Receding Horizon Control with Linear Models

María M. Seron

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Centre for Complex Dynamic Systems and Control
Outline

1. Problem Formulation

2. Quadratic Programming
   - Objective Function Handling
   - Constraint Handling

3. Observers and Integral Action
Up to this point we have considered rather general nonlinear receding horizon optimal control problems, for which the algorithms are relatively complex.

However, remarkable simplifications occur if we specialise to the case of linear systems subject to linear inequality constraints.

We will show how a fixed horizon optimal control problem for linear systems with a quadratic objective function and linear constraints can be set up as a quadratic program.

We then discuss some practical aspects of the controller implementation, such as the use of observers to estimate states and disturbances.
Problem Formulation

We consider a system described by the following linear, time-invariant model:

\[ x_{k+1} = Ax_k + Bu_k, \]  
\[ y_k = Cx_k + d_k, \]

where \( x_k \in \mathbb{R}^n \) is the state, \( u_k \in \mathbb{R}^m \) is the control input, \( y_k \in \mathbb{R}^m \) is the output, and \( d_k \in \mathbb{R}^m \) is a time-varying output disturbance.

We assume that \((A, B, C)\) is stabilisable and detectable and that one is not an eigenvalue of \(A\).
We consider the problem where the output $y_k$ in (2) is required to track a constant reference $y^*$ in the presence of the disturbance $d_k$.

That is, we wish to regulate, to zero, the output error

$$e_k \triangleq y_k - y^* = Cx_k + d_k - y^*. \quad (3)$$

The linear model for RHC is then:
Problem Formulation

We consider the following steady state values:

- $\bar{d}$ is the steady state value of the output disturbance $d_k$:
  \[ \bar{d} \triangleq \lim_{k \to \infty} d_k. \]

- $y_s = y^*$ is the output setpoint, or desired steady state value.

- $u_s$ and $x_s$ are the setpoints, or desired steady state values for $u_k$ and $x_k$, respectively.

We then have that

\[ y_s = y^* = Cx_s + \bar{d} \]

and hence, given $\bar{d}$ and $y^*$, we can compute $u_s$ and $x_s$ as

\[ u_s = [C(I - A)^{-1}B]^{-1}(y^* - \bar{d}), \quad (4) \]

\[ x_s = (I - A)^{-1}Bu_s. \quad (5) \]
The above steady state values will be used in the calculations.

Here we assume knowledge of the disturbance sequence \(\{d_k, d_{k+1}, \ldots, d_{k+N-1}\}\) for all \(k\), and the current state measurement \(x_k = x\).

The scheme for RHC is as follows:

\[
\begin{align*}
 u_k & \quad \xrightarrow{\text{RHC}} \quad x_{k+1} = Ax_k + Bu_k \\
 & \quad \xrightarrow{x_k} \quad \{d_k, d_{k+1}, \ldots, d_{k+N-1}\} \\
 & \quad \xrightarrow{\bar{d}, y^*} \quad \text{RHC}
\end{align*}
\]

Without loss of generality, we take the current time as zero.
The RHC is based on the following finite horizon optimisation problem:

Find the sequences \( \{u_0, \ldots, u_{M-1}\} \), \( \{x_0, \ldots, x_N\} \) and \( \{e_0, \ldots, e_{N-1}\} \) that minimise the objective function:

\[
V_{N,M}(\{x_k\}, \{u_k\}, \{e_k\}) \triangleq \frac{1}{2}(x_N - x_s)^T P(x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k
\]

\[
+ \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s),
\]

subject to (1)–(5), where \( P \succeq 0 \), \( Q \succeq 0 \), \( R > 0 \). \( N \) is the prediction horizon and \( M \leq N \) is the control horizon.

The control is set equal to its setpoint after \( M \) steps, that is, \( u_k = u_s \) for all \( k \geq M \).
We now introduce inequality constraints into the problem formulation.

Magnitude and rate constraints on the plant input and output can be expressed as follows:

\[
\begin{align*}
    u_{\text{min}} &\leq u_k \leq u_{\text{max}}, & k &= 0, \ldots, M - 1, \\
    y_{\text{min}} &\leq y_k \leq y_{\text{max}}, & k &= 1, \ldots, N - 1, \\
    \delta u_{\text{min}} &\leq u_k - u_{k-1} \leq \delta u_{\text{max}}, & k &= 0, \ldots, M - 1,
\end{align*}
\]  

(7)

where \(u_{-1}\) is the input used in the previous step of the receding horizon implementation, which has to be stored for use in the current fixed horizon optimisation.
More generally, we may require to impose state constraints of the form

$$x_k \in X_k \quad \text{for } k = 1, \ldots, N,$$

where $X_k$ is a polyhedral set of the form

$$X_k = \{ x \in \mathbb{R}^n : L_k x \leq W_k \}.$$

For example, the constraint $x_N \in X_f$, where $X_f$ is a set satisfying certain properties, is useful to establish closed loop stability, as we will see later today.
Problem Formulation

Summarising, given the disturbance sequence \(\{d_0, \ldots, d_{N-1}\}\), the current state \(x_0 = x\), the previous input \(u_{-1}\), and the steady state values \(y^*, \bar{d}\), and \(u_s\) and \(x_s\) computed from (4) and (5), the RHC is based on the problem:

\[
\begin{align*}
\text{minimise} & \quad \frac{1}{2}(x_N - x_s)^TP(x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k \\
& \quad + \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s),
\end{align*}
\]

subject to

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \quad k = 0, \ldots, N - 1, \\
e_k &= Cx_k + d_k - y^*, \quad k = 0, \ldots, N - 1, \\
u_k &= u_s, \quad k = M, \ldots, N - 1, \\
u_{\min} &\leq u_k \leq u_{\max}, \quad k = 0, \ldots, M - 1, \\
y_{\min} &\leq y_k \leq y_{\max}, \quad k = 1, \ldots, N - 1, \\
\delta u_{\min} &\leq u_k - u_{k-1} \leq \delta u_{\max}, \quad k = 0, \ldots, M - 1, \\
x_k &\in \mathcal{X}_k, \quad k = 1, \ldots, N.
\end{align*}
\]
Summarising, given the disturbance sequence \( \{d_0, \ldots, d_{N-1}\} \), the current state \( x_0 = x \), the previous input \( u_{-1} \), and the steady state values \( y^* \), \( \bar{d} \), and \( u_s \) and \( x_s \) computed from (4) and (5), the RHC is based on the problem:

\[
\begin{align*}
\text{minimise} & \quad \frac{1}{2} (x_N - x_s)^T P (x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k \\
& \quad + \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s),
\end{align*}
\]

subject to

\[
\begin{align*}
x_{k+1} &= A x_k + B u_k, \quad k = 0, \ldots, N - 1, \\
e_k &= C x_k + d_k - y^*, \quad k = 0, \ldots, N - 1, \\
u_k &= u_s, \quad k = M, \ldots, N - 1, \\
u_{\min} &\leq u_k \leq u_{\max}, \quad k = 0, \ldots, M - 1, \\
y_{\min} &\leq y_k \leq y_{\max}, \quad k = 1, \ldots, N - 1, \\
\delta u_{\min} &\leq u_k - u_{k-1} \leq \delta u_{\max}, \quad k = 0, \ldots, M - 1, \\
x_k &\in X_k, \quad k = 1, \ldots, N.
\end{align*}
\]
Summarising, given the disturbance sequence \( \{d_0, \ldots, d_{N-1}\} \), the current state \( x_0 = x \), the previous input \( u_{-1} \), and the steady state values \( y^* \), \( \bar{d} \), and \( u_s \) and \( x_s \) computed from (4) and (5), the RHC is based on the problem:

\[
\begin{align*}
\text{minimise} & \quad \frac{1}{2} (x_N - x_s)^T P (x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k \\
& \quad + \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s), \\
\text{subject to} & \quad x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N - 1, \\
& \quad e_k = C x_k + d_k - y^*, \quad k = 0, \ldots, N - 1, \\
& \quad u_k = u_s, \quad k = M, \ldots, N - 1, \\
& \quad u_{\min} \leq u_k \leq u_{\max}, \quad k = 0, \ldots, M - 1, \\
& \quad y_{\min} \leq y_k \leq y_{\max}, \quad k = 1, \ldots, N - 1, \\
& \quad \delta u_{\min} \leq u_k - u_{k-1} \leq \delta u_{\max}, \quad k = 0, \ldots, M - 1, \\
& \quad x_k \in X_k, \quad k = 1, \ldots, N.
\end{align*}
\]
Problem Formulation

Summarising, given the disturbance sequence \( \{d_0, \ldots, d_{N-1}\} \), the current state \( x_0 = x \), the previous input \( u_{-1} \), and the steady state values \( y^*, \bar{d} \), and \( u_s \) and \( x_s \) computed from (4) and (5), the RHC is based on the problem:

\[
\text{minimise} \quad \frac{1}{2} (x_N - x_s)^T P (x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k \\
+ \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s),
\]

subject to

\[
\begin{align*}
x_{k+1} &= A x_k + B u_k, \quad k = 0, \ldots, N - 1, \\
e_k &= C x_k + d_k - y^*, \quad k = 0, \ldots, N - 1, \\
u_k &= u_s, \quad k = M, \ldots, N - 1, \\
u_{\min} \leq u_k \leq u_{\max}, & \quad k = 0, \ldots, M - 1, \\
y_{\min} \leq y_k \leq y_{\max}, & \quad k = 1, \ldots, N - 1, \\
\delta u_{\min} \leq u_k - u_{k-1} \leq \delta u_{\max}, & \quad k = 0, \ldots, M - 1, \\
x_k \in X_k, & \quad k = 1, \ldots, N.
\end{align*}
\]
Problem Formulation

The above fixed horizon minimisation problem is solved at each time step for the current state and disturbance values.

Then, the first move of the resulting control sequence is used as the current control, and the procedure is repeated at the next time step in a RHC fashion, as described in the previous lecture.
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Problem Formulation

The above fixed horizon minimisation problem is solved at each time step for the current state and disturbance values.

Then, the first move of the resulting control sequence is used as the current control, and the procedure is repeated at the next time step in a RHC fashion, as described in the previous lecture.
The fixed horizon optimisation problem described above can be transformed into a *quadratic program* [QP].

We show below how this is accomplished.

We will separate the analysis in:

(i) Objective Function Handling

(ii) Constraint Handling
Objective Function Handling

We first substitute
\[ e_k = Cx_k + d_k - y^* \quad \text{ (} y^* = Cx_s + \bar{d} \text{)} \]
\[ = C(x_k - x_s) + (d_k - \bar{d}) \]
into the objective function
\[
V_{N,M} = \frac{1}{2} (x_N - x_s)^T P(x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} e_k^T Q e_k + \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s)
\]
to obtain
\[
V_{N,M} = \frac{1}{2} (x_N - x_s)^T P(x_N - x_s) + \frac{1}{2} \sum_{k=0}^{N-1} (x_k - x_s)^T C^T QC (x_k - x_s) \\
+ \sum_{k=0}^{N-1} (x_k - x_s)^T CQ (d_k - \bar{d}) + \frac{1}{2} \sum_{k=0}^{N-1} (d_k - \bar{d})^T Q (d_k - \bar{d}) \\
+ \frac{1}{2} \sum_{k=0}^{M-1} (u_k - u_s)^T R (u_k - u_s).
\]
Objective Function Handling

We next show how (10) can be transformed into an objective function of the form used in QP. We start by writing, from (1) with $x_0 = x$, and using the constraint that $u_k = u_s$ for all $k \geq M$, the following set of equations:

\begin{align*}
    x_1 &= Ax + Bu_0, \\
    x_2 &= A^2x + ABu_0 + Bu_1, \\
    &\vdots \\
    x_M &= A^Mx + A^{M-1}Bu_0 + \cdots + Bu_{M-1}, \\
    x_{M+1} &= A^{M+1}x + A^MBu_0 + \cdots + ABu_{M-1} + Bu_s, \\
    &\vdots \\
    x_N &= A^Nx + A^{N-1}Bu_0 + \cdots + A^{N-M}Bu_{M-1} + \sum_{i=0}^{N-M-1} A^iBu_s. \\
\end{align*}

(11)
Using $x_s = Ax_s + Bu_s$ (from (5)) recursively, we can write a similar set of equations for $x_s$ as follows:

\[
  x_s = Ax_s + Bu_s,
\]

\[
  x_s = A^2 x_s + ABu_s + Bu_s,
\]

\[
  \vdots
\]

\[
  x_s = A^M x_s + A^{M-1} Bu_s + \cdots + Bu_s,
\]

\[
  x_s = A^{M+1} x_s + A^M Bu_s + \cdots + ABu_s + Bu_s,
\]

\[
  \vdots
\]

\[
  x_s = A^N x_s + A^{N-1} Bu_s + \cdots + A^{N-M} Bu_s + \sum_{i=0}^{N-M-1} A^i Bu_s.
\]
We now subtract the set of equations (12) from the set (11), and rewrite the resulting difference in vector form to obtain

$$\mathbf{x} - \mathbf{x}_s = \Gamma(\mathbf{u} - \mathbf{u}_s) + \Omega(x - x_s), \quad (13)$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{x}_s \triangleq \begin{bmatrix} x_s \\ x_s \\ \vdots \\ x_s \end{bmatrix}, \quad \mathbf{u} \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix}, \quad \mathbf{u}_s \triangleq \begin{bmatrix} u_s \\ u_s \\ \vdots \\ u_s \end{bmatrix}, \quad (14)$$

($$x_s$$ is an $$nN \times 1$$ vector, and $$\mathbf{u}_s$$ is an $$mM \times 1$$ vector),
and where

\[
\Gamma \triangleq \begin{bmatrix}
    B & 0 & \ldots & 0 & 0 \\
    AB & B & \ldots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    A^{M-1}B & A^{M-2}B & \ldots & AB & B \\
    A^MB & A^{M-1}B & \ldots & A^2B & AB \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    A^{N-1}B & A^{N-2}B & \ldots & \ldots & A^{N-M}B
\end{bmatrix}, \quad \Omega \triangleq \begin{bmatrix}
    A \\
    A^2 \\
    \vdots \\
    A^N
\end{bmatrix}.
\] (15)

Thus,

\[
x - x_s = \Gamma(u - u_s) + \Omega(x - x_s)
\]

is a linear equality constraint on the vector of control moves \(u\).
We also define the disturbance vector

\[ \mathbf{d} \triangleq \begin{bmatrix} (d_1 - \bar{d})^T & (d_2 - \bar{d})^T & \cdots & (d_{N-1} - \bar{d})^T & 0_{1 \times m} \end{bmatrix}^T, \quad (16) \]

and the matrices

\[ \mathbf{Q} \triangleq \text{diag}\{C^TQC, \ldots, C^TQC, P\}, \]
\[ \mathbf{R} \triangleq \text{diag}\{R, \ldots, R\}, \]
\[ \mathbf{Z} \triangleq \text{diag}\{C^TQ, C^TQ, \ldots, C^TQ\}, \quad (17) \]

where \( \text{diag}\{A_1, A_2, \ldots, A_p\} \) denotes a block diagonal matrix having the matrices \( A_i \) as its diagonal blocks.
Objective Function Handling

We now rewrite the objective function (10) using the vector notation (14), (17) and (16), as follows:

\[ V_{N,M} = \frac{1}{2} e_0^\top Q e_0 + \frac{1}{2} (x - x_s)^\top Q (x - x_s) + \frac{1}{2} (u - u_s)^\top R (u - u_s) \]

\[ + (x - x_s)^\top Z d + \frac{1}{2} d^\top \text{diag}\{Q, Q, \ldots, Q\} d. \]  

(18)

Next, we substitute the equality constraint (13) into (18) to yield

\[ V_{N,M} = \bar{V} + \frac{1}{2} u^\top (\Gamma^\top Q \Gamma + R) u + u^\top \Gamma^\top Q \Omega (x - x_s) \]

\[ - u^\top (\Gamma^\top Q \Gamma + R) u_s + u^\top \Gamma^\top Z d, \]

\[ \triangleq \bar{V} + \frac{1}{2} u^\top H u + u^\top [F(x - x_s) - H u_s + D d], \]

where \( \bar{V} \) is independent of \( u \) and

\[ H \triangleq \Gamma^\top Q \Gamma + R, \quad F \triangleq \Gamma^\top Q \Omega, \quad D \triangleq \Gamma^\top Z. \]  

(19)

Note that \( H \) in (19) is positive definite because \( R > 0 \).
Summarising, we have:

- expressed the variables in vector form, and
- substituted the equality constraints given by the state equations into the objective function

To obtain the function

\[
V_{N,M} = \bar{V} + \frac{1}{2} u^T H u + u^T [F(x - x_s) - H u_s + D d],
\]

(20)

which is a quadratic function on the vector of control moves \( u \).
From (20) it is clear that, if the problem is *unconstrained*, \( V_{N,M} \) is minimised by taking

\[
\mathbf{u} = \mathbf{u}^{\text{OPT}}_{\mathbf{uc}} \triangleq -H^{-1} [F(x - x_s) - H\mathbf{u}_s + D\mathbf{d}].
\] (21)

The vector formed by the first \( m \) components of (21), \( u^{\text{OPT}}_{0,\mathbf{uc}} \), has a linear time-invariant feedback structure of the form

\[
u^{\text{OPT}}_{0,\mathbf{uc}} = -K(x - x_s) + \mathbf{u}_s + K_d \mathbf{d},
\]

where \( K \) and \( K_d \) are defined as the first \( m \) rows of the matrices \( H^{-1}F \) and \( -H^{-1}D \), respectively.
Hence, the control law

$$u_{0,uc}^{OPT} = -K(x - x_s) + u_s + K_d d,$$

(22)

is the control used by the RHC algorithm if the problem is unconstrained.

By appropriate selection of the weightings in the objective function (6), the resulting $K$ is such that the matrix $(A - BK)$ is Hurwitz, that is, all its eigenvalues have moduli smaller than one.

More interestingly, even in the constrained case, the optimal RHC solution has the form (22) in a region of the state space that contains the steady state setpoint $x = x_s$. This point will be discussed in detail on Day 3.
Hence, the control law

\[ u_{0,uc}^{\text{OPT}} = -K(x - x_s) + u_s + K_d d, \]  

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is the control used by the RHC algorithm if the problem is unconstrained.

By appropriate selection of the weightings in the objective function (6), the resulting \( K \) is such that the matrix \((A - BK)\) is Hurwitz, that is, all its eigenvalues have moduli smaller than one.

More interestingly, even in the constrained case, the optimal RHC solution has the form (22) in a region of the state space that contains the steady state setpoint \( x = x_s \). This point will be discussed in detail on Day 3.
We now consider the input and output constraints

\[
\begin{align*}
    u_{\min} &\leq u_k \leq u_{\max}, & k &= 0, \ldots, M - 1, \\
    y_{\min} &\leq y_k \leq y_{\max}, & k &= 1, \ldots, N - 1, \\
    \delta u_{\min} &\leq u_k - u_{k-1} \leq \delta u_{\max}, & k &= 0, \ldots, M - 1,
\end{align*}
\]

(23)

and the state constraints

\[
\begin{align*}
x_k &\in X_k, & k &= 1, \ldots, N, \\
X_k &= \{ x \in \mathbb{R}^n : L_k x \leq W_k \}.
\end{align*}
\]

(24)

(25)

We require that the setpoint \( y_s = y^* \) and the corresponding input and state setpoints \( u_s \) and \( x_s \) satisfy the required constraints.

For example, in the case of the constraints given in (23), we assume that

\[
u_{\min} \leq u_s \leq u_{\max} \quad \text{and} \quad y_{\min} \leq y^* \leq y_{\max}.
\]
The constraints (23)–(25) can be written as linear constraints on \( \mathbf{u} \) of the form

\[
L \mathbf{u} \leq W.
\]

To see how this is done, let us start with an example with \( N = M = 2 \) and constraints

\[
\begin{align*}
&u_{\text{min}} \leq u_0 \leq u_{\text{max}}, \\
&u_{\text{min}} \leq u_1 \leq u_{\text{max}}, \\
&L_1 x_1 \leq W_1, \\
&L_2 x_2 \leq W_2.
\end{align*}
\]

In this case \( \mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \).
For the input constraints we have

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \leq \begin{bmatrix} u_{\max} \\ u_{\max} \end{bmatrix},
\]

\[
- \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \leq - \begin{bmatrix} u_{\min} \\ u_{\min} \end{bmatrix}.
\]

For the state constraints we have

\[
L_1 x_1 \leq W_1, \quad \text{where} \quad x_1 = A x_0 + B u_0,
\]

\[
L_2 x_2 \leq W_2, \quad \text{where} \quad x_2 = A^2 x_0 + A B u_0 + B u_1,
\]

which leads to

\[
\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} B & 0 \\ A B & B \end{bmatrix} \mathbf{u} \leq - \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ A^2 & B \end{bmatrix} x_0 + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}.
\]
In a similar way, all constraints (23)–(25) can be written as *linear* constraints on $u$ of the form

$$Lu \leq W,$$  \hspace{1cm} (26)

where

$$L = \begin{bmatrix} I_{Mm} \\ \Psi \\ E \\ -I_{Mm} \\ -\Psi \\ -E \\ \tilde{L} \end{bmatrix}, \quad W = \begin{bmatrix} u_{\max} \\ y_{\max} \\ \delta u_{\max} \\ u_{\min} \\ y_{\min} \\ \delta u_{\min} \end{bmatrix}. \hspace{1cm} (27)$$

In (27), $I_{Mm}$ is the $Mm \times Mm$ identity matrix (where $M$ is the control horizon and $m$ is the number of inputs).
\[ \Psi, \ E \text{ and } \tilde{L} \text{ are the following matrices:} \]

\[
\Psi = \begin{bmatrix}
CB & 0 & \ldots & 0 & 0 \\
CAB & CB & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
CA^{M-1}B & CA^{M-2}B & \ldots & CAB & CB \\
CA^MB & CA^{M-1}B & \ldots & CA^2B & CAB \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
CA^{N-2}B & CA^{N-3}B & \ldots & \ldots & CA^{N-M-1}B \\
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
I_m & 0 & \ldots & 0 \\
-l_m & l_m & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -l_m & l_m \\
\end{bmatrix}, \quad \tilde{L} \triangleq \text{diag}\{L_1, L_2, \ldots, L_N\}\Gamma,
\]

where \( L_1, \ldots, L_N \) are the constraint matrices given in (9) and \( \Gamma \) is given in (15).
The vectors forming $W$ in (27) are as follows

$$u_{\text{max}} = \begin{bmatrix} u_{\text{max}} \\ \vdots \\ u_{\text{max}} \end{bmatrix}, \quad u_{\text{min}} = \begin{bmatrix} -u_{\text{min}} \\ \vdots \\ -u_{\text{min}} \end{bmatrix},$$

$$\delta u_{\text{max}} = \begin{bmatrix} u_{-1} + \delta u_{\text{max}} \\ \delta u_{\text{max}} \\ \vdots \\ \delta u_{\text{max}} \end{bmatrix}, \quad \delta u_{\text{min}} = \begin{bmatrix} -u_{-1} - \delta u_{\text{min}} \\ -\delta u_{\text{min}} \\ \vdots \\ -\delta u_{\text{min}} \end{bmatrix},$$

where $u_{\text{max}}, u_{\text{min}}, \delta u_{\text{max}}$ and $\delta u_{\text{min}}$ are the vectors of input magnitude and rate constraint limits defined in (7).
Constraint Handling

\[ y_{\text{max}} = \begin{bmatrix} 
    y_{\text{max}} - CAx - d_1 \\
    \vdots \\
    y_{\text{max}} - CA^M x - d_M \\
    y_{\text{max}} - CA^{M+1} x - d_{M+1} - CBu_s \\
    \vdots \\
    y_{\text{max}} - CA^{N-1} x - d_{N-1} - \sum_{i=0}^{N-M-2} CA^i Bu_s 
\end{bmatrix}, \]

\[ y_{\text{min}} = \begin{bmatrix} 
    -y_{\text{min}} + CAx + d_1 \\
    \vdots \\
    -y_{\text{min}} + CA^M x + d_M \\
    -y_{\text{min}} + CA^{M+1} x + d_{M+1} + CBu_s \\
    \vdots \\
    -y_{\text{min}} + CA^{N-1} x + d_{N-1} + \sum_{i=0}^{N-M-2} CA^i Bu_s 
\end{bmatrix}, \]

where \( x \) is the initial state, and \( y_{\text{max}} \) and \( y_{\text{min}} \) are the vectors of output magnitude constraint limits defined in (7).
Finally, for the state constraints we have

\[ \tilde{W} \triangleq - \text{diag}\{L_1, L_2, \ldots, L_N\} \]

\[
\begin{bmatrix}
Ax \\
\vdots \\
A^{M}x \\
A^{M+1}x + Bu_s \\
A^{M+2}x + ABu_s \\
\vdots \\
A^{N}x + \sum_{i=0}^{N-M-1} A^iBu_s
\end{bmatrix} + \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_N
\end{bmatrix},
\]

where \( x \) is the initial state and \( L_1, \ldots, L_N, W_1, \ldots, W_N \) are the matrices and vectors of the state constraint polyhedra (9).
The QP Problem

Using the above formalism, we can express the problem of minimising (6) subject to the inequality constraints (7)–(9) as the QP problem of minimising (20) subject to (26), that is,

\[
\min_u \frac{1}{2} u^T H u + u^T [F(x - x_s) - H u_s + D d],
\]

subject to:

\[ L u \leq W. \]  

(28)

Note that the term \( \bar{V} \) in (20) has not been included in (28) since it is independent of \( u \).

The optimal solution \( u^{\text{opt}}(x) \) to (28) is then:

\[
 u^{\text{opt}}(x) = \arg \min_{L u \leq W} \frac{1}{2} u^T H u + u^T [F(x - x_s) - H u_s + D d].
\]  

(29)
The matrix $H$ is called the *Hessian* of the QP.

If the Hessian is positive definite, the QP is convex. This is indeed the case for $H$ given (19), which, as already mentioned, is positive definite since $R > 0$ in the objective function (6).

On Day 4 we will investigate the structure of the Hessian in detail and formulate numerically stable ways to compute it from the problem data.

Standard numerical procedures (called *QP algorithms*) are available to solve the above optimisation problem.
Once the QP problem (29) is solved, the RHC algorithm applies, at the current time $k$, only the first control move, formed by the first $m$ components of the optimal vector $u^{\text{opt}}(x)$ in (29).

This yields a control law of the form

$$u_k = \mathcal{K}(x_k, \bar{d}, y^*, d),$$

(30)

where $x_k = x$ is the current state, and where the dependency on $\bar{d}$ and $y^*$ is via $u_s$, $x_s$ and $d$ (see (4), (5), and (16)) as data for the optimisation (29).

Then the whole procedure is repeated at the next time instant, with the optimisation horizon kept constant.
RHC using QP

\[ x_{k+1} = Ax_k + Bu_k \]

\[ K(x_k, \bar{d}, y^*, d) \]

\[ \{d_k, \ldots, d_{k+N-1}\} \]

\[ [\bar{d}, y^*] \]

\[ u_k \]

\[ u^\text{OPT}_0 \]

\[ [I \, 0 \ldots 0] \]

\[ \text{QP} \]

\[ x_k \]

\[ \{d_k, \ldots, d_{k+N-1}\} \]

\[ [\bar{d}, y^*] \]

Figure: Receding horizon control using QP
Observers and Integral Action

The above development has assumed that the system evolves in a deterministic fashion and that the full state (including disturbances) is measured.

When the state and disturbances are not measured, it is possible to obtain combined state and disturbance estimates via an observer.

Those estimates can then be used in the control algorithm by means of the certainty equivalence [CE] principle.

The CE principle consists of designing the control law assuming knowledge of the states and disturbances, and then using their estimates as if they were the true ones when implementing the controller.
In practice, it is also important to ensure that the true system output reaches its desired steady state value, or setpoint, despite the presence of unaccounted constant disturbances and modelling errors.

In linear control, this is typically achieved by the inclusion of integrators in the feedback loop; hence, we say that a control algorithm that achieves this property has *integral action*. 
In the context of constrained control, there are several alternative ways in which integral action can be included into a control algorithm.

For example, using CE, the key idea is to include a model for constant disturbances at the input or output of the system and design an observer for the composite model including system and disturbance models.

Then the control is designed to reject the disturbance, assuming knowledge of states and disturbance.

Finally, the control is implemented using CE.

The resulting observer-based closed loop system has integral action, as we will next show.
We will consider a *model* of the system of the form (1)-(2) with a constant output disturbance. (One could equally assume a constant input disturbance.) This leads to a composite model of the form

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \\
d_{k+1} &= d_k = \bar{d}, \\
y_k &= Cx_k + d_k.
\end{align*}
\] (31)

We do not assume that the *model* (31) is a correct representation of the *real* system.

In fact, we do not assume knowledge of the real system at all, but we assume that we can measure its output, which we denote $y_k^{\text{REAL}}$. 
The RHC algorithm is now applied to the model (31) to design a controller for rejection of the constant output disturbance (and tracking of a constant reference) assuming knowledge of the model state and disturbance measurements.

At each $k$ the algorithm consists of solving the QP problem (29), that is,

$$u^{\text{opt}}(x) = \arg \min_{\sum u \leq W} \frac{1}{2} u^T H u + u^T [F(x - x_s) - H u_s], \quad (32)$$

for the current state $x_k = x$, with $u_s, x_s$ computed from (4), (5).

Note that we have set $\bar{d} = 0$ (this follows from (16), since $d_k = \bar{d}$ for all $k$).
To apply the CE principle, we use the model (31) and the real system output $y_k^{\text{REAL}}$ to construct an observer of the form

$$\dot{x}_{k+1} = A\hat{x}_k + Bu_k + L_1[y_k^{\text{REAL}} - C\hat{x}_k - \hat{d}_k],$$
$$\dot{d}_{k+1} = \hat{d}_k + L_2[y_k^{\text{REAL}} - C\hat{x}_k - \hat{d}_k],$$

where $L_1$ and $L_2$ are determined via any observer design method (such as the Kalman filter) that ensures that the matrix

$$\begin{bmatrix}
A - L_1 C & -L_1 \\
-L_2 C & I - L_2
\end{bmatrix}$$

is Hurwitz.
Then we simply use the estimates \((\hat{x}_k, \hat{d}_k)\) given by (33) in the RHC algorithm as if they were the true states, that is, \(\hat{x}_k\) replaces \(x\) (which is the current state) and \(\hat{d}_k\) replaces \(\bar{d}\).

Specifically, the QP problem (32) is solved at each \(k\) for \(x = \hat{x}_k\), \(d = 0\) and with \(u_s = u_{s,k}\) and \(x_s = x_{s,k}\) computed as

\[
\begin{align*}
    u_{s,k} &\triangleq [C(I - A)^{-1} B]^{-1}(y^* - \hat{d}_k), \\
    x_{s,k} &\triangleq (I - A)^{-1} Bu_{s,k}.
\end{align*}
\]

(34) (35)

Note that now \(u_{s,k}\) and \(x_{s,k}\) are time-varying variables (compare with (4) and (5)).
Thus, the resulting CE control law has the form (30) evaluated at $x_k = \hat{x}_k$, $\bar{d} = \hat{d}_k$, $y^* = y^*$, and $d = 0$, that is,

$$u_k = \mathcal{K}(\hat{x}_k, \hat{d}_k, y^*, 0).$$

(36)
Thus, the resulting CE control law has the form (30) evaluated at $x_k = \hat{x}_k$, $\bar{d} = \hat{d}_k$, $y^* = y^*$, and $d = 0$, that is,

$$u_k = \mathcal{K}(\hat{x}_k, \hat{d}_k, y^*, 0).$$  \hspace{1cm} (36)
We will next show how integral action is achieved.

We make the following assumptions.

**Assumption**

*We assume that the real system in closed loop with the (constrained) control law (36) reaches a steady state in which no constraints are active and where \{\hat{y}_k^{\text{REAL}}\} and \{\hat{d}_k\} converge to the constant values \(\hat{\bar{y}}^{\text{REAL}}, \bar{d}\).*

**Assumption**

*The matrix \((A - BK)\) in (22) \((u_{0,uc}^{\text{OPT}} = -K(x - x_s) + u_s + K_d d)\) is Hurwitz, that is, all its eigenvalues have moduli smaller than one.*
Note that the assumption of no constraints being active when steady state is achieved implies that the control law, in steady state, must satisfy equation (22) (with $d = 0$) evaluated at the steady state values, that is,

$$
\bar{u} = -K(\bar{x} - \bar{x}_s) + \bar{u}_s, \tag{37}
$$

where $\bar{u} \triangleq \lim_{k \to \infty} u_k$, $\bar{x} \triangleq \lim_{k \to \infty} \hat{x}_k$, $\bar{x}_s \triangleq \lim_{k \to \infty} x_{s,k}$, and $\bar{u}_s \triangleq \lim_{k \to \infty} u_{s,k}$. In (37), $\bar{u}_s$ and $\bar{x}_s$ satisfy, from (34) and (35),

$$
\bar{u}_s = [C(I - A)^{-1} B]^{-1}(y^* - \tilde{d}), \tag{38}
$$

$$
\bar{x}_s = (I - A)^{-1} B\bar{u}_s, \tag{39}
$$

where $\tilde{d} \triangleq \lim_{k \to \infty} \hat{d}_k$. 
We then have the following result.

**Lemma**

*Under Assumptions 4.1 and 4.2, the real system output converges to the desired setpoint $y^*$, that is*

$$\bar{y}^{\text{REAL}} = y^*. \quad (40)$$
From the observer equations (33) in steady state we have

\[(I - A) \ddot{x} = B\ddot{u}, \quad (41)\]

\[\ddot{y}^{\text{REAL}} = C\dddot{x} + \dddot{d}. \quad (42)\]

Substituting (37) \((\ddot{u} = -K(\dddot{x} - \dddot{x}_s) + \dddot{u}_s)\) in (41) and using \((I - A)\ddot{x}_s = B\ddot{u}_s\) from (39), we obtain

\[(I - A) \dddot{x} = B\ddot{u}_s - BK\dddot{x} + BK\dddot{x}_s = (I - A)\dddot{x}_s - BK\dddot{x} + BK\dddot{x}_s.\]
Proof (continued):

Reordering terms in the above equation yields

\[ (I - A + BK)\ddot{x} = (I - A + BK)\dot{x}_s, \quad \text{or} \quad \ddot{x} = \dot{x}_s, \quad (43) \]

since \((A - BK)\) is Hurwitz by Assumption 4.2.

We then have, from (43), (39) and (38), that

\[ C\ddot{x} = C\dot{x}_s = y^* - \ddot{d}. \quad (44) \]

Thus, the result (40) follows upon substitution of (44) into (42). \(\square\)
Note that we have not shown (and indeed it will not be true in general) that $\tilde{d}$ is equal to the true output disturbance.

In fact, the disturbance could actually be at the system input. Moreover, since we have not assumed that the model is correct, there need be no connection between $\hat{x}$ and the states of the real system.

The Lemma just proved is then an important result since it shows that, subject to the assumption that a steady state is achieved, the required output setpoint can be achieved despite uncertainty of different sources.