

# Maximal unidirectional perturbation bounds for stability of polynomials and matrices \*

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**Abstract:** Given a stable polynomial or matrix, we consider a class of unidirectional perturbations. In this note, we provide a closed form for the maximal perturbation bounds under which stability is preserved. The results are then used to derive a closed form for the maximal stability box around the coefficients of a nominally stable polynomial.

**Keywords:** Robustness, Algebraic stability tests, Perturbation bounds.

## 1. Introduction

The motivation for this note is derived from ‘stability robustness’ considerations. Given a stable characteristic polynomial or state matrix associated with a linear system, it is of interest to determine the ‘maximal’ perturbation bounds under which stability is preserved. Such bounds are frequently called *measures of robustness*. To this end, Barmish [1] uses Kharitonov’s Theorem [2] to obtain maximal bounds for so-called interval polynomials. In Soh and et al. [3], the  $L_2$ -norm is used to measure coefficient perturbations and once again, maximal bounds are obtained for both continuous-time and discrete-time systems. In the matrix case, an  $L_2$ -norm stability bound is obtained using a Lyapunov matrix associated with the ‘nominal’ system; e.g., see Patel and Toda [4]. This result is later improved by Yedavalli [5], Lee [6] and Zhou and Khargonekar [7] for handling both structured and unstructured perturbations.

For the case when complex variations in the entries of a matrix are allowed, a maximal  $L_2$ -norm stability bound is given in Qiu and Davison [8], Hinrichsen and Pitchard [9] and Martin and Hewer [10]. For the case of real variations in the entries of a matrix, however, maximal perturbation bounds are known only for rather special cases. For example, if the matrix remains symmetric under perturbation, then a maximal  $L_2$  bound is given in Siljak [11] and a maximal  $L_\infty$  bound can be found using the results in Shi and Gao [12].

In this note, we consider another class of matrix perturbations for which a maximal bound can be given – the class of unidirectional perturbations. Indeed, given an  $n \times n$  Hurwitz matrix  $A_0$  (a ‘nominal’ system) and an  $n \times n$  matrix perturbation direction  $A_1$ , we consider the problem of finding the largest interval  $(r_{\min}, r_{\max})$  such that

$$A_r = A_0 + rA_1 \tag{1}$$

is strictly stable for all  $r \in (r_{\min}, r_{\max})$ . A special case of the problem above arises in the generation of a

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Root Locus diagram when one seeks to find the maximal range of a gain  $k$ , around its nominal value  $k_0$ , such that a unity feedback system with open loop transfer function

$$G(s) = \frac{kg(s)}{f(s)}$$

remains stable. In this case, letting  $r = k - k_0$ ,  $p_0(s) = f(s) + k_0g(s)$  and  $p_1(s) = g(s)$ , the associated polynomial problem is to find maximal bounds on a scalar  $r$  such that

$$p_0(s) + rp_1(s) \tag{2}$$

remains stable.

For the class of unidirectional perturbations as in (1) or (2), results on stability are given in Bialas [13] and Marden [14]. In [13], necessary and sufficient conditions are given for the stability of all convex combinations of two stable polynomials or matrices. The difference between the convex combination problem in [13] and the unidirectional perturbation problem associated with (1) or (2) is that the perturbation bounds are not specified in the latter case. To obtain  $r_{\min}$  and  $r_{\max}$  using the results in [13], one must execute an iterative process which involves successively increasing the bound of  $r$ . Here, however, we provide closed forms for  $r_{\min}$  and  $r_{\max}$ . A special case of the unidirectional perturbation problem for polynomials is discussed in [14]: A formula is given for the maximal *symmetric perturbation*; i.e., with the added constraint that  $r_{\min} + r_{\max} = 0$ .

The objectives of this note are two-fold: First we provide closed forms for  $r_{\min}$  and  $r_{\max}$  for the matrix case. Second, for the polynomial case, we also give forms for  $r_{\min}$  and  $r_{\max}$  without assuming symmetric perturbations. The formulae for the polynomial case are then used to provide a closed form description of the maximal stability box around the coefficients of a nominally stable polynomial. This result amounts to an improvement over the result in Barmish [1] where the maximal stability box is found by an iterative process. After introducing definitions and notation in Section 2, we present the main results in Section 3. The proof of the main results are given in Section 4 and the conclusion is provided in Section 5.

## 2. Definitions, notation and a basic lemma

All polynomials and matrices are assumed real and the *Hurwitz matrix* of an  $n$ -th order polynomial

$$p(s) = a_0s^n + a_1s^{n-1} + \cdots + a_n$$

has dimension  $n \times n$  and is denoted by

$$H(p) \doteq \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & & \\ a_0 & a_2 & a_4 & \cdots & & \\ 0 & a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 & a_4 & \cdots & \\ & & & & \ddots & \\ & & & & & a_n \end{bmatrix}$$

$\lambda_{\max}^+(M)$  will denote the maximum positive (real eigenvalue of a square matrix  $M$ . In case  $M$  has no positive eigenvalues, we adopt the convention  $\lambda_{\max}^+(M) = 0^+$ . Similarly,  $\lambda_{\min}^-(M)$  will denote the minimum negative (real) eigenvalue of a square matrix  $M$  and if  $M$  has no negative eigenvalues, we take  $\lambda_{\min}^-(M) = 0^-$ . A set of polynomials  $\mathcal{P}$  (or a set of matrices  $\mathcal{M}$ ) is said to be *strictly stable* if every member of  $\mathcal{P}$  (or  $\mathcal{M}$ ) is strictly stable. Similarly, a set of polynomials  $\mathcal{P}$  (or a set of matrices  $\mathcal{M}$ ) is said to be *nonsingular* if every member of  $\mathcal{P}$  (or  $\mathcal{M}$ ) is nonsingular. For notational convenience, we take the zero polynomial (i.e., the polynomial with all zero coefficients) to be unstable. A linear mapping  $T(\cdot) : \mathbf{R}^{n \times n} \rightarrow$

$\mathbb{R}^{n \times n}$  is said to transform the stability problem into a nonsingularity problem if, for every  $M \in \mathbb{R}^{n \times n}$ ,  $T(M)$  has at least one real eigenvalue and

$$\max\{\text{Re}(\lambda) : \lambda \text{ is an eigenvalue of } M\}$$

and

$$\max\{\nu : \nu \text{ is a real eigenvalue of } T(M)\}$$

have the same sign, or both of them are equal to zero.

The basic lemma to follow makes it clear why a mapping  $T(\cdot)$  having the stated properties above transforms the stability problem into a nonsingularity problem.

**Lemma 2.1.** *Given two  $n \times n$  matrices  $M_0$  and  $M_1$  with  $M_0$  strictly stable, consider the family of matrices*

$$\mathcal{M} \doteq \{M_r = M_0 + rM_1 : r \in (r_1, r_2)\}$$

where  $r_1 < 0$  and  $r_2 > 0$  are specified. Then given any linear mapping  $T(\cdot)$  which transforms the stability problem into a nonsingularity problem, it follows that  $\mathcal{M}$  is strictly stable if and only if

$$T(\mathcal{M}) \doteq \{T(M) : M \in \mathcal{M}\}$$

is nonsingular.

**Proof.** The necessity is obvious since stability of  $M_r$  implies that every eigenvalue of  $T(M_r)$  must have negative real part. Similarly, the sufficiency is immediate from the properties of  $T(\cdot)$ . That is, sufficiency follows from the nonsingularity of every  $T(M_r)$  and the continuity of eigenvalues of  $M_r$  with respect to  $r$ . More specifically, the nonsingularity of every  $T(M_r)$  implies that the real part of every eigenvalue of  $M_r \in \mathcal{M}$  must be non-zero. Notice that the real part of every eigenvalue of  $M_0$  is negative (by stability of  $M_0$ ) and that the eigenvalues of  $M_r$  continuously depend on  $r$ . Therefore, every eigenvalue of  $M_r$  must have negative real part, i.e.,  $M_r$  is strictly stable for all  $M_r \in \mathcal{M}$ .  $\square$

We conclude this section by providing some examples of mappings which transform the stability problem into a nonsingularity problem. The first example is obtained by mapping a matrix  $M$  into the Kronecker sum of  $M$  with itself. Indeed, with  $m = n^2$ , let

$$T(M) = \text{diag}\{\underbrace{M, M, \dots, M}_n\} + [m_{ij}I].$$

To verify that  $T(\cdot)$  has the required property, notice that if  $\lambda_i, i = 1, 2, \dots, n$ , are the eigenvalues of an  $n \times n$  matrix  $M$ , then the eigenvalues of  $T(M)$  are given by  $\lambda_i + \lambda_j, i, j = 1, 2, \dots, n$ ; see [16]. The second example is the linear mapping  $T(\cdot)$  provided in [13] which has  $m = \frac{1}{2}n(n + 1)$ . To illustrate, for a  $3 \times 3$  matrix  $M$ ,

$$T(M) = \begin{bmatrix} m_{11} & m_{12} & 0 & m_{13} & 0 & 0 \\ m_{21} & m_{11} + m_{22} & m_{12} & m_{23} & m_{13} & 0 \\ 0 & m_{21} & m_{22} & 0 & m_{23} & 0 \\ m_{31} & m_{32} & 0 & m_{11} + m_{33} & m_{12} & m_{13} \\ 0 & m_{31} & m_{32} & m_{21} & m_{22} + m_{33} & m_{23} \\ 0 & 0 & 0 & m_{31} & m_{32} & m_{33} \end{bmatrix}$$

is a  $6 \times 6$  matrix. Again, it can be shown that the requirements on  $T(M)$  are satisfied; see [13]. Finally, for the case when  $M$  is symmetric, notice that the identity mapping  $T(M) = M$  satisfies the requirements on  $T(\cdot)$  since the eigenvalues of a symmetric matrix are all real.

### 3. Main results

In this section, we provide a solution to the unidirectional perturbation problem posed in Section 1. Theorem 3.1 below provides formulae for  $r_{\min}$  and  $r_{\max}$  for the matrix case. Theorem 3.2 is the analogue of Theorem 3.1 for the polynomial case. The proofs of the results are given in Section 4.

**Theorem 3.1.** *Let  $T(\cdot): \mathbf{R}^{n \times n} \rightarrow \mathbf{R}^{m \times m}$  be any linear mapping which transforms the stability problem into a nonsingularity problem. Then, given two  $n \times n$  matrices  $M_0$  and  $M_1$  with  $M_0$  strictly stable, consider the family of matrices*

$$\mathcal{M} \doteq \{M_r = M_0 + rM_1: r \in (r_{\min}, r_{\max})\}. \quad (3)$$

Then, the maximal range  $(r_{\min}, r_{\max})$  for  $\mathcal{M}$  to be strictly stable is given by

$$r_{\min} = \frac{1}{\lambda_{\min}^-(-T(M_0)^{-1}T(M_1))}, \quad r_{\max} = \frac{1}{\lambda_{\max}^+(-T(M_0)^{-1}T(M_1))}. \quad (4)$$

**Theorem 3.2.** *Given an  $n$ -th order polynomial  $p_0(s)$  and an  $m$ -th order polynomial  $p_1(s)$  with  $m \leq n$  and  $p_0(s)$  strictly stable, consider the family of  $n$ -th order polynomials*

$$\mathcal{P} \doteq \{p_r(s) = p_0(s) + rp_1(s): r \in (r_{\min}, r_{\max}); \deg p_r(s) = n\}. \quad (5)$$

Then, the maximal range  $(r_{\min}, r_{\max})$  for  $\mathcal{P}$  to be strictly stable is given by

$$r_{\min} = \max \left\{ r_{\min}^*, \frac{1}{\lambda_{\min}^-(-H(p_0)^{-1}H(p_1))} \right\}, \quad r_{\max} = \min \left\{ r_{\max}^*, \frac{1}{\lambda_{\max}^+(-H(p_0)^{-1}H(p_1))} \right\}, \quad (6)$$

where

$$r_{\min}^* \doteq \begin{cases} -\frac{a_0(p_0)}{a_0(p_1)} & \text{if } m = n \text{ and } a_0(p_0)a_0(p_1) > 0, \\ -\infty & \text{otherwise,} \end{cases} \quad (7)$$

$$r_{\max}^* \doteq \begin{cases} -\frac{a_0(p_0)}{a_0(p_1)} & \text{if } m = n \text{ and } a_0(p_0)a_0(p_1) < 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (8)$$

and (for the purpose of conformability of matrix multiplication),  $H(p_1)$  is obtained by treating  $p_1(s)$  as an  $n$ -th order polynomial.

**Corollary 3.3.** *Consider a strictly stable polynomial  $p_0(s)$  given by*

$$p_0(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s + a_n \quad (9)$$

and a set of nonnegative weights  $\omega_i^L, \omega_i^U, i = 1, 2, \dots, n$ . For any  $r > 0$ , we define an interval polynomial

$$\mathcal{P}_r \doteq \left\{ s^n + \sum_{i=1}^n \gamma_i s^{n-1}: a_i - \omega_i^L r < \gamma_i < a_i + \omega_i^U r, i = 1, 2, \dots, n \right\} \quad (10)$$

Then, the maximal  $r$ , call it  $r_{\max}$ , such that all polynomials in  $\mathcal{P}_r$  are strictly Hurwitz for all  $r \leq r_{\max}$  is given by

$$r_{\max} = \min \left\{ \frac{1}{\lambda_{\max}^+(-H(p_0)^{-1}H(p_i))}: i = 1, 2, 3, 4 \right\} \quad (11)$$

where  $H(p_i)$ ,  $i = 1, 2, 3, 4$ , are the Hurwitz matrices of the following four Kharitonov-like polynomials:

$$\begin{aligned} p_1(s) &= \omega_1^U s^{n-1} + \omega_2^U s^{n-2} - \omega_3^L s^{n-3} - \omega_4^L s^{n-4} + \omega_5^U s^{n-5} + \omega_6^U s^{n-6} + \dots, \\ p_2(s) &= \omega_1^U s^{n-1} - \omega_2^L s^{n-2} - \omega_3^L s^{n-3} + \omega_4^U s^{n-4} + \omega_5^U s^{n-5} - \omega_6^L s^{n-6} + \dots, \\ p_3(s) &= -\omega_1^L s^{n-1} - \omega_2^L s^{n-2} + \omega_3^U s^{n-3} + \omega_4^U s^{n-4} - \omega_5^L s^{n-5} - \omega_6^L s^{n-6} + \dots, \\ p_4(s) &= -\omega_1^L s^{n-1} + \omega_2^U s^{n-2} + \omega_3^U s^{n-3} - \omega_4^L s^{n-4} - \omega_5^L s^{n-5} + \omega_6^U s^{n-6} + \dots, \end{aligned}$$

respectively.

#### 4. Proofs of the main results

The following lemma is instrumental in the proofs of Theorems 3.1 and 3.2.

**Lemma 4.1.** Suppose  $M_0$  and  $M_1$  are two  $n \times n$  matrices with  $M_0$  nonsingular. Let  $\mathcal{M}$  be the family of perturbed matrices given by (3). Then, the maximal range  $(r_{\min}, r_{\max})$  for  $\mathcal{M}$  to be nonsingular is given by

$$r_{\min} = \frac{1}{\lambda_{\min}^-(-M_0^{-1}M_1)}, \quad r_{\max} = \frac{1}{\lambda_{\max}^+(-M_0^{-1}M_1)}. \quad (12)$$

**Proof.** Notice that for  $r \neq 0$  and  $\delta_r = 1/r$ ,

$$M_r = rM_0 \left( \frac{1}{r}I + M_0^{-1}M_1 \right) = rM_0 (\delta_r I - (-M_0^{-1}M_1)).$$

Therefore,  $M_r$  is nonsingular for all  $r \in (r_{\min}, r_{\max})$  if and only if  $\delta_r$  is not an eigenvalue of  $-M_0^{-1}M_1$  for all  $\delta_r < 1/r_{\min}$  and  $\delta_r > 1/r_{\max}$ . Hence, the maximal range  $(r_{\min}, r_{\max})$  for every  $M_r \in \mathcal{M}$  to be nonsingular is given in (12).  $\square$

**Proof of Theorem 3.1.** From Lemma 2.1, we know that  $\mathcal{M}$  is strictly stable if and only if  $T(\mathcal{M})$  is nonsingular. Then, we obtain (4) by observing that

$$T(\mathcal{M}) = \{T(M_0) + rT(M_1) : r \in (r_{\min}, r_{\max})\}$$

and applying Lemma 4.1.  $\square$

**Proof of Theorem 3.2.** By definition, it follows that  $r_{\min} \geq r_{\min}^*$  and  $r_{\max} \leq r_{\max}^*$  in order to guarantee that every  $p_r(s) \in \mathcal{P}$  is  $n$ -th order. Without loss of generality, we assume  $r_{\min} \geq r_{\min}^*$  and  $r_{\max} \leq r_{\max}^*$  in the rest of the proof. Similar to the proof of Theorem 3.1, we first show that  $\mathcal{P}$  is strictly stable if and only if  $H(p_r)$  is nonsingular for all  $p_r(s) \in \mathcal{P}$ . Indeed, the necessity is immediate from the Routh–Hurwitz criteria. To establish sufficiency, we proceed by contradiction. Suppose there exists some  $\alpha \in (r_{\min}, r_{\max})$  such that  $p_\alpha(s)$  is not strictly stable; we need to show that there exists some  $\beta \in (r_{\min}, r_{\max})$  such that  $H(p_\beta)$  is singular. Since the zeros of  $p_0(s)$  all have negative real parts and the zeros of  $p_r(s)$  continuously depend on  $r$ , we conclude that there must exist some  $\beta$  between 0 and  $\alpha$  (implying that  $\beta \in (r_{\min}, r_{\max})$ ) such that either  $a_n(p_\beta) = 0$  or  $p_\beta(s)$  has a pair of purely imaginary zeros  $\pm j\omega$ ,  $\omega \geq 0$ . In either case, we claim that  $H(p_\beta)$  is singular. To establish this claim we apply the Orlando Formula (see, for example, pp. 190–197 of [18]) and obtain

$$\det H(p_\beta) = (-1)^{n(n-1)/2} a_0^{n-1}(p_\beta) a_n(p_\beta) \prod_{1 \leq i < k \leq n} (z_i + z_k)$$

where  $z_j$  are the zeros of  $p_\beta(s)$ ,  $j = 1, 2, \dots, n$ . From the formula above, it is clear that  $\det H(p_\beta)$  must vanish. Now, we simply apply Lemma 4.1 to  $H(p_0)$  and  $H(p_1)$  to obtain (6).  $\square$

**Proof of Corollary 3.3.** For  $i = 1, 2, 3, 4$ , let

$$r_{\max}^i \doteq \min\{r^i \geq 0: p_0(s) + r^i p_i(s) \text{ is not strictly stable}\}, \quad i = 1, 2, 3, 4.$$

Using the results in Barmish [1], we have

$$r_{\max} = \min\{r_{\max}^i: i = 1, 2, 3, 4\}.$$

Now, Theorem 3.2 can be used to determine  $r_{\max}^i$ . Since  $a_0(p_i) = 0$  for  $i = 1, 2, 3, 4$ ,  $r_{\max}^{i*} = +\infty$ . Therefore,

$$r_{\max}^i = \frac{1}{\lambda_{\max}^+(-H(p_0)^{-1}H(p_i))}$$

and we obtain (11).  $\square$

## 5. Conclusion

In this note, formulae were given for the maximal unidirectional perturbation bounds for a strictly stable polynomial or matrix. It should be noted that the results for the polynomial case (Theorem 3.2) can be easily extended to handle discrete-time systems via a bilinear transformation. That is, the linear coefficient structure is preserved under such a transformation. However, the results for the matrix case do not readily extend to discrete-time systems. Another limitation of the results is that they only handle unidirectional perturbations; see [19] for a discussion of problems encountered in the multidimensional case. Note, however, that even for multidimensional perturbations, it is still possible to transform the stability problem into a nonsingularity problem. If it turns out that the nonsingularity problem is more tractable than the stability problem, then more general results will be attainable in the future. There is a tradeoff, however, because transformation of the stability problem into a nonsingularity problem usually increases the dimension of the matrices; see the examples in Section 2.

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