Performance Analysis for Subband Identification

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Abstract

The so-called *subband identification* method has been introduced recently as an alternative method for identification of finite-impulse response systems with a large tap size. It is known that this method can be more numerically efficient than the classical system identification method. However, no results are available to quantify the advantages of the subband method. This paper offers a rigorous study of the performance of the subband method. More precisely, we aim to compare the performance of the subband identification method with the classical (full-band) identification method. The comparison is done in terms of asymptotic residual error, asymptotic convergence rate and computational cost. It is shown that, by properly choosing the filterbanks, the number of parameters in each subband, the number of subbands and the downsampling factor, the two identification methods can have compatible asymptotic residual errors and convergence rate. However, for applications where a high order model is required, the subband method is more numerically efficient. We study two types of subband identification schemes, one using critical-sampling and another one using oversampling. The former is simpler to use and easier to understand, whereas the latter involves more design problems but give further computational savings.

I. Introduction

This paper studies the use of the so-called *subband identification* method for identification of linear time-invariant systems. This is a relatively new approach and is intended to replace the classical linear system identification technique for applications where the system model is an finite-impulse-response (FIR) filter with a large tap size.

The key idea of the subband identification method is to subdivide the given input-output signals of the system into a number of subbands in the frequency domain by using filterbanks and down-samplers and identify subband models of the system. There are two main types of subband identification schemes, one using critical-sampling and another using oversampling. Critical-sampling refers to the scheme where the number of subbands is equal to the down-sampling factor, whereas in the oversampling scheme the number of subbands is more than the down-sampling factor. For comparative purposes, we will refer the classical system identification method as full-band identification. The reader is referred to [1], [2] for an introduction to full-band identification. In comparison with the full-band identification method, the subband method offers two main advantages:

- Lower computational cost: This is mainly because of the fact that the signal rate for each subband is much slower and the each subband model requires a much smaller tap size. These features are partly counter-balanced by the extra computation required for forming subband signals. But
through a careful choice of design parameters (including the number of subbands, filterbanks, and subband models), significant computational saving can be achieved using subband identification.

- Better numerical properties: Each subband model requires a much smaller tap size compared with the full-band model. As a result of it, much better numerical stability can be obtained in the subband case for the identified model parameters.

These advantages have been recognized in various ways in the literature. Subband identification has been used in speech signal processing applications where long FIR models are often required. See, for example, [3], [4], [5], [6], [7], [8], [9]. In general, there is crossing of aliases between subband channels due to filter overlaps; see [3]. There are two main approaches to cope with this problem. The first approach uses critical sampling by applying non-overlapping filterbanks which result in spectral gaps between subbands; see [4]. In order to cope with this problem, the paper [5] used auxiliary channels, with the corresponding extra computational cost. Finally, the paper [6] introduced the use of adaptive cross-terms between subbands. However, these cross-terms increase the computational cost and the slow the convergence rate. The second approach uses oversampling. For example, the paper [7] analyses the existence of exact solutions of the identification problem without cross-terms. The paper [8] use the gabor expansion to design the filterbanks, which restrict the flexibility for the filterbanks. Finally, [9] analyzes the convergence rate for the oversampling case under a number of simplifying assumptions.

However, the available research work in the literature is far from enough to quantify the advantages of the subband identification method. Since there are a number of design parameters available for subband identification (as pointed out earlier) and there are a number of performance indices (such as computational cost, asymptotic estimation error and convergence rate), the subband method may or may be outperform the full-band method when measured on a particular performance index. As a result of it, exact comparison with the full-band method is rather difficult. To make the problem worse, it is quite non-trivial to know how to optimize the design parameters.

The purpose of this paper is to deal with the aforementioned matters. That is, we aim to quantify the advantages of the subband identification method and study the problem of how to optimize the subband design parameters. To this end, we first give an introduction to the subband identification approach. We will show why this approach can be more numerically efficient than the full-band approach for applications where a high order model is required. Secondly, we
compare the behavior of the subband identification technique with the traditional full-band identification technique in terms of the asymptotic residual error, asymptotic convergence rate and computational cost. This comparison is used to demonstrate the potential power of the subband technique. We study the two types of subband identification schemes, i.e. critical-sampling and oversampling. The former scheme is simpler to use and easier to understand, whereas the latter scheme involves more design problems but give further computational savings.

This paper is a companion paper of [10] where we studied the asymptotic properties of a subband identification scheme. The results of [10] are used in this paper. Although we have tried to make this paper independent of [10], readers who are interested in detailed studies may find it necessary to read [10] as well. This paper is also partly based on our previous work [11], [12], [13]. In [11] we studied the critical sampling case, in [12] we studied the design of the filterbanks that minimize the asymptotic residual error in both cases (i.e. critical-sampling and oversampling) and [13] is an earlier and simpler version of this paper.

The outline of the paper is as follows. In section II, we give a review of the full-band identification technique. We also discuss results in our previous paper [10] that are relevant to subband identification. In section III, we introduce the subband identification method, together with the different analysis approaches. In section IV we deal with the conditions for the subband to be decoupled. In section V, we introduce the formal assumptions. In section VI we introduce expressions for the performance indexes mentioned above (i.e. asymptotic residual error, asymptotic convergence rate and computational cost). In section VII we consider design issues to optimize the performance indexes in the critical-sampling case, and in section VIII, we do the same for the oversampling case. Finally, in section IX we give an example to illustrate the performance of the subband identification technique.

II. FULL-BAND IDENTIFICATION

In this section, we give a short review of the well-known theory of linear system identification [1], [2], which we call the full-band identification approach. The setting of the identification problem is illustrated in figure 1, where \( u(t) \) is the input signal, \( w(t) \) is the output of the system, \( y(t) \) is the measured output, \( v(t) \) is the process noise, \( g(q) \) (\( q \) is the forward shift operator, i.e. \( qz(t) = x(t + 1) \)) is the transfer function of the system, \( \hat{g}(q, \theta) \) is the model of the system, and \( \hat{e}(t, \theta) \) is the prediction error, \( \theta \in \mathbb{R}^n \) represents the parameters of the model.
A. Some Definitions on Random Processes

The following definitions are needed for our development. See [10] for a more detailed exposition.

Convention 1: All the random processes and linear systems considered are assumed to be scalar and complex, unless explicitly specified. The superscript $^*$ denotes complex conjugate. $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set of positive integers.

Definition 1: Let $p \in \{1, 2\}$ and let $\{x(t)\}$ and $\{y(t)\}, t \in \mathbb{Z}$ be two random processes. They are said to be strongly ergodic of $p$-th order if the following holds:
1. They have uniformly bounded $4p$-th moments.
2. For any $A, B \in \mathbb{N}$, there exists $C > 0$ such that,
\[
\mathcal{E} \left\{ C_{xy}^{2p}(A, B, a, b, T) \right\}^{1/p} \leq \frac{C}{T}, \quad \forall a, b \in \mathbb{Z}
\]
where
\[
C_{xy}(A, B, a, b, T) = \frac{1}{T} \sum_{t=1}^{T} x^*(At + a)y(Bt + b) - \mathcal{E} \left\{ x^*(At + a)y(Bt + b) \right\}
\]

Definition 2: Let $\{x(t)\}$ and $\{y(t)\}, t \in \mathbb{Z}$, be two random process. They are said to be jointly quasi-stationary if
1. They have uniformly bounded second moments (i.e., there exists $M_x > 0$ such that $\mathcal{E}\{x(t)^2\}^{1/2} < M_x, \forall t \in \mathbb{Z}$ and similarly for $\{y(t)\}$).
2. For all $\tau \in \mathbb{Z}$, the following limit exists
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E} \{ x^*(t)y(t + \tau) \}
\]
If they further satisfy that, for all $A, B \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, the following limit exists
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E} \{ x(At + a)y^*(Bt + b) \}
\]
then they are said to be \textit{jointly quasi-stationary by phases}. If they further satisfy that, for all $D \in \mathbb{N}$ and $\tau \in \mathbb{Z}$,

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}\{x^*(Dt)y(Dt + \tau)\} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}\{x^*(t)y(t + \tau)\}
$$

then they are said to be \textit{almost stationary}. Finally, if they further satisfy that, for all $t, \tau \in \mathbb{Z}$,

$$
\mathcal{E}\{x^*(t)y(t + \tau)\} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}\{x^*(t)y(t + \tau)\}
$$

then they are said to be \textit{stationary}.

\textit{Definition 3}: A random process is strongly ergodic (or quasi-stationary, etc.) if it is jointly strongly ergodic (or jointly quasi-stationary, etc.) with itself. Also, a collection of random processes is strongly ergodic (or quasi-stationary, etc.) if every two random processes in the collection (including a random process with itself) are jointly strongly ergodic (or jointly quasi-stationary, etc.).

\textit{Definition 4}: Let $\{x(t)\}, t \in \mathbb{Z}$, be a quasi-stationary random process. The \textit{auto-correlation} of $\{x(t)\}$ is defined by

$$
R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}\{x^*(t)x(t + \tau)\}
$$

(4)

The \textit{power} of $\{x(t)\}$ is defined by

$$
S_x = R_x(0)
$$

(5)

and the \textit{power spectrum} of $\{x(t)\}$ is defined by

$$
\Phi_x(\omega) = \sum_{\tau=-\infty}^{\infty} R_x(\tau) e^{-j\omega \tau}
$$

(6)

provided the infinite sum exists.

\textbf{B. Formal Assumptions}

\textit{Assumption 1}: The signals $\{u(t)\}, \{w(t)\}$ and $\{v(t)\}$ satisfy:

1. The collection formed by the signals $\{u(t)\}, \{w(t)\}$ and $\{v(t)\}$ is strongly ergodic of second order and quasi-stationary.
2. $\{v(t)\}$ is independent of $\{u(t)\}$ and $\{w(t)\}$.
3. $\{v(t)\}$ is stationary and has zero mean.
4. $\tau R_u(\tau), \tau R_v(\tau) \in l_1(\mathbb{Z})$. 
5. There exists $\varepsilon > 0$ such that $\Phi_u(\omega) \geq \varepsilon$, $\forall \omega \in [-\pi, \pi]$.

**Assumption 2:** The model $\tilde{g}(q, \theta)$ is an FIR model of tap size $n_f$. The set of parameters is assumed to satisfy $\theta \in \mathcal{D} \subset \mathbb{R}^{n_f}$, where $\mathcal{D}$ is assumed to be compact (i.e. close and bounded). Define

$$V_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} |\hat{\phi}(t, \theta)|^2$$

(7)

It is further required that there exists $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$,

$$\arg \min_{\theta \in \mathbb{R}^{n_f}} V_N(\theta) \in \text{int}(\mathcal{D})$$

where $\text{int}(\mathcal{D})$ denotes the interior of $\mathcal{D}$.

Denote

$$\theta_N = \arg \min_{\theta \in \mathcal{D}} V_N(\theta)$$

(8)

Then, $\theta_N$ is computed using the least-squares (LS) algorithm:

$$\theta_N = [R_N]^{-1} \frac{1}{N} \sum_{t=1}^{N} \varphi^*(t)y(t)$$

(9)

where

$$R_N = \frac{1}{N} \sum_{t=1}^{N} \varphi^*(t) \varphi(t)$$

(10)

$$\varphi(t) = [u(t), u(t-1), ..., u(t - (n_f - 1))]$$

(11)

The superscript $^*$ above denotes transpose conjugate.

C. **Performance Indexes**

**Notation 1:** Define the prediction error $\tilde{\omega}(t, \theta) = w(t) - \hat{w}(t, \theta)$ and denote its power by $S_{\tilde{\omega}}(\theta)$. Let

$$S_{\tilde{\omega}:\text{opt}} = \min_{\theta \in \mathcal{D}} S_{\tilde{\omega}}(\theta)$$

**Asymptotic residual error:** We know from [10], that, under the assumptions 1 and 2,

$$\lim_{N \to \infty} S_{\tilde{\omega}}(\theta_N) = S_{\tilde{\omega}:\text{opt}} \text{ w.p.1}$$

(13)

In this paper, we are not interested in a bound on $S_{\tilde{\omega}:\text{opt}}$, but in a bound on the difference between the errors of the full-band and the subband methods. This is done in section VI.

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Asymptotic convergence rate: From [10], we have the following result: Suppose assumptions 1 and 2 are satisfied. If \( n_f \) and \( N \) are large enough, and \( S_{\hat{w}, \text{opt}} \) is small enough, then

\[
\mathcal{E} \{ S_{\hat{w}, \text{dir}}(N) \} \approx \frac{n_f}{N} S_v
\]

where \( S_{\hat{w}, \text{dir}}(N) = S_{\hat{w}}(\theta_N) - S_{\hat{w}, \text{opt}} \) and \( S_v \) is the power of the noise signal \( v(t) \).

Computational cost: We assume that the LS solution (9) is implemented using a recursive least-squares (RLS) algorithm. The computational cost depends on the particular RLS algorithm used, ranging from \( O(n_f) \) to \( O(n_f^2) \). Fast algorithms should be used for applications with a high order model, but they tend to be difficult to implement and sensitive numerically; see [14] for a summary of RLS algorithms. For comparative purposes, we consider a reasonably efficient algorithm in [2]. For that algorithm, the computational cost, measured in terms of number of multiplications per sample, is given by

\[
\Psi = 9n_f
\]

III. Subband Identification

A. Overview of the Method

The scheme of subband identification is depicted in figure 2. As we mentioned in Introduction, the idea of subband identification is to split both signals \( u(t) \) and \( y(t) \) into \( M \) subbands using analysis filterbanks \( l(q) = [l_1(q), ..., l_M(q)]^T \) and \( h(q) = [h_1(q), ..., h_M(q)]^T \), respectively. These subband signals are down-sampled and the results are denoted by two vector signals \( U(t) = [U_1(t), ..., U_M(t)]^T \) and \( Y(t) = [Y_1(t), ..., Y_M(t)]^T \). The subband parametric model \( \hat{G}(q, \theta) = [\hat{G}_{ij}(q, \theta)]_{i,j=1}^M \) is identified in order to reconstruct \( \hat{W}(t, \theta) = [\hat{W}_1(t, \theta), ..., \hat{W}_M(t, \theta)]^T \) which is the subband equivalent of \( \hat{w}(t, \theta) \). The prediction error \( \hat{V}(t, \theta) = [\hat{V}_1(t, \theta), ..., \hat{V}_M(t, \theta)]^T = Y(t) - \hat{W}(t, \theta) \) is then formed. Finally, an up-sampler and a synthesis filterbank \( f(q) = [f_1(q), ..., f_M(q)]^T \) is used to reconstruct \( \hat{w}(t, \theta) \).

For analysis purposes, we denote by \( W(t) = [W_1(t), ..., W_M(t)]^T \) the downsampled version of \( h(q)w(t) \). We define \( \hat{W}(t, \theta) = W(t) - \hat{W}(t, \theta) = [\hat{W}_1(t, \theta), ..., \hat{W}_M(t, \theta)]^T \) and denote by \( \hat{w}(t, \theta) \), the signal obtained by upsampling \( \hat{W}(t, \theta) \) and then filtering it with \( f(q) \). We also denote the downsampled version of \( h(q)v(t) \) by \( V(t) = [V_1(t), ..., V_M(t)]^T \).

We now can give some intuition why subband identification can be advantageous compared to full-band identification. To this end, we consider the three key properties as mentioned above: asymptotic residual error, convergence rate, and computational cost. For simplicity, we assume
the critical-sampling case where $D = M$. First, we point out that in each subband, the frequency response of the system is much smoother. Therefore, each subband model requires a much lower tap size, compared to the full-band model. It turns out that it is reasonable to take the tap size for each subband model to be $n_s = n_f / M$ plus a small constant which we will ignore here. This assures that the subband identification has a negligible asymptotic residual error. Secondly, we note that the number of samples in each subband is reduced by a factor of $M$. This means that the convergence rate remains roughly the same. Thirdly, the computational cost will be $M$ times cheaper because both the tap size and the number of samples in each subband is reduced by a factor of $M$ and that there are $M$ subbands. For RLS algorithms with complexity more than $O(n_f)$, there will be more savings offered by the subband approach. The analysis above shows clearly the advantage of the subband approach. However, we have not taken into account the extra computation required for forming the subband signals. As we will see later that this is a major design issue which determines the efficiency of the subband approach. Nevertheless, we have seen the possibility of using subbands to save computations.

B. Analysis Methods for Subband Identification

There are three methods for analyzing the subband scheme, when the signals are assumed to belong to $l_2(\mathbb{Z})$. The first one is the direct representation and corresponds to the scheme in figure 2. The other two equivalent representations are the alias representation and the polyphase representation [15]. We will just use the direct representation and the alias representation.

**Direct representation:** The scheme is depicted in figure 2. We will simplify the analysis of this representation. A more complete analysis involves the use of frames [16], [17]. Let the
input signals $u(t), v(t) \in l^2(\mathbb{Z})$ and the impulse response of the filters $l(t), h(t), f(t) \in l^2_2(\mathbb{Z})$. We define the following linear maps

$$T_l : l^2(\mathbb{Z}) \rightarrow l^2_2(\mathbb{Z}) : u(t) \mapsto U(t)$$

$$T_h : l^2(\mathbb{Z}) \rightarrow l^2_2(\mathbb{Z}) : g(t) \mapsto Y(t)$$

$$T_f^* : l^2_2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z}) : \hat{V}(t, \theta) \mapsto \hat{v}(t, \theta)$$

where the superscript * denotes the adjoint operator. We denote their induced norms by

$$\|T_l\| := \sup_{\|x(t)\|_2 = 1} \|T_l x(t)\|_2$$

$$\|T_h\| \text{ and } \|T_f^*\|$$ are defined in a similar way.

It is straightforward to verify that the conditions for the subband identification scheme to satisfy $\hat{v}(t, \theta) = v(t)$, for all $u(t), v(t) \in l^2(\mathbb{Z})$, are

$$T_G(\theta)T_l = T_h T_f^*$$

(16)

$$T_f^* T_h = I$$

(17)

where $T_f$ and $T_G(\theta)$ are the operators associated with the transfer functions $g(q)$ and $\hat{G}(q, \theta)$ respectively and $I$ is the identity operator.

**Alias representation:** In this representation, we will use $z$-transform representations for systems and signals. We will also use the notation

$$\Omega = e^{-j\frac{2\pi}{D}}$$

(18)

The scheme is depicted in figure 3, where

$$U_A(z) = [u(z) \ u(\Omega z) \ \cdots \ u(\Omega^{D-1} z)]^T$$

is the alias representation of the signal $u(z)$ (similar representations for $v(z)$, $w(z)$ and $y(z)$),

$$H_A(z) = \frac{1}{D} \begin{bmatrix}
    h_1(z) & h_1(\Omega z) & \cdots & h_1(\Omega^{D-1} z) \\
    h_2(z) & h_2(\Omega z) & \cdots & h_2(\Omega^{D-1} z) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_M(z) & h_M(\Omega z) & \cdots & h_M(\Omega^{D-1} z)
\end{bmatrix}$$

is the so-called alias matrix of the analysis filterbank $h(z)$ (similar notation for $l(z)$) and

$$G_A(z) = \text{diag}\{g(z), g(\Omega z), \ldots, g(\Omega^{D-1} z)\}$$
can be interpreted as the alias representation of the plant model \( g(z) \).

In this approach, conditions (16) and (17) can be expressed as

\[
\hat{G}(z^D, \theta) L_A(z) = H_A(z) G_A(z) \\
\hat{f}^T(z) H_A(z) = [1, 0, \ldots, 0]
\]

IV. Decoupling Condition

In general, the condition (16) requires the subband model \( \hat{G}(q, \theta) \) to be a full matrix. This complicates the identification process substantially. To simplify the computation, the filterbank \( h(q) \) should be designed to reduce the number of non-zero terms in \( \hat{G}(q, \theta) \). In the following, we analyze the ideal case where the subband channels are decoupled which implies that \( \hat{G}(q, \theta) \) is a diagonal matrix. We have the following result:

**Theorem 1:** Consider the subband identification scheme of figure 2. Let \( h(q) \) be defined such that, for each \( m = 1, \ldots, M \), \( h_m(e^{j\omega}) \) has its support on a (possibly disjoint) segment \( \sigma^h_m \subset [-\pi, \pi] \), i.e.

\[
h_m(e^{j\omega}) \begin{cases} 
\neq 0 & \omega \in \sigma^h_m \\
= 0 & \omega \notin \sigma^h_m
\end{cases}
\]

Let \( \ell(q) \) be defined in a similar way such that \( \sigma^h_m \subset \sigma^l_m \). If there exists a branch \( \beta^D_m(e^{j\omega}) \) (see appendix A for definition) such that \( \beta^D_m(e^{j\omega}) \) maps \([\pi, \pi] \) into a segment \( \sigma_m \subset [-\pi, \pi] \) satisfying

\[
\sigma^h_m \subset \sigma^l_m \subset \sigma_m
\]
then all the diagonal matrices \( \tilde{G}(q, \theta) \) that satisfy (16) are given by \( \tilde{G}(q) \), where

\[
\tilde{G}(q) = \text{diag}\{\tilde{G}_m(q), (m = 1, \ldots, M)\}
\]

\[
\tilde{G}_m(e^{j\omega}) = \begin{cases} 
\frac{h_m(e^{j\omega})}{h_m(\beta_m e^{j\omega})} g(\beta_m e^{j\omega}) & \omega : \beta_m e^{j\omega} \in \sigma_m^h \\
0 & \omega : \beta_m e^{j\omega} \in \sigma_m^l \setminus \sigma_m^h \\
\text{undefined} & \omega : \beta_m e^{j\omega} \in \sigma_m \setminus \sigma_m^l
\end{cases}
\]

The converse is also true, i.e., if the subband identification scheme yields a diagonal solution, then the analysis filterbank \( h(q) \) satisfies the conditions above, and therefore, all the diagonal solutions satisfy (22)-(23).

This result is a slight generalization of the result in [7]. The only modification we have is that in [7], the filterbanks \( l(q) \) and \( h(q) \) are assumed to be identical. See appendix II for proof.

**Corollary 1:** If \( h_m(q) = l_m(q) \), and \( \sigma_m = \sigma_m^h = \sigma_m^l \), then

\[
\tilde{G}_m(t) = (\Gamma_m(\tau) * g(\tau))(Dt)
\]

where \( \Gamma_m(q) \) is the ideal bandpass filter

\[
\Gamma_m(e^{j\omega}) = \begin{cases} 
D & \omega \in \sigma_m \\
0 & \omega \notin \sigma_m
\end{cases}
\]

See appendix II for proof.

**The support of the ideal analysis filterbanks:**

From (21), it is clear that decoupling of subbands implies that the supports \( \sigma_m^h \) and \( \sigma_m^l \) have measure less or equal to \( 2\pi/D \). The filters \( h_m(e^{j\omega}), m = 1, \ldots, M, \) are typically approximated using FIR filters. Since the tap size has a negative influence on the computational cost, it is desirable to minimize it. In order to do that, it is required that \( \sigma_m^h = \sigma_m^l = \sigma_m \) and that \( \sigma_m \) be a connected subset of \( [-\pi, \pi] \), of measure \( 2\pi/D \). Also, (20) requires that the union of \( \sigma_m \) equals to \( [-\pi, \pi] \). In summary, the filters \( h(q) \) and \( l(q) \) need to meet the following conditions:

C1: \( \sigma_m^h = \sigma_m^l = \sigma_m \), for each \( m = 1, \ldots, M \).

C2: \( \sigma_m \) is a connected subset of \( [-\pi, \pi] \), for each \( m = 1, \ldots, M \).

C3: \( \sigma_m \) has a measure equal to \( 2\pi/D \), for each \( m = 1, \ldots, M \).

C4: \( \bigcup_{m=1}^{M} \sigma_m = [-\pi, \pi] \).

If \( D = M \), these three requirements imply that \( h_m(q) \) are simply a set of non-overlapping ideal bandpass filters with the passband having a length of \( 2\pi/D \). In the case where \( D < M \), the filters \( h_m(q) \) are still ideal bandpass filters with the same bandwidth for the pass band, but in this case...
their supports are allowed to overlap. It is clear from this discussion that the \( h_m(q) \) require lower order FIR filters to approximate in the oversampling case than the critical-sampling case.

Remark 1: Since the sets \( \sigma_m \) are connected sets of measure \( 2\pi/D \), it follows that the filter \( h(q) \) and \( l(q) \) are complex. For the critical sampling case \( (D = M) \), the sets \( \sigma_m \) can be split in two symmetric sets of measure \( \pi/D \), so as to allow the use of real filters. In this case, in terms of computational cost, the extra length of the real filters, will be compensated by the fact that the filtering consists of multiplications of real numbers instead of complex. But it is not possible to use real filters in the oversampling case, since the sets \( \sigma_m \) need to be the range of some branch \( \beta_m^D(e^{j\omega}) \) and it can be verified that 2 sets of symmetric branches cannot have partial overlapping (as needed for oversampling).

V. FORMAL ASSUMPTIONS

In this section we will introduce the formal assumptions for the analysis of the subband identification method.

Assumption 3: The signals \( \{u(t)\} \) and \( \{v(t)\} \) satisfy assumption 1 but are strengthened as follows:

1. The collection formed by the signals \( \{u(t)\} \) and \( \{v(t)\} \) is quasi-stationary by phases.
2. The signal \( \{u(t)\} \) is almost stationary.

Assumption 4: The two analysis filterbanks are the same (i.e. \( l(q) = h(q) \)). The filters \( h_m(q) \) are FIR with the same tap size \( l_h \) and the filters \( f_m(q) \) are FIR with the same tap size \( l_f \). The synthesis filterbank \( f(q) \) satisfies the perfect reconstruction condition (17). The impulse responses satisfy \( \text{th}_m(t), f_m(t) \in l_1(\mathbb{Z}), m = 1,...,M \). There exists \( \alpha > 0 \) such that

\[
\frac{1}{D} \sum_{d=0}^{D-1} \left| \hat{h}_m(\Omega^d e^{j\frac{\pi}{D}}) \right|^2 \geq \alpha, \quad \forall m = 1,...,M, \forall \omega \in [-\pi, \pi]
\]

where \( \Omega \) is given by (18).

Assumption 5: The subband model is a diagonal matrix \( \hat{G}(q,\theta) = \text{diag}\{\hat{G}_m(q,\theta_m), \quad m = 1,...,M\} \), where \( \theta = [\theta_1^T,...,\theta_M^T]^T \). Each \( \hat{G}_m(q,\theta_m) \) satisfies assumption 2, i.e. \( \hat{G}_m(q,\theta_m) \) is an FIR model of tap size \( n_s \) with \( \theta_m \in \mathcal{D} \subset \mathbb{R}^{n_s} \), where \( \mathcal{D} \) is assumed to be compact. It is further required that there exists \( N_0 \in \mathbb{N} \) such that, for all \( N \geq N_0 \),

\[
\arg \min_{\theta_m \in \mathbb{R}^{n_s}} V_m, N(\theta_m) \in \text{int}(\mathcal{D})
\]

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where
\[ V_{m,N}(\theta) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} |\hat{V}_m(t, \theta_m)|^2 \]

The optimal value for \( \theta_m \) is computed using the RLS implementation with computing cost \( 9n_s \) per subband sampling time (see (15)).

**Assumption 6**: The system \( g(q) \) is linear and time-invariant with impulse response satisfying \( tg(t) \in l_2(\mathbb{Z}) \).

**Remark 2**: Note that assumption 6 is satisfied if \( g(z) \) is rational and stable.

**VI. Performance Indexes**

As said in the introduction, we will evaluate the performance of the subband identification method by comparing it with the full-band identification method. The comparison will be based on three performance indexes: asymptotic residual error, asymptotic convergence rate and computational cost. In this section, we will provide expressions for all of them. These are general expressions in the sense that they are valid for both critical-sampling and oversampling. In the next two sections, we will provide the final expressions of the performance indexes for each particular case.

**Asymptotic Residual Error:**

**Notation 2**: For each subband, define the prediction error \( \hat{W}_m(t, \theta) = W_m(t) - \hat{W}_m(t, \theta_m) \), denote its power by \( S_{\hat{W}}(\theta) \) and let
\[
\theta_{m,\text{opt}} \in \arg\min_{\theta_m \in \mathcal{D}} S_{\hat{W}_m}(\theta_m) \tag{26}
\]

\[
S_{\hat{W}_m,\text{opt}} = S_{\hat{W}_m}(\theta_{m,\text{opt}})
\]

For the overall performance, define the prediction error \( \hat{w}(t, \theta) = w(t) - \hat{w}(t, \theta) \), denote its power by \( S_{\hat{w}}(\theta) \) and let
\[
\theta_{\text{opt}} \in \arg\min_{\theta \in \mathcal{D}} S_{\hat{w}}(\theta)
\]

\[
S_{\hat{w},\text{opt}} = S_{\hat{w}}(\theta_{\text{opt}})
\]

In the full-band method, the sequence of random variables \( S_{\hat{w}}(\theta_N) \), converges, with probability one to the deterministic constant \( S_{\hat{w},\text{opt}} \), which is the global minimum of \( S_{\hat{w}}(\theta) \). In the subband method, \( S_{\hat{w}}(\theta_N) \) still converges, with probability one, to a deterministic constant \( S_{\hat{w},\text{lim}} \), but it does not equal \( S_{\hat{w},\text{opt}} \) in general. The following theorem states this fact formally and its corollary gives the conditions that guarantee \( S_{\hat{w},\text{lim}} = S_{\hat{w},\text{opt}} \).
Theorem 2 (Theorem 5 in [10]) Consider the subband identification scheme of figure 2, together with assumptions 3, 4, 5 and 6. Then, there exists a deterministic constant $S_{\tilde{g}, \bar{u}, \lim} \geq 0$, such that

$$\lim_{N \to \infty} S_{\tilde{g}}(\theta_N) = S_{\tilde{g}, \bar{u}, \lim} \quad w.p.1$$

Corollary 2: If $D = M$ (critical-sampling case), the analysis filterbank $h(q)$ is paraunitary (i.e. $T_h^* T_h = c I$), and the synthesis filterbank $f(q)$ is given by

$$f(t) = \frac{1}{c} h^*(-t)$$

(i.e. $T_f = \frac{1}{c} T_h$), then,

$$S_{\tilde{g}, \bar{u}, \lim} = S_{\tilde{g}, 0 \text{pt}} = \min_{\theta \in D} S_{\tilde{g}}(\theta)$$

Proof: See appendix III.

As said in section II, we are not interested in a bound on $S_{\tilde{g}, \bar{u}, \lim}$, but in a bound on the difference between $S_{\tilde{g}, \bar{u}, \lim}$ and the asymptotic residual error of the full-band method $S_{\tilde{g}}^{FB}$. In order to do that, we will provide a bound of $S_{\tilde{g}, \bar{u}, \lim}$ when the system to be identified is the FIR model $\hat{g}(q, \theta_{0 \text{pt}})$ obtained by identifying the system $g(q)$ using the full-band method. For technical simplicity, we will provide this bound under the assumption that the input signal $\{u(t)\}$ is a white noise.

Theorem 3: Consider the subband identification scheme of figure 2, together with assumptions 3, 4 and 5. Let the system $g(q) = \hat{g}(q, \theta_{0 \text{pt}})$ (i.e. $g(q)$ is linear time-invariant and FIR, whose tap size is denoted by $n_f$). Let the FIR subband models $\hat{G}_m(q, \theta), \ m = 1, \ldots, M$ be given by

$$\hat{G}_m(q, \theta) = \sum_{i=-n_d}^{n_d} \hat{G}_{m,i} q^i$$

where $n_p = \left[ \frac{n_f - 1}{D} \right] + 1$ and $n_d > 0$. In the above, $[x]$ denotes the smallest integer greater than or equal to $x$.

If $\{u(t)\}$ is white, then

$$S_{\tilde{g}, \bar{u}, \lim} \leq \|T_f\|^2 \|T_h\|^2 (J + R^2 K) \|g(t)\|^2 I \|S_u + S_{\tilde{g}}^{FB}\|_I$$

where

$$J = \frac{4}{D} M C^2; \ K = \frac{M}{D} C^2$$

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and

\[ R = \frac{1}{\sqrt{D}} \max_{1 \leq m \leq M} \left\{ \| h_m(z) \|_\infty \right\} \]

\[ F = \frac{1}{\sqrt{D}} \max_{1 \leq m \leq M} \left\{ \frac{1}{2\pi} \int_{-\pi,\pi} |h_m(e^{j\omega})|^2 d\omega \right\}^{1/2} \]

\[ C = \left( \sum_{m \notin \{-n_d, \ldots, n_p + n_a - 1\}} \text{sinc}^2 \left( t - \frac{1}{2} \right) \right)^{1/2} \]

where \( \sigma_m \) is defined in theorem 1 and \( \text{sinc}(t) = \sin(\pi t)/(\pi t) \).

**Proof:** See appendix III. ■

**Corollary 3:** In order to minimize \( S_{\hat{q}, \text{lim}} \), the analysis filterbank \( \hat{h}(q) \) needs to be the paramatery filterbank that minimizes \( F \), and the synthesis filterbank has to be given by (27).

**Proof:** See appendix III. ■

**Remark 3:** From (28), the tap size of the subband models is given by

\[ n_s = n_p + 2n_d \]

where the first \( n_d \) parameters are non-causal. Note that this non-causality can be removed in implementation by inserting delays in the subband models.

**Remark 4:** From theorem 1 and corollary 1, we know that, if the filterbanks satisfy conditions C1-C4, and the subband models are given by \( \hat{G}(q, \theta) = \hat{G}(q) \), (where \( \hat{G}(q) \) is given by (24)), then the asymptotic residual error \( S_{\hat{q}, \text{lim}} \) is zero. However, these filterbanks and subband models have an infinite tap size. Since we will use FIR approximations, (assumptions 4 and 5), (29) indicates that the bound of \( S_{\hat{q}, \text{lim}} - S_{\hat{q}, \text{lim}}^{\text{FIR}} \) is made of two components:

E1: Error due to FIR approximation of the subband filters \( \| T_f \| \| T_h \| J \| g(t) \| ^2 S_u \).

E2: Error due to FIR approximation in the diagonal subband terms \( \| T_f \| \| T_h \| R^2 K ^2 \| g(t) \| ^2 S_u \).

**Asymptotic Convergence Rate:**

As in the full-band case, we define \( S_{\hat{q}, \text{diff}}(N) = S_{\hat{q}, \theta_N} - S_{\hat{q}, \text{lim}} \).

**Theorem 4** (Theorem 7 in [10]) Consider the subband identification scheme of figure 2, together with assumptions 3, 4, 5 and 6. Then for large \( n_s \) and \( N \), and for small \( S_{\hat{q}, \text{lim}} \),

\[ \mathcal{E}\{S_{\hat{q}, \text{diff}}(N)\} \lesssim \frac{Dn_s}{N} \| T_f \| \| T_h \| S_v \]

where \( S_v \) is the power of the noise signal \( v(t) \).
Corollary 4: In order to maximize the convergence rate, the analysis filterbank \( h(q) \) needs to be paraunitary, and the synthesis filterbank has to be given by (27). In this case, for large \( n_s \) and \( N \), and for small \( S_{\tilde{d}, \text{lim}} \),

\[
\mathcal{E}\{S_{\tilde{d}, \text{dif}}(N)\} \lesssim \frac{Dn_s}{N} S_v
\]  

(36)

\textbf{Proof:} The proof is similar to the proof of corollary 3.

\textbf{Computational cost:}

Recall that we use a RLS implementation for computing \( \theta_{m,N} \). Then, from (15) we have that the computational cost measured in the amount of multiplications per full-band sample is

\[
\Psi = \frac{M}{D} [2l_h + l_f + 9n_s]
\]  

(37)

In the following two sections we will use the expressions of the performance indexes given in this section in order to provide design issues for the two cases (critical-sampling and oversampling). In each case, we need to provide:

1. The ideal analysis \( h(z) \) and synthesis \( f(z) \) filterbanks.
2. Issues for practical implementation: optimal values for \( M, D, n_s, l_h \) and \( l_f \), and optimization criteria for the design of the FIR filterbanks.

\textbf{VII. The Critical-sampling Case}

In the critical-sampling case \((M = D)\), the number of subbands is minimal. This eliminates the information redundancy, which saves computational cost. Also, in view of corollary 2, the minimum asymptotic residual error is guaranteed to be achieved.

\textbf{The ideal analysis \( h(z) \) and synthesis \( f(z) \) filterbanks:}

Since \( M = D \), given an analysis filterbank \( h(q) \), the option for the synthesis filterbank \( f(q) \) that satisfies the perfect reconstruction condition (17) is unique. Therefore, we just need to give a choice for the analysis filterbank \( h(q) \).

From corollaries 3 and 4, in order to minimize the asymptotic residual error and maximize the asymptotic convergence rate, the filterbank \( h(q) \) has to be paraunitary and minimize \( F \), and \( f(q) \) has to be given by (27) (which in this case is the only possibility). Ideally, we want to choose \( h(q) \) such that \( F = 0 \). This is achieved if \( h(q) \) satisfies conditions C1-C4.

\textbf{Proposition 1:} Consider the subband identification scheme of figure 2. Let \( h_m(q) \), \( m = 1, \ldots, M \),
satisfy conditions C1-C4. Then \( h(q) \) is paraunitary if and only if, there exists \( c > 0 \), such that

\[
\sum_{m=1}^{M} |h_m(e^{i\omega})|^2 = c
\]

(38)

Proof: Conditions C1-C4 imply that, in lemma 2, \( \beta(d) = 0, \ d = 1, \cdots, D - 1 \), and the result follows since \( h(q) \) being paraunitary is equivalent to \( T_h \) being scaled unitary.

In view of proposition 1, under the conditions C1-C4, \( h(q) \) being scaled paraunitary is equivalent to (38). It is straightforward to verify that both conditions (conditions C1-C4 and (38)) are satisfied by the filters in figure 4.

Fig. 4. Analysis filterbank for the critical-sampling case

**Issues for practical implementation:**

The ideal filters of figure 4 will be approximated using linear phase FIR filters, whose frequency response can not have zero amplitude in their stop-band, and therefore \( F \neq 0 \). It follows that \( h(q) \) has to be the paraunitary filterbank that minimizes \( F \), and \( f(q) \) be given by (27).

The number of parameters of the subband models \( n_s \) is taken as is (34).

Since the filters \( h_m(q), m = 1, \ldots, M, \) will be approximations of the filters in figure 4, we have that, in (29), \( R \approx 1 \) and \( \|T_f\| \|T_h\| \approx 1 \), then, if \( \{u(t)\} \) has a flat power spectral density,

\[
S_{g,\lim} \lesssim (J + K) \|g(t)\|_2^2 S_u + S_{g,\lim}^{\text{FB}}
\]

From (36) and (34), for large \( n_f \) and \( N \) and small \( S_{g,\lim} \) and if \( n_f/D \gg n_d \),

\[
\mathcal{E}\{(S_{g,\text{dir}}(N)) \approx \frac{n_f}{N} S_c
\]

(39)

For given \( D \) and \( l_h \), the required \( h(q) \) and \( f(q) \) are uniquely specified. Then, we can express \( F(D,l_h) \) as a function of \( D \) and \( l_h \). From (33), and since \( n_f/D \gg n_d \), we have that

\[
C \approx C(n_d) = \left( \sum_{t=n_d}^{\infty} \sin^2 \left( t - \frac{1}{2} \right) \right)^{1/2}
\]
From (30), we see that if we choose some desired values for \( J \) and \( K \), the required tap size \( l_h(D, J) \) and subband parameters \( n_d(K) \) can be computed numerically. Also, from (27), we have that \( l_f = l_h \). Then, (37) becomes

\[
\Psi = 3l_h(D, J) + 9 \left( \frac{n_f - 1}{D} \right) + 1 + 2n_d(K) \tag{40}
\]

Let us summarize the analysis above. We can see that the convergence rate (39) is independent of the design parameters and is the same as in the full-band case. The choices of \( J \) and \( K \) influence the asymptotic residual error for the subband identification. Hence, they need to be small. However, they should not be too small or the computational cost will be too high. Then, for fix values of \( J \) and \( K \), we can numerically optimize \( D \) so as to minimize the computational cost while having the asymptotic residual error and convergence rate compatible with the full-band method.

**VIII. The Oversampling Case**

The disadvantage of critical-sampling is that the filters \( h(q) \) need a sharp transition band. Consequently, the required filter length is considerably long, which contributes negatively to the computational cost. The idea of oversampling is to increase the value of \( M \) so as to allow filters that are easier to approximate. Of course, more computational cost is required to cope with the extra subbands caused by oversampling. But it turns out that this cost can be overweighted by the savings on the filterbank approximations.

**The Ideal Analysis \( h(z) \) and Synthesis \( f(z) \) Filterbanks:**

In contrast to the critical-sampling case, for a given analysis filterbank \( h(q) \), there are infinite possible synthesis filterbanks that satisfy the perfect reconstruction condition (17). So we can take advantage of this property.

Following the reasoning introduced for the critical-sampling case, in order to minimize the asymptotic residual error, maximize the asymptotic convergence rate and make \( F = 0 \), \( h(q) \) needs to satisfy the conditions C1-C4 and (38), and \( f(q) \) has to be given by (27). In view of proposition 1, it can be verified that the two conditions on \( h(q) \) are satisfied by the filters in figure 5, where the shape of the transition bands is proportional to \( \sqrt{\omega} \).

**Issues for practical implementation:**

As in the previous section, the ideal analysis filters \( h(q) \) will be approximated by linear phase FIR filters that are the best scaled paraunitary filters in the sense that they minimize \( F \).
synthesis filters $f(q)$ are given by (27). We also take $n_s$ as in (34).

As in the critical-sampling case, we also have that, if \( \{u(t)\} \) has a flat power spectral density,

\[
S_{\tilde{u},\text{lim}} \lesssim (J + K) \|g(t)\|_1^2 S_u + S_{\tilde{u},\text{FB}}^E
\]

Further, for large $n_f$ and $N$, small $S_{\tilde{u},\text{lim}}$, and $n_f/D \gg n_d$,

\[
\mathcal{E}\{S_{\tilde{u},\text{dir}}(N)\} \simeq \frac{n_f}{N} S_v
\]

Also in this case, we can numerically evaluate the required tap size $l_h(M, D, J)$ and subband parameters $n_d(M, D, K)$. Then, (37) becomes

\[
\Phi = \frac{M}{D} \left( 3l_h(M, D, J) + 9 \left( \left\lfloor \frac{n_f - 1}{D} \right\rfloor + 1 + 2n_d(M, D, K) \right) \right)
\] (41)

Therefore, for given values of $J$ and $K$ we can numerically optimize $M$ and $D$ so as to minimize the computational cost while having the asymptotic residual error and convergence rate compatible with the full-band case.

IX. Simulations

In sections VII and VIII, we state that both the critical sampling and over-sampling subband methods can have the same performance as the full-band method (in terms of asymptotic residual error and convergence rate) at a less computational cost. Further, we expect that the computational savings are more significant in the oversampling case since it includes the critical-sampling case as a particular case. In order to illustrate these points, we identify a linear, time-invariant system using the tree methods.

The transfer function of the system is shown in figure 6, with $\|g(t)\|_1 = 141$. The power of the input signal is $S_v = 1$.

We used a tap size of $n_f = 200$ for the full-band method. This choice of $n_f$ means that the full-band model ignores the part of the impulse response after $t = 200$, resulting in an asymptotic
residual error of approximately $S_{\tilde{u},\text{opt}} = 0.05$. In order to bound the error of the subband method, we adopted $J = 0.001$ and $K_2 = 0.05$.

With these parameters, we optimize the values of $M$ and $D$ to minimize the computational cost for the critical-sampling case (equation (40)) and the oversampling case (equation (41)). The computational cost as a function of $M$ is shown in figure 7. The plot of the computational cost of the oversampling case, is obtained by using the optimal value of $D$ for each value of $M$.

![Fig. 6. Plant Impulse Response](image)

The optimal values are $M = D = 12$ for the critical-sampling case and $M = 12, D = 10$ for the oversampling case. With these values, we have, for the critical-sampling case: $l_h = 354, n_s = 19, n_d = 1$ and for the oversampling case: $l_h = 63, n_s = 22, n_d = 1$.

The evolution of $S_{\tilde{u}}(N)$ is shown in figure 8. We see that the asymptotic residual error (0.042 for the full-band method, 0.037 for the critical-sampling subband method and 0.039 for the oversampling subband method) and asymptotic convergence rate are compatible. However, the computational costs are 1800 [multiplications per full-band sample] for the full-band method, 1233 for the critical-sampling subband method and 464 for the oversampling subband method.

![Fig. 7. Computational Cost vs. M](image)
Fig. 8. Evolution of the identification error power

X. CONCLUSION

In this work, we have analyzed the performance of the decoupled subband identification method, in both critical-sampling and oversampling cases, by comparing them with the classical time-domain identification method (the full-band method). The comparison is based on three performance indexes: asymptotic residual error, asymptotic convergence rate and computational cost. We provided selection criteria for the filterbanks, the number of parameters in each subband model, the number of subbands and the downsampling factor. We have shown that, with these criteria, the asymptotic residual error and asymptotic convergence rate of both subband methods are compatible with that of the full-band method. However, if the impulse response of the system to be identified is large enough, the computational costs of the subband methods are smaller. Further, more computational savings can be achieved in the oversampling case. We expect that subband identification methods to find applications in various acoustic and speech signal processing problems as well as wideband communications problems where high order FIR models are required.

APPENDIX

I. THE ROOTS OF $z$ AND ITS BRANCHES

While using the alias representation, sometimes we meet the expression $z^{1/D}$ (the $D$-root of $z$). We know that if $z \in \mathbb{C}$ is given by $z = re^{j\theta}$, $r > 0$, $-\pi \leq \theta < \pi$, then there are $D$ values for $z^{1/D}$, that are given by

$$z^{1/D} = r^{1/D}e^{j\frac{2\pi d}{D}}, \quad d = 0, \ldots, D - 1$$

In order to avoid ambiguities, we need the notion of branches.

Definition 5: Let $\gamma : [-\pi, \pi) \rightarrow \{0, \ldots, D - 1\}$ be a map. Then, the $\gamma$-branch of the $D$-root of...
$z = re^{i\theta}$ is the map $\beta_D^j : \mathbb{C} \to \mathbb{C}$ defined by

$$\beta_D^j(z) = r^\frac{i}{D} e^{\frac{2\pi i (\theta + \phi)}{D}}$$

So a branch is a map that sends every $z$ to one of the $D$ possible $D$-roots, and the decision is based on the angle $\theta$. We have the following two useful equalities.

**Proposition 2:** Let $f : \mathbb{C} \to \mathbb{C}$. If $\beta_D^j_1, \beta_D^j_2$ are two branches of $z^{1/D}$, then

$$\sum_{d=0}^{D-1} f(\Omega^d \beta_D^j_1(z)) = \sum_{d=0}^{D-1} f(\Omega^d \beta_D^j_2(z)) \quad (42)$$

$$\sum_{d=0}^{D-1} f(\Omega^d \beta_D^j(z)) = \sum_{d=0}^{D-1} f(\Omega^d \beta_D^j_1(z))^D = \sum_{d=0}^{D-1} f(\Omega^d z) \quad (43)$$

where $\Omega = e^{-i\frac{2\pi}{D}}$.

Equation (42) means that if we are adding over all the $D$ $D$-roots, the addition is independent of the branch adopted. Equation (43) means that, in the same kind of addition, we can replace $(zD)^{1/D}$ by $z$, whatever the branch $z^{1/D}$ takes (note that $(zD)^{1/D} = z$ is not true in general).

**Notation 3:** In view of (42), when the expression is valid for any branch, we will denote the $D$-root of $z$ by $z^{1/D}$.

II. PROOFS FOR SECTION IV

**Proof of theorem 1.** Denote $\hat{G}(q, \theta)$ by $\hat{G}(q)$ because its dependence on $\theta$ is irrelevant in this theorem. Let $\hat{G}(q) = [\hat{G}_{ij}(q)]_{i,j=1}^M$. By using the alias representation (figure 3) we have that, for $m = 1, \ldots, M$,

$$W_m(z) = \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega^d z^{1/D})g(\Omega^d z^{1/D})u(\Omega^d z^{1/D}) \quad (44)$$

$$\hat{W}_m(z) = \sum_{i=1}^M \hat{G}_{mi}(z) \frac{1}{D} \sum_{d=0}^{D-1} h_i(\Omega^d z^{1/D})u(\Omega^d z^{1/D}) \quad (45)$$

Assume (21). Recall that by using $z^{1/D}$ we implicitly mean that the expression is independent of the branch used (in view of (42)). This means that, in equations (44) and (45) we can replace $z^{1/D}$ by the branch $\beta^D_m(z)$. Then

$$W_m(z) = \frac{1}{D} g(\beta^D_m(z))h_m(\beta^D_m(z))u(\beta^D_m(z)) \quad (46)$$

$$\hat{W}_m(z) = \frac{1}{D} \sum_{i=1}^M \hat{G}_{mi}(z)h_i(\beta^D_m(z))u(\beta^D_m(z)) \quad (47)$$

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If we impose the subband model \( \hat{G}(z) \) to be diagonal matrix, then (47) becomes

\[
\hat{W}_m(z) = \frac{1}{D} \hat{G}_m(z) h_m(\beta_m^D(z)) u(\beta_m^D(z))
\]

where we used the notation \( \hat{G}_m(z) = \hat{G}_{mm}(z) \). By comparing (46) and (48), we conclude that, if we want \( W_m(e^{j\omega}) = \hat{W}_m(e^{j\omega}), \forall \omega \in [-\pi, \pi] \), then all the possible diagonal solutions are given by (23).

For the converse, suppose that there exists a diagonal \( \hat{G}(z) \) such that \( W_m(z) = \hat{W}_m(z) \), \( m = 1, \ldots, M \). Then, from (44) and (45), we have

\[
\frac{1}{D} \sum_{d=0}^{D-1} g(\Omega_d z^{1/D}) h_m(\Omega_d z^{1/D}) u(\Omega_d z^{1/D}) = \hat{G}_m(z) \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega_d z^{1/D}) u(\Omega_d z^{1/D})
\]

which can be true only if (21) holds.

**Proof of corollary 1.** If \( h_m(q) = l_m(q) \) and \( \sigma_m^\beta = \sigma_m^h (= \sigma_m^l) \), then

\[
\hat{G}_m(e^{j\omega}) = g(\beta_m^D(e^{j\omega}))
\]

Also, from [15, eq. (4.1.4), p. 102], we know that, for a given \( f(t) \in l_2(\mathbb{Z}) \),

\[
Z \{ f(Dt) \} = \frac{1}{D} \sum_{d=0}^{D-1} f(\Omega_d z^{1/D})
\]

Then,

\[
\{ Z \{ (G_m(\tau) * g(\tau))(Dt) \} = \frac{1}{D} \sum_{d=0}^{D-1} G_m(\Omega_d^D \beta_m^D(z)) g(\Omega_d^D \beta_m^D(z)) = \frac{1}{D} \sum_{d=0}^{D-1} G_m(\Omega_d^D \beta_m^D(z)) g(\Omega_d^D \beta_m^D(z)) = g(\beta_m^D(z))
\]

Hence (24) follows from (49) and (50).

### III. Proofs for Section VI

**Lemma 1** (Lemma 8 in [10]) Let \( \{ X(t) = [X_1(t), \ldots, X_M(t)]^T \}, t \in \mathbb{Z} \) be an array of quasi-stationary random processes and let \( H(q) = [H_{ij}(q)], i, j = 1, \ldots, M \) satisfy \( H_{ij}(t) \in l_1(\mathbb{Z}) \). Let \( \hat{X}(t) = [\hat{X}_1(t), \ldots, \hat{X}_M(t)]^T \) be generated from \( \{ X(t) \} \) by upsampling by a factor of \( U \). Let \( Y(t) = [Y_1(t), \ldots, Y_M(t)]^T \) be defined by \( Y(t) = \sum_{k=-\infty}^{\infty} H(k) \hat{X}(Dt - k) \) (i.e. \( \{ Y(t) \} \) is generated from \( \{ \hat{X}(t) \} \) by filtering followed by downsampling by a factor \( D \)). Let \( x(t) = [x_1(t), \ldots, x_M(t)]^T \in \)
$l_2^M(\mathbb{Z})$ and let $y(t)$ be generated from $x(t)$ in the same way as \{Y(t)\} is generated from \{X(t)\}. If there exists $T > 0$ such that $\|y(t)\|_2 \leq T \|x(t)\|_2$, where $\|x(t)\|_2 = \sum_{m=1}^{M} \|x_m(t)\|_2$, then

$$S_Y \leq \frac{D}{T^2} S_X$$

where $S_X = \sum_{m=1}^{M} S_{X_m}$. Further, if $\|y(t)\|_2 = T \|x(t)\|_2$, then $S_Y = \frac{D}{T^2} S_X$.

**Proof of corollary 2.** Since $h(g)$ is paramitary, then $\frac{1}{\sqrt{e}} T h$ and $\sqrt{e} T f$ are isometric isomorphisms. Then, lemma 1 and (13), imply that

$$S_{\tilde{W},\lim} = c S_{\hat{W},\lim} = c \sum_{m=1}^{M} S_{\tilde{W}_m,\lim} = c \sum_{m=1}^{M} \min_{\theta \in \Theta} S_{\tilde{W}_m}(\theta) = \min_{\theta \in \Theta} c S_{\tilde{W}_m}(\theta) = \min_{\theta \in \Theta} S_{\hat{W}_m}(\theta)$$

**Proof of theorem 3.** From [10, theorem 4], in every subband,

$$\lim_{N \to \infty} S_{\tilde{W}_m}(\theta_{m,N}) = S_{\tilde{W}_m,\text{opt}} \text{ w.p.1}$$

Therefore, in the context of this proof, we will assume that the set of parameters $\theta$ is given by $\theta = [\theta_{m,\text{opt}}^T, \ldots, \theta_{M,\text{opt}}^T]^T$, and we will eliminate from the notation the dependence on the set of parameters. We split the proof into 6 steps.

**Step 1:** Assume for a moment that $u(t) \in l_2(\mathbb{Z})$. In view of the alias representation (figure 3) we have that, for $m = 1, \ldots, M$,

$$W_m(z) = \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega_d z^{1/D}) g(\Omega_d z^{1/D}) u(\Omega_d z^{1/D})$$

$$\hat{W}_m(z) = \hat{G}_m(z) \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega_d z^{1/D}) u(\Omega_d z^{1/D})$$

then

$$W_m(z) = W_m(z) - \hat{W}_m(z)$$

$$= \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega_d z^{1/D}) \left( g(\Omega_d z^{1/D}) - \hat{G}_m(z) \right) u(\Omega_d z^{1/D})$$

$$= \frac{1}{D} \sum_{d=0}^{D-1} h_m(\Omega_d z^{1/D}) \left( g(\Omega_d z^{1/D}) - \hat{G}_m(\Omega_d z^{1/D}) \right) u(\Omega_d z^{1/D})$$

So, in the time domain,

$$\tilde{W}_m(t) = \downarrow_D \left\{ u_m^\#(t) \right\}$$

$$u_m^\#(t) = h_m(t) \ast \left( g(t) - \uparrow_D \left\{ \hat{G}_m(t) \right\} \right) \ast u(t)$$

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where $\downarrow D, \uparrow D : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$ denote the downsampling and upsampling operators, respectively.

**Step 2:** Now let $\{u(t)\}$ be the random process satisfying assumption 3. Since $\{u(t)\}$ is almost stationary, it is straightforward to verify that $\{u^m_n(t)\}$ is also almost stationary. It follows

$$R_{W_m} (\tau) = R_{u^m_n} (D\tau)$$  \hfill (52)

From [15, eq. (4.1.4), p. 102], we know that, if $f(t) \in l_2(\mathbb{Z})$ and $F(\omega) = \mathcal{F}\{(\cdot)\} \in \mathcal{L}_{\in [-\pi, \pi]}$, then

$$\mathcal{F}\{(D\cdot)\} = \frac{D}{\pi} \sum_{l=0}^{D-\infty} \mathcal{F}(\pi \frac{D}{\pi} + \omega)$$

From (52) and the fact that $\{u(t)\}$ is white, we get

$$\Phi_{W_m}(\omega) = \mathcal{F}\{R_{W_m}(\tau)\} = \frac{D}{\pi} \sum_{l=0}^{D-\infty} \Pi_{D_l} (\pi \frac{D}{\pi} + \omega)$$

$$= \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d e^{i\omega}) \right|^2 \left| g (\Omega^d e^{i\omega}) - \tilde{G}_m (\Omega^d e^{i\omega}) \right|^2 \Phi_u (2\pi \frac{D}{\pi} + \omega)$$

$$= \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d e^{i\omega}) \right|^2 \left| g (\Omega^d e^{i\omega}) - \tilde{G}_m (\Omega^d e^{i\omega}) \right|^2 S_u$$

and from (42), we have

$$\Phi_{W_m}(\omega) = \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d \beta^D_m (e^{i\omega})) \right|^2 \left| g (\Omega^d \beta^D_m (e^{i\omega})) - \tilde{G}_m (\Omega^d \beta^D_m (e^{i\omega})) \right|^2 S_u$$

$$= \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d \beta^D_m (e^{i\omega})) \right|^2 \left| g (\Omega^d \beta^D_m (e^{i\omega})) - \tilde{G}_m (e^{i\omega}) \right|^2 S_u$$  \hfill (53)

where $\beta^D_m (z)$ is the branch defined in theorem 1.

**Step 3:** Consider the ideally required subband (IIR) model $\tilde{G}_m (e^{i\omega})$ given by (24), and suppose we truncate it to have the same support than $\tilde{G}_m (t)$. Denote this truncated version by $\tilde{G}_m^{tr}(t)$.

From (51) and (53), we have that

$$S_{W_m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d \beta^D_m (e^{i\omega})) \right|^2 \left| g (\Omega^d \beta^D_m (e^{i\omega})) - \tilde{G}_m (e^{i\omega}) \right|^2 S_u \right) d\omega$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{D} \sum_{d=0}^{D-1} \left| h_m (\Omega^d \beta^D_m (e^{i\omega})) \right|^2 \left| g (\Omega^d \beta^D_m (e^{i\omega})) - \tilde{G}_m^{tr} (e^{i\omega}) \right|^2 S_u \right) d\omega$$
Then,

\[
S_{W_m} \leq \frac{1}{D} \sum_{d=0}^{D-1} \left\| h_m (\Omega^d \beta_m (\mathbf{e}^j \omega)) \left( g(\Omega^d \beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right) \right\|_2^2 S_u
\]

\[
\leq \frac{1}{D} \left\| h_m (\beta_m (\mathbf{e}^j \omega)) \right\|_\infty^2 \left\| g(\beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right\|_2^2 S_u + \frac{1}{D} \sum_{d=1}^{D-1} \left\| h_m (\Omega^d \beta_m (\mathbf{e}^j \omega)) \right\|_2^2 \left\| g(\Omega^d \beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right\|_\infty^2 S_u
\]

(54)

**Step 4:** Since \( \tilde{G}_m (\mathbf{e}^j \omega) = g(\beta_m (\mathbf{e}^j \omega)) \), we have that

\[
\left\| g(\beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right\|_2^2 = \left\| \tilde{G}_m (t) - \tilde{G}_m^{tr} (t) \right\|_2^2 = \sum_{t \notin \{ -n_d, \ldots, n_p + n_d - 1 \}} \left| (\Gamma_m * g)(Dt) \right|^2
\]

(55)

where \( \Gamma_m (t) \) is given by (25). Now, by Holder’s inequality

\[
\left| (\Gamma_m * g)(Dt) \right| \leq \sum_{\tau=0}^{n_j-1} |g(\tau)\Gamma_m(Dt-\tau)| \leq \sum_{\tau=0}^{n_j-1} |g(\tau)|\Gamma_m(Dt-\tau)^2)^{1/2} \left| g(\tau) \right|^{1/2}
\]

\[
\leq \left( \sum_{\tau=0}^{n_j-1} |g(\tau)| \right)^{1/2} \left( \sum_{\tau=0}^{n_j-1} |\Gamma_m(Dt-\tau)|^2 \right)^{1/2}
\]

\[
= \| g(t) \|_1^{1/2} \left( \sum_{\tau=0}^{n_j-1} |g(\tau)| |\Gamma_m(Dt-\tau)|^2 \right)^{1/2}
\]

From (55),

\[
\left\| g(\beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right\|_2^2 \leq \| g(t) \|_1 \sum_{t \notin \{ -n_d, \ldots, n_p + n_d - 1 \}} \sum_{\tau=0}^{n_j-1} |g(\tau)| |\Gamma_m(Dt-\tau)|^2
\]

\[
= \| g(t) \|_1 \sum_{\tau=0}^{n_j-1} |g(\tau)| \sum_{t \notin \{ -n_d, \ldots, n_p + n_d - 1 \}} |\Gamma_m(Dt-\tau)|^2
\]

\[
\leq C_m^2 \| g(t) \|_1^2
\]

where \( C_m^2 = \max_{0 \leq \tau \leq n_j-1} \left( \sum_{t \notin \{ -n_d, \ldots, n_p + n_d - 1 \}} |\Gamma_m(Dt-\tau)|^2 \right) \). In view of conditions C1-C4, the only possibility for \( \Gamma_m (t) \) is \( \Gamma_m(t) = \text{sinc}(t/D)e^{j\omega_m} \) for some \( \omega_m \). It is straightforward to show that

\[
C_m^2 \leq C^2 = \sum_{t \notin \{ -n_d, \ldots, n_p + n_d - 1 \}} \text{sinc}^2 \left( t - \frac{1}{2} \right), \quad m = 1, \ldots, M
\]

(56)

Finally,

\[
\left\| g(\beta_m (\mathbf{e}^j \omega)) - \tilde{G}_m^{tr} (\mathbf{e}^j \omega) \right\|_2^2 \leq C^2 \| g(t) \|_1^2
\]

(57)

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Step 5: Since, for $m = 1, \ldots, M$, the identified model $\hat{G}_m^r(e^{j\omega})$ is a truncation of the ideally required model $\hat{G}_m(e^{j\omega}) = g(\beta_m D(e^{j\omega}))$, we can take the following approximation:

$$
\left\| g(\Omega^d \beta_m^D(e^{j\omega})) - \hat{G}_m^r(e^{j\omega}) \right\|_\infty \simeq \left\| g(\Omega^d \beta_m D(e^{j\omega})) - g(\beta_m^D(e^{j\omega})) \right\|_\infty \leq 2 \left\| g(e^{j\omega}) \right\|_\infty \leq 2 \left\| g(t) \right\|_1
$$

(58)

Step 6: Finally, putting together (54), (57), (58) and (32), we have

$$
S_{\hat{W}} \leq \|T_h\|^2 (4F^2 + R^2 C^2) \left\| g(t) \right\|_1^2 S_v
$$

where

$$
R_m = \frac{\|h_m(z)\|_\infty}{\sqrt{D} \|T_h\|}
$$

Then,

$$
S_{\hat{W}} = \sum_{m=1}^M S_{\hat{W}_m} \leq M \|T_h\|^2 (4F^2 + R^2 C^2) \left\| g(t) \right\|_1^2 S_v
$$

We know that, if $\hat{W}(t) \in l_2^M(Z)$, then $\left\| \hat{w}(t) \right\|_2 \leq \|T_f\| \left\| \hat{W}(t) \right\|_2$. Then, from lemma 1,

$$
S_{\hat{w}} \leq \frac{M}{D} \|T_f\|^2 \|T_h\|^2 (4F^2 + R^2 C^2) \left\| g(t) \right\|_1^2 S_v = \|T_f\|^2 \|T_h\|^2 (J + R^2 K) \left\| g(t) \right\|_1^2 S_v
$$

Lemma 2: Consider the subband identification scheme in figure 2. Then, for all $y(t) \in l_2(Z)$,

$$
\frac{\|T_h y(t)\|^2}{\|y(t)\|_2^2} \geq A := \frac{1}{D} \left[ \inf_{|l|=1} \left\{ \sum_{m=1}^M |h_m(z)|^2 \right\} - \sum_{d=1}^{D-1} \left( \beta(-d) \beta(d) \right)^{1/2} \right]
$$

(59)

$$
\frac{\|T_h y(t)\|^2}{\|y(t)\|_2^2} \leq B := \frac{1}{D} \left[ \sup_{|l|=1} \left\{ \sum_{m=1}^M |h_m(z)|^2 \right\} + \sum_{d=1}^{D-1} \left( \beta(-d) \beta(d) \right)^{1/2} \right]
$$

(60)

with equality if $\beta(d) = 0, d = 1, D - 1$, and where

$$
\beta(d) = \sup_{|l|=1} \left\{ \sum_{m=1}^M |h_m(z)| \left| h_m \left( \Omega^d z \right) \right| \right\}
$$

Proof: The proof follows the proof of [16, section 3.3.2, p. 67].

Proof of corollary 3. We will assume that $u(t) \in l_2(Z)$. Then, we will look at the analysis $h(q)$ and synthesis $f(q)$ filterbanks whose associated operators $T_h$ and $T_f$ minimize the 2-norm of $\hat{w}(t)$. In view of lemma 1, this is equivalent to minimize the power $S_{\hat{w}}$. As in the proof of theorem 3, we will eliminate from the notation the dependence on $\theta$.  

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The perfect reconstruction condition (16) can be written as

\[ T^*_f T_h = I \]

so \( T^*_f \) must be a left inverse of \( T_h \). In order to minimize the 2-norm of \( \hat{v} \), we need that \( T^*_f \) cancels the orthogonal complement of the range of \( T_h \). It follows that

\[ T^*_f = T^+_h \]  \hspace{1cm} (61)

where

\[ T^+_h = (T^*_h T_h)^{-1} T^*_h \]  \hspace{1cm} (62)

From (59) and (60), \( \|T_h\| \leq B^{1/2} \) and \( \|T_f\| \leq A^{-1/2} \). Also, \( A \leq B \), then, from (29), to minimize the asymptotic residual error, we need that \( A = B \). This in turn means that \( T_h \) is a scaled isometry, or equivalently, that \( h(q) \) is paraunitary. Also, from (29), we have that \( h(q) \) needs to minimize \( F \).

Finally, from (61) and (62), \( T_f = \frac{1}{\epsilon} T_h \), or equivalently, \( f(t) = \frac{1}{\epsilon} h^*(-t) \).

REFERENCES


