities and a non-convex coupling condition, is related to the SOFS problem in the sense that the latter can be transformed into the former.

In this paper, we consider the problem of static output feedback pole placement (SOFPP): Given a linear time-invariant system and a set of desired poles, determine if there exists a static output feedback controller such that the closed-loop system contains poles at these desired locations. For some special cases where the numbers of inputs and outputs are very small, constructive methods are available for SOFPP; see [5]. It is also known that generic pole placement using static output feedback is not possible; see, e.g., [6]. The difficulty, however, is that it is not clear whether one can easily determine whether the SOFPP problem is solvable for a given system and a given set of desired poles. Our result shows that this problem is, unfortunately, NP-hard.

II. MAIN RESULT
The SOFPP problem can be formally stated as follows: Given an $n$th order linear time-invariant system:

$$\delta x = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where $\delta x = \dot{x}$ for the continuous-time case, or $\delta x = x(t+1)$ for the discrete-time case, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{r \times n}$, and a set of desired eigenvalues $\lambda_i, i = 1, 2, \cdots, q \leq n$, determine if there exists a static output feedback controller such that $\lambda_i, i = 1, 2, \cdots, q$, are the closed-loop eigenvalues. Without loss of generality, $(A, B)$ and $(A, C)$ are assumed to be controllable and observable pairs.

Remark 1: Note that we have set $q \leq n$. This is because it is generally impossible to assign all $n$ poles arbitrarily. Obviously, this happens when $mr < n$. Even when $mr \geq n$, arbitrary pole placement may not be possible. This is because the closed-loop characteristic polynomial, $p(s)$, is multi-affine in $K$, implying that the domain of the mapping from $K$ to $p(s)$ may not cover all the $n$th order polynomials.

Denote the open-loop transfer function by $G(s) = C(sI - A)^{-1}B$, it is clear that the SOFPP problem above is equivalent to finding $K$ such that

$$\det(I + KG(\lambda_i)) = 0, \ i = 1, 2, \cdots, q$$

Theorem 1: The SOFPP problem is NP-hard.

The proof of the result above is done by transforming a known NP-complete problem, the so-called (0,1)-Knapsack problem, into the SOFPP problem. See, e.g., [4] for definitions and examples of P, NP-complete and NP-hard problems.

The (0,1)-Knapsack Problem [4]: Given an integer vector $c = (c_1, \cdots, c_p)^T$, determining if there exists a binary vector $x = (x_1, \cdots, x_p)^T \in \{-1, 1\}^p$ such that $c^T x = 0$.

Lemma 1: Denote two $2 \times 2$ matrices

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

and a scalar function $f(X, H) = \det(I + XH)$. Define

$$H_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \ H_2 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},$$
$$H_3 = \frac{1}{3} \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}, \ H_4 = \begin{bmatrix} -1 & -1 \\ -2 & -4 \end{bmatrix}$$

Then, the following simultaneous equations:

$$f(X, H_j) = 0, j = 1, 2, 3, 4$$

have only two possible solutions:

$$x \in \{-1, 1\}, \ y = -x, \ z = 1 + x, \ w = 1 + x$$

Proof: It is straightforward to verify that

$$f(X, H) = 1 + ax + cy + bz + dw + (xw - yz)(ad - bc)$$
Substituting $H_1$ and $H_2$ into the above, respectively, we obtain
\[ z = 1 + x \text{ and } w = 1 - y. \]
Simplifying $f(X, H)$ gives
\[
f(X, H) = (1 + b + d) + (a + b + ad - bc)x + (c - d - ad + bc)y - 2(ad - bc)xy
\]
Substituting $H_3$ and $H_4$ into the above, respectively, we obtain
\[ x + y = 0 \text{ and } xy = -1. \]
Hence, the only solutions are as given in (7).

Proof of Theorem 1: We first note that, if we allow $c$ to be a vector of rational numbers and replace $c^T x = 0$ by $c^T x + 1 = 0$, the modified $(0,1)$-Knapsack problem is still NP-complete. Indeed, it is obvious that the modified $(0,1)$-Knapsack problem is in P. If it is not NP-complete, then the original problem can be transformed into two modified problems: We set $x_p = 1$ and normalize the given integer vector $c$ by dividing it by $x_p c_p$ (assumed to be non-zero, without loss of generality). Then we obtain the modified $(0,1)$-Knapsack problem with $p - 1$ variables $x_1, \cdots , x_{p-1}$. Similarly, set $x_p = -1$ and repeat the above to obtain another modified $(0,1)$-Knapsack problem. Hence, the modified $(0,1)$-Knapsack problem must be NP-complete.

Next, given a rational vector $c \in \mathbb{R}^p$ for the modified $(0,1)$-Knapsack problem, we want to construct a SOFPP problem such that its solution coincides with the solution of the former. To do so, we choose the matrix $K \in \mathbb{R}^{2p \times 2p}$ as follows:
\[ K = [X_1 \ X_2 \ \cdots \ X_p] \quad (8) \]
where each $X_i$ is a $2 \times 2$ matrix as in (4). Next, we choose $4p + 1$ eigenvalues $\lambda_{i,j}$, $i = 1, \cdots , p$, $j = 1, \cdots , 4$, and $\lambda_{4p+1}$ to be any distinct, real, non-zero rational values and require $G(s)$ to satisfy the following constraints:
\[ G(\lambda_{i,j}) = \begin{bmatrix} 0 & \cdots & 0 & H_{i,j}^T & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^{2p \times 2} \quad (9) \]
for $i = 1, \cdots , p$, $j = 1, \cdots , 4$, where $H_{i,j}$ is the same as the $H_j$ in (5). That is, $G(\lambda_{i,j})$ starts with $(i - 1) \ 2 \times 2$ zero matrices, followed by $H_{i,j}$ and some more $2 \times 2$ zero matrices. Also, $G(\lambda_{4p+1})$ is constructed as follows: The $(2i - 1)$th element in the first column equals to $c_i$, $i = 1, \cdots , p$, and all other elements are zero.

We then obtain $G(s)$ by interpolation as follows: Set $q = 4p + 1$ and order the eigenvalues $\lambda_{i,j}$ as $\lambda_1, \lambda_2, \cdots , \lambda_{q-1}$. Then,
\[ G(s) = \sum_{i=1}^{q} \frac{\lambda_i^q}{s^q} \prod_{j \neq i} \frac{s - \lambda_j}{s - \lambda_i} G(\lambda_i) \quad (10) \]
which is a simple polynomial interpolation. The extra term $\lambda_i^q/s^q$ is used to ensure the strict properness of $G(s)$. Obviously, $n \geq q$ because each entry of $G(s)$ has order $q$.

By construction of $G(\lambda_{i,j})$ and Lemma 1, we know that $\det(I + KG(\lambda_{i,j})) = 0$ at all $i,j$ and if and only if $x_i \in \{-1, 1\}$. Further, it is easy to verify that
\[ \det(I + KG(\lambda_{4p+1})) = c^T x + 1 \]
Hence, we have transformed the given modified $(0,1)$-Knapsack problem into a SOFPP. It is easy to verify that this transformation is done in a polynomial time.

### III. Conclusion
We have given a negative result for the static output feedback pole placement problem. This explains why there is lack of effective solutions to such a fundamental problem. However, it does not imply that the SOFS problem is NP-hard. In fact, it seems that neither problem implies the other in an obvious way. Nevertheless, it seems to strengthens the conjecture that the SOFS problem is NP-hard too.

### REFERENCES