Abstract: This paper is concerned with the problem of robust filtering for uncertain linear discrete-time descriptor systems. The matrices of the system state-space model are uncertain and supposed to belong to a given polytope. A linear matrix inequality based method is proposed for designing linear stationary strictly proper filters that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function.

Keywords: Filtering, robust LMS filtering, descriptor systems, uncertain systems, discrete-time systems, parameter-dependent Lyapunov functions.

1. INTRODUCTION

Descriptor systems (also known as singular systems or differential-/difference-algebraic systems) are an important class of dynamic system models from both a theoretical and practical point of view due to their capacity in describing algebraic constraints between physical variables (Lewis, 1986; Dai, 1989a). Descriptor models are encountered naturally in many areas of applications, such as economical systems, electrical networks, robotic systems, chemical processes, etc. (Dai, 1989c; Hill and Marceis, 1990; Lewis, 1986; Takaba and Katayama, 1997; Verghese et al., 1981).

In the last decade, a great deal of interest has been devoted to signal estimation methods based on Kalman filtering for discrete-time descriptor systems; see, for instance, Bianco et al. (2005), Dai (1989b), Deng and Liu (1999), Ishihara et al. (2004a), Nikoukhah et al. (1992), Nikoukhah et al. (1999) and Zhang et al. (1999). These methods are based on the knowledge of a perfect model for the signal to be estimated and thus they may fail to provide a guaranteed error variance when only an approximate model is available.

On the other hand, the design of robust Kalman filters for standard linear discrete-time state space models has been attracting increasing attention and both norm-bounded and polytopic parameter uncertainties have been considered; see, for instance, Barbosa et al. (2002), Petersen and McFarlane (1996), Petersen and Savkin (1999).
and the reference therein. Very recently, a robust Kalman filter for linear discrete-time descriptor systems subject to norm-bounded parameter uncertainty using a Riccati equation approach has been proposed in Ishihara et al. (2004b). To the best of the authors’ knowledge, to-date the problem of robust Kalman filtering for linear discrete-time descriptor systems with convex bounded uncertainty has not yet been investigated.

This paper addresses the design of robust filters for linear discrete-time descriptor systems. The matrices of the system state-space model are uncertain and supposed to belong to a given polytope. A linear matrix inequality (LMI) based method is developed for designing linear stationary strictly proper filters that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function.

Notation. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices, $I_n$ is the $n \times n$ identity matrix, $\text{Tr}[\cdot]$ denotes matrix trace, and $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix. For a real matrix $S$, $S^T$ denotes its transpose, $\text{Her}\{S\}$ stands for $S + S^T$ and $S > 0$ means $S$ is symmetric and positive definite. For a symmetric block matrix, the symbol $\Sigma$ denotes the transpose of the symmetric blocks outside the main diagonal block. Finally, $\mathbb{E}[\cdot]$ stands for mathematical expectation.

2. PROBLEM FORMULATION

Consider the following linear descriptor system

$$
\begin{align*}
E \zeta(k+1) &= A_0 \zeta(k) + B_0 w(k) \\
y(k) &= C_0 \zeta(k) + D_0 w(k) \\
s(k) &= L_0 s(k)
\end{align*}
$$

(1)

where $\zeta \in \mathbb{R}^{n_\zeta}$ is the state, $w(k) \in \mathbb{R}^{n_w}$ is a zero-mean white noise signal (including process and measurement noises) with an identity covariance matrix and uncorrelated with $\zeta(0)$ for $k = 0, 1, \ldots$, $y(k) \in \mathbb{R}^{n_y}$ is the measurement, $s(k) \in \mathbb{R}^{n_s}$ is the signal to be estimated, and $E, A_0, B_0, C_0, D_0$ and $L_0$ are real constant matrices of appropriate dimensions with $\text{rank}(E) = n < n_\zeta$.

It is assumed that system (1) satisfies the following assumptions:

**A1** System (1) is regular, i.e. $\det(ze - A) \neq 0$.

**A2** $\text{rank}(E) = \text{deg} \det(ze - A)$.

**A3** $\text{Im}\{B_0\} \subseteq \text{Im}\{E\}$.

Note that, assumption **A1** guarantees the existence and uniqueness of solution to (1) for any initial condition, **A2** ensures that the system is impulse-free and implies the causality of (1), whereas **A3** implies that the noise signal $w(k)$ does not influence the algebraic constraints in the system.

It is well known that under assumptions **A1** – **A3**, system (1) can be transformed into an equivalent difference-algebraic system of the following SVD normal form (Bender and Laub, 1987; Takaba and Katayama, 1997):

$$
\begin{align*}
S: \quad x(k+1) &= A_1 x(k) + A_2 \phi(k) + B w(k) \\
o &= H_1 x(k) + H_2 \phi(k) \\
y(k) &= C_1 x(k) + C_2 \phi(k) + D w(k) \\
s(k) &= L_1 x(k) + L_2 \phi(k)
\end{align*}
$$

(2)

where $x(k) \in \mathbb{R}^n$ is the dynamic state, $\phi(k) \in \mathbb{R}^{n_\phi}$, $n_\phi = n_\zeta - n_i$, is the algebraic state, $s(k)$, $w(k)$ and $y(k)$ are as before and $A_1, C_1, H_1, L_1, i = 1, 2, B$ and $D$ are real matrices of appropriate dimensions with $H_2$ nonsingular. Note that $w(k)$ is uncorrelated with $x(0)$ and $\phi(0)$ for $k = 0, 1, \ldots$

Hereafter, it is assumed that system (1) is already given in the difference-algebraic form (2) and the underlying matrices are unknown but assumed to belong to the following polytope:

$$
\Omega = \left\{ \Pi : \Pi = \sum_{i=1}^\ell \alpha_i \Pi_i, \alpha_i \geq 0 : \sum_{i=1}^\ell \alpha_i = 1 \right\}
$$

(3)

where

$$
\Pi = \begin{bmatrix}
A_1 & A_2 & B \\
H_1 & H_2 & 0 \\
C_1 & C_2 & D
\end{bmatrix}, \quad \Pi_i = \begin{bmatrix}
A_{i1} & A_{i2} & B_i \\
H_{i1} & H_{i2} & 0 \\
C_{i1} & C_{i2} & D_i
\end{bmatrix}
$$

(4)

and where $A_{ki}, C_{ki}, H_{ki}, L_{ki}, k = 1, 2, B_i$ and $D_i$ are given real constant matrices with $H_{2i}$ nonsingular.

This paper is aimed at designing a stationary asymptotically stable linear filter $F$ which provides an estimate $\hat{s}$ of the signal $s$ with a guaranteed performance in the mean square sense, irrespective of the uncertainty. We seek a filter of order $n$ with a state-space realization

$$
\begin{align*}
F: \quad \hat{x}(k+1) &= A_f \hat{x}(k) + B_f y(k), \quad \hat{x}(0) = 0 \\
\hat{s}(k) &= C_f \hat{x}(k)
\end{align*}
$$

(5)

where the matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times n_y}$ and $C_f \in \mathbb{R}^{n_s \times n}$ are to be found. The filter $F$ is required to ensure an optimized upper-bound on the worst-case asymptotic variance of the estimation error over $\Omega$, namely:

$$
\max_{\Omega} \left\{ \var\{e\} \right\},
$$

$$
\var\{e\} := \lim_{k \to \infty} \mathbb{E}\left\{ \left[ s(k) - \hat{s}(k) \right]^T \left[ s(k) - \hat{s}(k) \right] \right\}.
$$
Observe that considering (2) and (5), the dynamics of the estimation error \( e \) can be described by the following state-space model:

\[
S_e: \ 
\begin{cases}
\dot{\xi}(k+1) = \tilde{A}_1 \xi(k) + \tilde{A}_2 \phi(k) + \tilde{B}w(k) \\
\dot{e}(k) = \tilde{L}_1 \xi(k) + \tilde{L}_2 \phi(k)
\end{cases}
\]

where

\[
\xi = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ B_f C_1 & A_f \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_2 \\ B_f C_2 \end{bmatrix}
\]

\[
\tilde{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \quad \tilde{H}_1 = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \quad \tilde{H}_2 = H_2
\]

\[
\tilde{L}_1 = \begin{bmatrix} L_1 & -C_f \end{bmatrix}, \quad \tilde{L}_2 = L_2.
\]

We conclude this section by recalling a version of Finsler’s lemma that will be used in the derivation of a result in the next section.

**Lemma 1.** (Boyd et al., 1994) Given matrices \( \Psi_i = \Psi_i^T \in \mathbb{R}^{n \times n} \) and \( N_i \in \mathbb{R}^{m \times n}, \ i = 1, \ldots, \nu \), then

\[
x_i^T \Psi_i x_i < 0, \ \forall x_i \in \mathbb{R}^n; \ N_i x_i = 0, \ x_i \neq 0; \ 
i = 1, \ldots, \nu
\]

if and only if there exist matrices \( L_i \in \mathbb{R}^{n \times m}, \ i = 1, \ldots, \nu \), such that

\[
\Psi_i + L_i N_i + N_i^T L_i^T < 0, \ \forall i = 1, \ldots, \nu.
\]

Note that the conditions of (11) remain sufficient for (10) to hold even when arbitrary constraints are imposed to the scaling matrices \( L_i \), including setting \( L_i = L, \ i = 1, 2, \ldots, \nu \).

3. ROBUST FILTER

In this section we shall deal with the filter design. In order to pave the way for deriving the robust filter, initially we consider the case where the system matrix \( \Pi \) is perfectly known, i.e. \( \ell = 1 \).

First, we derive a necessary and sufficient LMI condition which ensures a prescribed upper-bound on the asymptotic variance of the estimation error for a given filter (5).

**Lemma 2.** Consider system (2) with known matrices. Given a filter (5) and a scalar \( \gamma > 0 \), the estimation error system (6) is asymptotically stable and \( \text{var} \{ e \} < \gamma \) if and only if any of the following equivalent conditions hold:

(a) There exist matrices \( P > 0, \ \Xi, N_1 \) and \( N_2 \) satisfying the following LMIs:

\[
\begin{bmatrix}
-P + \text{Her} \{ N_1 \tilde{H}_1 \} & * & * & * \\
N_2 \tilde{H}_1 + \tilde{H}_2 \tilde{N}_1^T & \text{Her} \{ N_2 \tilde{H}_2 \} & * & * \\
\tilde{P} \tilde{A}_1 & \tilde{P} \tilde{A}_2 & -P & * \\
\tilde{L}_1 & \tilde{L}_2 & 0 & -I
\end{bmatrix} < 0
\]

(b) There exist matrices \( P > 0, \ G, \ \Xi, \ N_1 \) and \( N_2 \) satisfying the following LMIs:

\[
\begin{bmatrix}
- P + \text{Her} \{ N_1 \tilde{H}_1 \} & * & * & * \\
N_2 \tilde{H}_1 + \tilde{H}_2 \tilde{N}_1^T & \text{Her} \{ N_2 \tilde{H}_2 \} & * & * \\
G \tilde{A}_1 & G \tilde{A}_2 & P - G - G^T \gamma & * \\
\tilde{L}_1 & \tilde{L}_2 & 0 & -I
\end{bmatrix} < 0
\]

\[
\gamma - \text{Tr} \{ \Xi \} > 0.
\]

**Proof.** (a) First, since the matrix \( \tilde{H}_2 \) is nonsingular, the estimation error system (6) is equivalent to the following system without algebraic constraints:

\[
\begin{cases}
\dot{\xi}(k+1) = \tilde{A}_1 \xi(k) + \tilde{B}w(k) \\
\dot{e}(k) = \tilde{L}_1 \xi(k) + \tilde{L}_2 \phi(k)
\end{cases}
\]

where

\[
\tilde{A} = \tilde{A}_1 - \tilde{A}_2 \tilde{H}_2^{-1} \tilde{H}_1, \quad \tilde{L} = \tilde{L}_1 - \tilde{L}_2 \tilde{H}_2^{-1} \tilde{H}_1.
\]

It is well known that system (18), or equivalently (6), is asymptotically stable and \( \text{var} \{ e \} < \gamma \) if and only if the there exist matrices \( P > 0 \) and \( \Xi > 0 \) satisfying the following inequalities (Boyd et al., 1994):

\[
\begin{cases}
\tilde{A}^T P \tilde{A} - P + \tilde{L}^T \tilde{L} < 0 \\
\Xi - \tilde{B}^T \tilde{P} \tilde{B} > 0 \\
\gamma - \text{Tr} \{ \Xi \} > 0.
\end{cases}
\]

In view of (19), it can be readily verified that (20) is equivalent to:

\[
\eta^T \Lambda \eta < 0, \ \eta = \Phi x, \ \forall x \in \mathbb{R}^n, \ x \neq 0
\]

where

\[
\Lambda = \begin{bmatrix}
\tilde{A}^T P \tilde{A}_1 - P + \tilde{L}_1^T \tilde{L}_1 & \tilde{A}^T P \tilde{A}_2 + \tilde{L}_2^T \tilde{L}_2 \\
\tilde{A}^T P \tilde{A}_2 + \tilde{L}_2^T \tilde{L}_1 & \tilde{A}^T P \tilde{A}_2 + \tilde{L}_2^T \tilde{L}_2
\end{bmatrix}
\]

\[
\Phi = \begin{bmatrix}
I_n \\
\tilde{H}_2^{-1} \tilde{H}_1
\end{bmatrix}.
\]

Considering that

\[
\left\{ \eta : \eta = \Phi x, \ \forall x \in \mathbb{R}^n, \ x \neq 0 \right\} = \left\{ \eta : \tilde{H} \eta = 0, \ \eta \neq 0 \right\}
\]

where

\[
\tilde{H} = \begin{bmatrix}
\tilde{H}_1 & \tilde{H}_2
\end{bmatrix}
\]

it follows from (23) that (20) is also equivalent to:

\[
\eta^T \Lambda \eta < 0, \ \forall \eta \in \mathbb{R}^n : \tilde{H} \eta = 0, \ \eta \neq 0.
\]
By Lemma 1, (26) holds if and only if there exists a matrix $N$ of appropriate dimensions such that:

$$\Lambda + NH + H^TNT < 0. \quad (27)$$

Next, letting $N = [N_1^T \ N_2^T]^T$, it can be readily verified that (27) is identical to (12).

On the other hand, by applying Schur’s complement to (21), it follows that this inequality is equivalent to (13).

(b) The equivalence of conditions (a) and (b) will be established.

(a) $\Rightarrow$ (b): If there exist matrices $P$, $\Xi$, $N_1$ and $N_2$ satisfying (12)-(14), then it follows that (15)-(17) hold with $G = P$ and the same matrices $\Xi$, $N_1$ and $N_2$.

(b) $\Rightarrow$ (a): Pre- and post multiplying (15) by

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and its transpose, respectively, leads to

$$\begin{bmatrix} -P + \text{Her}\{N_1\tilde{H}_1\} + \tilde{L}_1^T\tilde{L}_1 & * \\ N_2\tilde{H}_1 + \tilde{H}_2^T N_1^T + \tilde{L}_2^T\tilde{L}_1 & \text{Her}\{N_2\tilde{H}_2\} + \tilde{L}_2^T\tilde{L}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{A}_1^TP\tilde{A}_1 \\ \tilde{A}_2^TP\tilde{A}_2 \end{bmatrix} < 0. \quad (28)$$

Next, pre- and post multiplying (16) by $[I - \tilde{B}^T]$ and its transpose, respectively, implies that

$$\Xi - \tilde{B}^TP\tilde{B} > 0. \quad (29)$$

As $P > 0$, by applying Schur’s complement to (28) and (29) one concludes that these inequalities are equivalent to (12) and (13), respectively.

It should be remarked that $V(\xi) = \xi^T(k)P^{-1}\xi(k)$, with $P$ satisfying part (a) or (b) of Lemma 2, is a Lyapunov function for the unforced system of $S_c$.

Note that as the inequalities (15)-(17) are affine in the system matrices, by convexity condition (b) of Lemma 3 can be readily extended to the case where the matrices of system (2) are uncertain and belong to the polytope $\Omega$. In such a situation, the LMs (15)-(17) need to be satisfied at all vertices of the polytope $\Omega$ for fixed matrices $G$, $N_1$ and $N_2$ and with different matrices $P$ and $\Xi$, namely $P_i$ and $\Xi_i$, $i = 1, \ldots, \ell$ for each of the $\ell$ vertices of $\Omega$. This robust variation analysis result is presented in the next lemma. Notice that since fixed matrices $G$, $N_1$ and $N_2$ are used for all vertices of $\Omega$, this lemma provides only sufficient conditions.

Lemma 3. Consider system (2) with uncertain matrices and let $\Omega$ be a given polytope of admissible system matrices $\Pi$. Given a filter (5) and a scalar $\gamma > 0$, the estimation error system (6) is asymptotically stable and $\text{var}\{e\} < \gamma$ over $\Omega$ if there exist matrices $P_i > 0$, $\Xi_i > 0$, $i = 1, \ldots, \ell$, $G$, $N_1$ and $N_2$ satisfying the following LMs for $i = 1, \ldots, \ell$:

$$[-P_i + \text{Her}\{N_1\tilde{H}_1\} \ * \ * \ *]$$

$$\begin{bmatrix} N_2\tilde{H}_1 + \tilde{H}_2^T N_1^T & \text{Her}\{N_2\tilde{H}_2\} \ * \ * \ \tilde{G}\tilde{A}_1 \ \tilde{G}\tilde{A}_2 \ P_i - G - G^T \ * \ \tilde{L}_1 \ \tilde{L}_2 \ 0 \ -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Xi_i & \tilde{B}_1^T G^T \\ GB_1 & G + G^T - P_i \end{bmatrix} > 0$$

$$\gamma - \text{Tr}\{\Xi_i\} > 0$$

where $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{L}_1$ and $\tilde{L}_2$ denote the matrices $A_1$, $A_2$, $B$, $\bar{H}_1$, $H_2$, $\bar{L}_1$ and $\bar{L}_2$, respectively, at the $i$-th vertex of the polytope $\Omega$.

In the sequel we shall develop LMI methods to design filters with an optimized asymptotic error variance for system (2). The next theorem deals with the case where the system matrix $\Pi$ is perfectly known and is derived from conditions (15)-(17) of Lemma 2(b) by using appropriate congruent transformations and parameterizations of the filter matrices and matrix $G$.

**Theorem 1.** Consider system (2) with known matrices. Given a scalar $\gamma > 0$, there exists a filter (5) such that the estimation error system (6) is asymptotically stable and $\text{var}\{e\} < \gamma$ if and only if there exist matrices $X > 0$, $\Xi > 0$, $F_1$, $F_2$, $Q$, $R$, $S$, $Z$ and $W$ satisfying the following LMs:

$$[-X + \text{Her}\{F_1\tilde{H}_1\} \ * \ * \ *]$$

$$\begin{bmatrix} F_2\tilde{H}_1 + \tilde{H}_2^TF_1^T & \text{Her}\{F_2\tilde{H}_2\} \ * \ * \ \tilde{A}_1 \ \tilde{A}_2 \ X - \Upsilon \ * \ \tilde{L}_1 \ \tilde{L}_2 \ 0 \ -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Xi & B^T \\ B & \Upsilon - X \end{bmatrix} > 0$$

$$\gamma - \text{Tr}\{\Xi\} > 0$$

where

$$\begin{bmatrix} RA_1 \\ SA_1 + YC_1 + Q \ Q \end{bmatrix}$$

$$\begin{bmatrix} RA_2 \\ SA_2 + YC_2 \ B \end{bmatrix}$$

$$\begin{bmatrix} L_1 - Z \ -Z \ \Upsilon \end{bmatrix}$$

$$\begin{bmatrix} R + R^T \ W^T + S^T \ W + S \ W + W^T \end{bmatrix}$$
Moreover, the filter with transfer function matrix
\[ H_{y_p}(z) = Z(zI - W^{-1}Q)^{-1}W^{-1}Y \]  
(39)
solves the filtering problem.

**Proof. Sufficiency:** In the sequel, it will be shown that if the LMI s of (33)-(35) hold, then the filter (39) ensures that the conditions (15)-(17) of Lemma 2 are satisfied. Initially, note that since \( X > 0 \), (33) ensures that \( T > 0 \), which implies that \( R \) and \( W \) are nonsingular matrices.

Let the \( 2n \times 2n \) matrix \( G \) in (15) and (16) be parameterized as follows:
\[ G = \begin{bmatrix} R^{-1} & R^{-1}V^T \\ -U^{-1}SR^{-1} & U^{-1}V^T - V^{-1}SR^{-1}V^T \end{bmatrix}^{-1} \]  
(40)
where \( U \) and \( V \) are \( n \times n \) nonsingular matrices such that \( UV = W \). Moreover, let the matrix
\[ T = \begin{bmatrix} R^T & S^T \\ 0 & U^T \end{bmatrix}. \]  
(41)

We now show that the matrix \( G \) as above is well defined and both \( T \) and \( G \) are nonsingular matrices. First, note that
\[ \hat{T} := G^{-1}T = \begin{bmatrix} I & 0 \\ V & V \end{bmatrix}. \]  
(42)
Since \( R \) and \( W \) are nonsingular matrices, it follows that \( U \) and \( V \) are well defined. Hence, \( T \) and \( G^{-1}T \) are nonsingular and thus \( G \) is a nonsingular matrix as well.

Consider the following state-space realization for the filter (39):
\[ A_f = V W^{-1}Q V^{-1}, \quad B_f = V W^{-1}Y, \quad C_f = Z V^{-1}. \]
By performing straightforward matrix manipulations, it can be readily verified that
\[ \hat{T}^T G \hat{A}_1 \hat{T} = A_1, \quad \hat{T}^T G \hat{A}_2 = A_2, \quad \hat{H}_1 \hat{T} = \tilde{H}_1 \]  
(43)
\[ \hat{L}_1 \hat{T} = \tilde{L}_1, \quad \hat{T}^T G \tilde{B} = B, \quad \hat{T}^T (G + G^T) \hat{T} = \tilde{Y}. \]  
(44)
Next, let the matrices
\[ P = \hat{T}^{-T} X \hat{T}^{-1}, \quad N_1 = \hat{T}^{-T} F_1, \quad N_2 = F_2. \]  
(45)

Pre- and post-multiplying (33) by \( T_1^T \) and \( T_1 \), respectively, where
\[ T_1 = \text{diag}\{ \hat{T}^{-1}, I_{n_2}, \hat{T}^{-1}, I_{n_1} \} \]
and considering (43)-(45), one gets (15). Thus, (15) is satisfied with the matrices \( A_f, B_f, C_f, G, P, N_1 \) and \( N_2 \) defined as above.

In the sequel, it will be shown that (34) implies the feasibility of (16). To this end, introduce the transformation matrix
\[ T_2 = \text{diag}\{ I_{n_2}, \hat{T}^{-1} \}. \]
Pre- and post-multiplying (34) by \( T_2^T \) and \( T_2 \), respectively, and considering (44) and (45) implies that (16) holds with the matrices \( G \) and \( P \) defined as above.

**Necessity:** Given a filter with state-space realization \( (A_f, B_f, C_f) \) such that \( \text{var}\{e\} < \gamma \), by Lemma 2(b), there exist matrices \( P > 0, G, \Xi, N_1 \) and \( N_2 \) such that the inequalities (15)-(17) hold. Note that (15) implies that the matrix \( G \) is nonsingular.

Let the matrix \( G^{-1} \) be partitioned as follows
\[ G^{-1} := \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \]
where all the blocks are \( n \times n \) matrices and \( G_1 \) and \( G_4 \) are nonsingular matrices. In addition, since the inequality of (15) is strict, without loss of generality, \( G_2 \) and \( G_3 \) can be assumed to be nonsingular; see, e.g. de Souza and Trofino (2000). This implies that \( G_4G_2^{-1}G_1 - G_3 \) is a nonsingular matrix as well.

Next, define the matrices
\[ R = G_1^{-1}, \quad U = (G_4G_2^{-1}G_1 - G_3)^{-1} \]
\[ V = G_4G_1^{-1}, \quad S = -UG_3G_1^{-1}, \quad W = UV \]
\[ Q = UA_fV, \quad Y = UB_f, \quad Z = C_fV \]
\[ T = \begin{bmatrix} R & 0 \\ S & U \end{bmatrix}^T, \quad \tilde{T} = G^{-T}T. \]

Note that in view of the latter definitions, it can be easily shown that the matrices \( G \) and \( \tilde{T} \) are of the form as in (40) and (42), respectively. Hence, the result can be readily obtained by inverting the arguments used in the sufficiency part of the proof.

**Remark 1.** Theorem 1 provides necessary and sufficient LMI conditions for the design of a filter \( \mathcal{F} \) for the difference-algebraic system (2) with known matrices that achieves a prescribed upper-bound on the asymptotic variance of the estimation error. Note that finding the filter \( \mathcal{F} \) with a minimum asymptotic error variance is a convex optimization problem with LMI constraints.

The next theorem presents the robust filter design method. In the light of Lemma 3 and considering that the inequalities (33)-(35) are affine in the matrices of system (2), the following result can be readily derived from Theorem 1.

**Theorem 2.** Consider system (2) with uncertain matrices and let \( \Omega \) be a given polytope of admissible system matrix \( \Pi \). Then there exists a filter (5) that minimizes the upper-bound on the asymptotic error variance of Lemma 3 if there exist matrices \( X_i > 0, i = 1, \ldots, \ell, F_1, F_2, \]
\( Q, R, S, Z \) and \( W \) solving the following convex optimization problem:
\[
\min \gamma \quad \text{subject to:}
\begin{bmatrix}
-X_i + \text{Her}(F_1 \tilde{H}_i) & * & * \\
F_2 \tilde{H}_i + H_2^i F_1^T & \text{Her}(F_2 H_2) & * & * \\
A_1 & A_2 & X_i - Y & * \\
1 & \mathcal{L}_1 & L_{2i} & 0 & -I
\end{bmatrix} < 0
\]
\[
[\Xi_i, B_i^T, 0 & \mathcal{L}_1, I]
\]

for \( i = 1, \ldots, \ell \), where \( A_1, A_2, B_i, \tilde{H}_i, \) and \( \mathcal{L}_1 \) denote the matrices \( A_1, A_2, B_i, \tilde{H}_i \), and \( \mathcal{L}_1 \) of Theorem 1 at the \( i \)-th vertex of the polytope \( \Omega \). Moreover, \( \text{var}\{\epsilon\} < \gamma \) and the filter transfer function matrix is as in (39).

4. CONCLUSIONS

This paper has addressed the design of minimum variance filters for uncertain linear discrete-time descriptor systems represented by a decoupled difference-algebraic state-space model. The matrices of the system state-space model are uncertain and assumed to belong to a given polytope. First, the case of systems without uncertainty is treated and necessary and sufficient LMI conditions are derived for the design of a linear stationary strictly proper filter that minimizes the asymptotic variance of the estimation error. This result is then extended for designing linear filters that guarantee the asymptotic stability of the estimation error and an optimized upper-bound on the asymptotic error variance, in spite of significant parameter uncertainty. The proposed robust filter design method is based on a parameter-dependent Lyapunov function.

REFERENCES


