Piecewise Lyapunov functions for robust stability of linear
time-varying systems\textsuperscript{1}

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Abstract

In this paper, we investigate the use of two-term piecewise quadratic Lyapunov functions for robust stability of linear time-varying systems. By using the so-called S-procedure and a special variable reduction method, we provide numerically efficient conditions for the robust asymptotic stability of the linear time-varying systems involving the convex combinations of two matrices. An example is included to demonstrate the usefulness of our results. © 1997 Elsevier Science B.V.

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1. Introduction

The quadratic stability approach is popularly used for robust stability analysis of time-varying uncertain systems. This approach, however, may lead to very conservative results. Alternatively, non-quadratic Lyapunov functions have been used to improve the estimate of robust stability (see [2, 1, 7–10]). The difficulty with non-quadratic Lyapunov functions is that the resulting optimization problem is typically non-convex. In this short paper, we investigate the use of two-term piecewise Lyapunov functions on time-varying linear systems involving the convex combination of two matrices. These Lyapunov functions are either the maximum or the minimum of two quadratic terms. By using the so-called S-procedure [11] and a variable reduction technique, we obtain necessary and sufficient conditions for establishing robust asymptotic stability of the uncertain system using such a piecewise Lyapunov function. The resulting optimization problem involves a set of linear matrix inequalities (LMIs) with two scaling parameters which can be numerically searched.

We show via an example that good improvement on the estimate of robust stability margin can be obtained by using these piecewise Lyapunov functions when compared with the quadratic stability technique.

2. Problem formulation

Consider the linear time-varying system

\[ \dot{x} = A(t)x, \quad A(t) \in \mathcal{A} = \text{Co}\{A_1, A_2\}, \]  

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where $\text{Co}\{A_1, A_2\}$ stands for the convex hull of $A_1$ and $A_2$. Our aim is to produce a test which is less conservative than quadratic stability result with reasonable computational cost. In particular, we use two kinds of piecewise Lyapunov functions as follows:

$$V(x) = \max\{x'P_1x, x'P_2x\}, \quad P_1 > 0, \ P_2 > 0$$

(2.2)

and

$$V(x) = \min\{x'P_1x, x'P_2x\}, \quad P_1 > 0, \ P_2 > 0.$$  

(2.3)

To check whether (2.2) proves the stability of (2.1) or not, we only need to check that

$$\frac{dV(x(t))}{dt} < 0$$

(2.4)

for all $x(t) \neq 0$ along the trajectory of system (2.1). Note that the derivative above is given by

$$\frac{dV(x(t))}{dt} = \begin{cases} 
    x'(t)(A'(t)P_1 + P_1A(t))x(t) & \text{when } x'P_1x \geq x'P_2x, \\
    x'(t)(A'(t)P_2 + P_2A(t))x(t) & \text{when } x'P_2x \geq x'P_1x.
\end{cases}$$

(2.5)

For the Lyapunov function in (2.3), the derivative in (2.5) should be replaced with

$$\frac{dV(x(t))}{dt} = \begin{cases} 
    x'(t)(A'(t)P_1 + P_1A(t))x(t) & \text{when } x'P_1x \leq x'P_2x, \\
    x'(t)(A'(t)P_2 + P_2A(t))x(t) & \text{when } x'P_2x \leq x'P_1x.
\end{cases}$$

(2.6)

In order to derive conditions for robust stability of (2.1) with the Lyapunov function (2.2) or (2.3), we need the lemma below.

**Lemma 2.1** (S-procedure lemma [11]). Let $F_0(x)$ and $F_1(x)$ be two arbitrary quadratic forms over $\mathbb{R}^n$. Then $F_0(x) < 0$ for all $x \in \mathbb{R}^n$ satisfying $F_1(x) \leq 0$ if and only if there exist $\tau \geq 0$ such that

$$F_0(x) - \tau F_1(x) \leq 0, \quad \forall x \in \mathbb{R}^n.$$ 

From the lemma above, the robust stability conditions (2.2)–(2.4) can be rewritten as in the following lemma.

**Lemma 2.2** (Boyd [3]). System (2.1) is robustly stable for all $A(t) \in \mathcal{A}$ with the Lyapunov function (2.2) if and only if there exist solutions to the following set of inequalities:

$$A_1'P_1 + P_1A_1 - \lambda_1(P_2 - P_1) < 0,$$

(2.7)

$$A_2'P_1 + P_1A_2 - \lambda_2(P_2 - P_1) < 0,$$

(2.8)

$$A_1'P_2 + P_2A_1 + \lambda_3(P_2 - P_1) < 0,$$

(2.9)

$$A_2'P_2 + P_2A_2 + \lambda_4(P_2 - P_1) < 0,$$

(2.10)

$$P_1 > 0, \quad P_2 > 0,$$

(2.11)

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_4 \geq 0.$$ 

(2.12)

Similarly, we can have a result for the Lyapunov function in (2.3).

**Lemma 2.3.** System (2.1) is robustly stable for all $A(t) \in \mathcal{A}$ with the Lyapunov function (2.3) if and only if there exist solutions to the following set of inequalities:

$$A_1'P_1 + P_1A_1 + \lambda_1(P_2 - P_1) < 0,$$

(2.13)
\[ A_2'P_1 + P_1A_2 + \lambda_2(P_2 - P_1) < 0, \quad (2.14) \]
\[ A_1'P_2 + P_2A_1 - \lambda_3(P_2 - P_1) < 0, \quad (2.15) \]
\[ A_2'P_2 + P_2A_2 - \lambda_4(P_2 - P_1) < 0, \quad (2.16) \]
\[ P_1 > 0, \quad P_2 > 0, \quad (2.17) \]
\[ \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_4 \geq 0. \quad (2.18) \]

**Remark 2.1.** The use of Lyapunov functions (2.2)–(2.3) and the S-procedure is not new [5, 3]. In fact, it is shown [7, 8] that the system (2.1) is robustly asymptotically stable if and only if there exists a piecewise quadratic Lyapunov function of the following form:

\[ V(x) = \max_{1 \leq i \leq p} \{ x'P_ix \}, \quad P_i > 0, \quad i = 1, \ldots, p \]

for some \( p \). But the stumbling block is that the subsequent optimization problem is non-convex. This is true even for (2.7)–(2.12) or (2.13)–(2.18). If we fix \( P_1 \) and \( P_2 \), then (2.7)–(2.12) and (2.13)–(2.18) are convex problems. This is also the case if we fix \( \lambda_i, \quad i = 1, \ldots, 4 \) (refer to [3] for example). However, (2.7)–(2.12) and (2.13)–(2.18) are not jointly convex in \( P_1, P_2 \) and \( \lambda_i \).

3. Main result

In this section, we are going to provide a variable reduction procedure which can reduce the number of variables \( \lambda_i \) in (2.7)–(2.18) from four to two. Also, the range of the new variables will be set to \([0, 1]\). The resulting problem involves a set of matrix inequalities which are linear in all variables except for two. Numerical searching of these two variables becomes a moderate problem. In particular, we can use a simple gridding scheme to find an estimate of them. The estimate can then be refined by using either further gridding or local searching methods such as Newton’s method.

**Theorem 3.1.** The system (2.1) is robustly stable with the Lyapunov function (2.2) if and only if there exist \( \delta_1, \delta_2 \in [0, 1] \) such that the following set of LMIs have a solution for \( H_1 \) and \( H_2 \):

\[ A_1'H_1 + H_1A_1 < 0, \quad A_2'H_2 + H_2A_2 < 0, \]
\[ (1 - \delta_2)(A_1'H_2 + H_2A_1) + \delta_2(H_2 - H_1) < 0, \]
\[ (1 - \delta_1)(A_2'H_1 + H_1A_2) - \delta_1(H_2 - H_1) < 0, \]
\[ 0 < H_1 < I, \quad 0 < H_2 < I. \quad (3.19) \]

If this is the case, then \( V(x) = \max \{ x'H_1x, x'H_2x \} \) is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (2.1).

**Theorem 3.2.** The system (2.1) is robustly stable with the Lyapunov function (2.3) if and only if there exist \( \delta_1, \delta_2 \in [0, 1] \) such that the following set of LMIs have a solution for \( H_1 \) and \( H_2 \):

\[ A_1'H_1 + H_1A_1 < 0, \quad A_2'H_2 + H_2A_2 < 0, \]
\[ (1 - \delta_2)(A_1'H_2 + H_2A_1) - \delta_2(H_2 - H_1) < 0, \]
\[ (1 - \delta_1)(A_2'H_1 + H_1A_2) + \delta_1(H_2 - H_1) < 0, \]
\[ 0 < H_1 < I, \quad 0 < H_2 < I. \quad (3.20) \]

If this is the case, then \( V(x) = \min \{ x'H_1x, x'H_2x \} \) is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (2.1).
Remark 3.1. To see the improvement given by (3.19) and (3.20) in comparison with (2.7)–(2.12) and (2.13)–(2.18), we look at the gridding method for finding $\delta_1$ and $\lambda_1$. If the number of grid points for each variable is $N$, then $N^2$ grid points are required for $(\delta_1, \delta_2)$ while $N^4$ points are needed for $(\lambda_1, \ldots, \lambda_4)$. For $N = 10$, the former is 100 while the latter is 10,000!

**Proof of Theorem 3.1 (Necessity).** Without loss of generality, we assume that $\lambda_2 \lambda_3 \neq \lambda_1 \lambda_4$. If $\lambda_2 \lambda_3 = \lambda_1 \lambda_4$, we can perturb one of these parameters slightly without violating (2.7)–(2.12). Define the transformation

$$
H_1 = \mu_1 (\lambda_1 P_2 + \lambda_3 P_1), \quad H_2 = \mu_2 (\lambda_2 P_2 + \lambda_4 P_1),
$$

where $\mu_1 = \delta_2 + \lambda_4$ and $\mu_2 = \delta_1 + \lambda_3$. Then (2.7)–(2.12) lead to the following:

$$
H_1 > 0, \quad H_2 > 0,
$$

$$
A_1^t H_1 + H_1 A_1 = \mu_1 \lambda_3 (A_1^t P_1 + P_1 A_1 - \lambda_1 (P_2 - P_1)) + \mu_1 \lambda_1 (A_1^t P_2 + P_2 A_1 + \lambda_3 (P_2 - P_1)) < 0,
$$

$$
A_2^t H_2 + H_2 A_2 = \mu_2 \lambda_4 (A_2^t P_1 + P_1 A_2 - \lambda_2 (P_2 - P_1)) + \mu_2 \lambda_2 (A_2^t P_2 + P_2 A_2 + \lambda_4 (P_2 - P_1)) < 0,
$$

$$
A_1^t H_2 + H_2 A_1 + \mu_2 (H_2 - H_1) = \mu_2 \lambda_4 (A_1^t P_1 + P_1 A_1 - \lambda_1 (P_2 - P_1)) + \mu_2 \lambda_2 (A_1^t P_2 + P_2 A_1 + \lambda_3 (P_2 - P_1)) < 0,
$$

$$
A_2^t H_1 + H_1 A_2 - \mu_1 (H_2 - H_1) = \mu_1 \lambda_3 (A_2^t P_1 + P_1 A_2 - \lambda_2 (P_2 - P_1)) + \mu_1 \lambda_1 (A_2^t P_2 + P_2 A_2 + \lambda_4 (P_2 - P_1)) < 0.
$$

Now, introducing the variable substitution

$$
\mu_1 = \frac{\delta_1}{1 - \delta_1}, \quad \mu_2 = \frac{\delta_2}{1 - \delta_2},
$$

we obtain (3.19). Note that the constraints $H_1 < I$ and $H_2 < I$ are harmless because we can always rescale $H_1$ and $H_2$ to satisfy them.

(Sufficiency): Assume that (3.19) holds. If we set $P_1 = H_1$, $P_2 = H_2$, $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \delta_1 / (1 - \delta_1)$ and $\lambda_3 = \delta_2 / (1 - \delta_2)$, then we find a solution to the system (2.7)–(2.12). □

The proof for Theorem 3.2 is very similar and therefore omitted.

4. Extension to discrete-time systems

The results in Theorems 3.1 and 3.2 are readily extendible to the discrete-time case. More specifically, we consider the following discrete-time system:

$$
x(k + 1) = A(k)x(k), \quad A(k) \in \mathcal{A} = \text{Co}\{A_1, A_2\}.
$$

The discrete-time counterpart of Theorems 3.1 and 3.2 is given below.

**Theorem 4.1.** The system (4.27) is robustly stable with the Lyapunov function (2.2) if and only if there exist $\delta_1, \delta_2 \in [0, 1]$ such that the following set of LMIs have a solution for $H_1$ and $H_2$:

$$
A_1^t H_1 A_1 - H_1 < 0, \quad A_2^t H_2 A_2 - H_2 < 0,
$$

$$
(1 - \delta_2)(A_1^t H_2 A_1 - H_2) + \delta_2 (H_2 - H_1) < 0,
$$

$$
(1 - \delta_1)(A_2^t H_1 A_2 - H_1) - \delta_1 (H_2 - H_1) < 0,
$$

$$
0 < H_1 < I, \quad 0 < H_2 < I.
$$

(4.28)
If this is the case, then $V(x) = \max \{x' H_1 x, x' H_2 x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (4.27).

**Theorem 4.2.** The system (4.27) is robustly stable with the Lyapunov function (2.3) if and only if there exist $\delta_1, \delta_2 \in [0, 1]$ such that the following set of LMIs have a solution for $H_1$ and $H_2$:

$$
A_1' H_1 A_1 - H_1 < 0, \quad A_2' H_2 A_2 - H_2 < 0, \\
(1 - \delta_2)(A_1' H_2 A_1 - H_2) - \delta_2 (H_2 - H_1) < 0, \\
(1 - \delta_1)(A_2' H_1 A_2 - H_1) + \delta_1 (H_2 - H_1) < 0,
$$

(4.29)

$$0 < H_1 < I, \quad 0 < H_2 < I.
$$

If this is the case, then $V(x) = \min \{x' H_1 x, x' H_2 x\}$ is also a valid Lyapunov function for establishing the robust asymptotic stability of the system (4.27).

## 5. Numerical example

In this section, we shall test our results on an example which has been used in [12]. Two tests will be carried out to show how the Lyapunov functions (2.2) and (2.3) can be used to obtain better estimation of robust stability margins.

**Example.** Consider the following system:

$$
\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - x_2 - u(t)x_1, \quad 0 \leq u(t) \leq k,
$$

(5.30)

where $u(t)$ is a time-varying uncertain parameter and $k$ is its bound. The goal is to find the largest $k$ for which the system still remains robustly asymptotically stable. This system is found to be quadratically stable for $k \leq 3.828$ only ($k \leq 3.82$ in [12]) and admit a fixed Lyapunov function $x' P x$, where

$$
P = \begin{pmatrix}
258.4972 & 32.9855 \\
32.9855 & 66.0536
\end{pmatrix}.
$$

In order to estimate the robust stability margin $k_{\text{max}}$, we use a bisection method. That is, we start with some initial $k_0$ and $k_1$ as lower and upper bounds for $k_{\text{max}}$, respectively, and then apply bisection to improve the bounds until their gap is sufficiently small.

**Test one:** Now, we apply Theorems 3.1 and 3.2 to (5.30). We simply use a gridding method to search for $\delta_1$ and $\delta_2$. With the grid size equal to 0.1, we find that (5.30) is robustly asymptotically stable with the Lyapunov function (2.2) for $k \leq 4.7$ and the Lyapunov function is given by

$$
\max \left\{ x' \begin{pmatrix}
0.7685 & 0.1326 \\
0.1326 & 0.1923
\end{pmatrix} x, \; x' \begin{pmatrix}
0.8199 & 0.0690 \\
0.0690 & 0.1795
\end{pmatrix} x \right\}.
$$

(5.31)

The corresponding parameters are $\delta_1 = 0.9$, $\delta_2 = 0.8$, respectively. Also, (5.30) is robustly asymptotically stable for $k \leq 4.4$ with the Lyapunov function

$$
\min \left\{ x' \begin{pmatrix}
0.7344 & 0.0737 \\
0.0737 & 0.1963
\end{pmatrix} x, \; x' \begin{pmatrix}
0.8252 & 0.1155 \\
0.1155 & 0.1772
\end{pmatrix} x \right\},
$$

(5.32)

corresponding to $\delta_1 = 0.8$ and $\delta_2 = 0.8$.

**Test two:** To further improve the robust stability margin, we use the following nonlinear transformation technique introduced in [4]:

$$
x_i^{[p]} = \sqrt{\left( \frac{P}{p_1} \left( \frac{P - p_1}{p_2} \right) \cdots \left( \frac{P - p_1 - \cdots - p_{p-1}}{p_p} \right) \right)} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}, \quad \sum_{i=1}^n p_i = p, \; p_i \geq 0,
$$

(5.33)
where the integer \( p \) is fixed, taking the value of 2, 3, ... and \( l \) is the index of the transformed state corresponding to a set \( \{ p_1, \ldots, p_n \} \) satisfying the constraints in (5.33). The transformed system can be expressed as

\[
\dot{x}^{[p]} = A_{[p]} x^{[p]}
\]

which is \((n+p-1)\)-dimensional. In our example, we use the transformation

\[
y_1 = x_1^2, \quad y_2 = x_1 x_2, \quad y_3 = x_2^2
\]

as done in [12], i.e. \( p = 2 \). This transformation changes (5.30) into

\[
\begin{align*}
\dot{y}_1 &= 2 y_2, \\
\dot{y}_2 &= -2 y_1 - y_2 + y_3 - u(t) y_1, \\
\dot{y}_3 &= -4 y_2 - 2 y_3 - 2 u(t) y_2, \quad 0 \leq u(t) \leq k.
\end{align*}
\]

(5.36)

Note that (5.36) is linear in \( u(t) \). It is well known that [4] the null solution of the system (5.30) is stable (asymptotically stable) if and only if the null solution of the system (5.36) is stable (asymptotically stable). Moreover, if all solutions of (5.30) are bounded by \( |x(t)| \leq M e^{-\delta t} \) then all solutions of (5.36) are bounded by \( |y(t)| \leq M_1 e^{-\delta_1 t} \). The stability of the transformed system is usually established using a quadratic Lyapunov function which leads to a 2-\( p \)-order Lyapunov function for the original system.

The system (5.36) is found to be quadratically stable for only \( k \leq 5.47 \) [12]. A better estimate \( k \leq 5.73 \) is also obtained by adding the quadratic constraint \( y_1 y_3 = y_3^2 \). However, by applying the Lyapunov functions (2.2) and Theorem 3.1 to (5.36), we have found that (5.36) is robustly asymptotically stable for \( k \leq 5.8 \) and admits a Lyapunov function

\[
\begin{align*}
\max \left\{ y' \begin{pmatrix} 0.8454 & 0.1246 & 0.0568 \\
0.1246 & 0.3679 & 0.0627 \\
0.0568 & 0.0627 & 0.0381 \end{pmatrix} y, \ y' \begin{pmatrix} 0.8098 & 0.1123 & -0.0077 \\
0.1123 & 0.4552 & 0.0530 \\
-0.0077 & 0.0530 & 0.0381 \end{pmatrix} y \right\} \\
= \max\{0.8454 x_1^4 + 0.2492 x_1^2 x_2 + 0.4815 x_1^2 x_2^2 + 0.1254 x_1 x_2^3 + 0.0381 x_2^4, \\
0.8098 x_1^4 + 0.2246 x_1^2 x_2 + 0.4398 x_1^2 x_2^2 + 0.1060 x_1 x_2^3 + 0.0381 x_2^4\}
\end{align*}
\]

(5.37)

with \( \delta_1 = 0.9, \ \delta_2 = 0.6 \).

Similarly, the use of (2.3) and Theorem 3.2 leads to \( k \leq 5.8 \) with the following Lyapunov function:

\[
\begin{align*}
\min \left\{ y' \begin{pmatrix} 0.7911 & 0.1106 & 0.0111 \\
0.1106 & 0.4285 & 0.0568 \\
0.0111 & 0.0568 & 0.0414 \end{pmatrix} y, \ y' \begin{pmatrix} 0.8393 & 0.1638 & 0.0232 \\
0.1638 & 0.4417 & 0.0575 \\
0.0232 & 0.0575 & 0.0393 \end{pmatrix} y \right\} \\
= \min\{0.7911 x_1^4 + 0.2212 x_1^2 x_2 + 0.4507 x_1^2 x_2^2 + 0.1136 x_1 x_2^3 + 0.0414 x_2^4, \\
0.8393 x_1^4 + 0.3276 x_1^2 x_2 + 0.4881 x_1^2 x_2^2 + 0.1150 x_1 x_2^3 + 0.0393 x_2^4\}
\end{align*}
\]

(5.38)

with \( \delta_1 = 0.8, \ \delta_2 = 0.8 \).

All the above computation is done using the LMI Control Toolbox [6]. The grid size for \( \delta_1 \) and \( \delta_2 \) is 0.1. If we use the grid size 0.05, we can obtain the result \( k = 5.9 \) and \( k = 5.8 \) for (2.2) and (2.3), respectively.

**Remark 5.1.** Further improvement can also be achieved by introducing the quadratic constraint

\[
y_2^2 = y_1 y_2
\]

(5.39)

as is done in [12]. This constraint is evident from the transformation (5.35). By using the S-procedure on this constraint, we can obtain \( k = 6.2 \) and \( k = 5.9 \) for (2.2) and (2.3), respectively. To explain how this is done
for (2.2), we rewrite the constraint (5.39) as

\[
y'Qy = y' \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} y = 0.
\]  

(5.40)

Then, (2.2) is modified by adding a term \( \tau_i Q \) to the left-hand side of each LMI, where \( \tau_i \) is a real scalar variable, different for each LMI. Using the S-procedure, a sufficient condition for robust stability of (5.36) is that this modified set of LMIs has a feasible solution. Although this is a sufficient condition, it is a better (weaker) condition than ignoring (5.40) because \( \tau_i \) are non-zero in general. The increase in computational complexity is very marginal because the resulting LMIs are linear in \( \tau_i \). The modification for (2.3) is very similar.

6. Conclusions

In this paper, we have investigated the use of piecewise Lyapunov functions for providing better estimation of robust stability. The Lyapunov functions we used are in the form (2.2) and (2.3). We have proposed necessary and sufficient conditions for robust stability of convex combinations of two matrices. The key contribution of the paper is a variable reduction technique which allows us to gain some computational efficiency. We show through an example that our method can produce better estimation than existing methods.

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