A Model-Based Feedback Control Strategy for Heap Bioleaching Processes

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Abstract—In this paper we propose a feedback control strategy to increase the copper recovery in a bioleaching process. The control strategy is implemented using Internal Model Control based on a multivariable linear state-space model obtained by the Maximum Likelihood method. The linear model describes the variations of defined outputs (average temperature and copper concentration) around a nominal trajectory. This trajectory is obtained as the response to nominal inputs that have been proved to work well in real leaching applications, and is computed using a comprehensive high complexity mathematical model developed by BHP Billiton Innovation. The results show that significant increments in copper extraction can be obtained using limited control actions with the proposed feedback strategy.

I. INTRODUCTION

This paper proposes a simple feedback control strategy to improve mineral extraction in heap bioleaching processes for copper sulphidic ores. Mineral heap leaching is a mining technology based on the dissolution of minerals by a percolating solution through large piles of crushed copper ore. In heap bioleaching the mineral extraction is enhanced by the catalytic action of naturally occurring bacteria. Bioleaching appears as a lower-cost and more environmentally friendly alternative to smelting for the production of high purity cathodic copper from low grade sulphidic ores [1].

Although heap bioleaching has been used for copper extraction in stand-alone facilities worldwide for many years, its efficiency has been marred by lower than expected production rates and longer than expected start up times [2]. Hence in recent years, the interest of mining companies in developing control and optimisation strategies to improve the technology has increased, leading to greater research efforts to understand the mechanisms that make the process work better [3], [4].

The present work aims to contribute to the optimisation of the bioleaching technology by utilising model-based feedback control methods—to the best of our knowledge, without precedents in the literature. On one hand, typical heap bioleaching facilities around the world operate in essentially an open-loop mode, with fixed set-points for the entire life of the process (sometimes of the order of two years long [5]). On the other hand, although there exist accurate models to describe important aspects of the process such as oxidation, bacterial growth, and temperature [6], [2], [4], such models are typically of a great mathematical complexity, which makes the design of model-based control strategies very hard.

In our approach we consider an important model estimation aspect well known in the area of identification for control: very often, a relatively simple model suffices as the basis for successful control design for complex processes, as long as essential dynamic aspects of the process are captured by the model [7]. Having this statement in mind, in Section II we focus on the estimation of a simple multivariable state-space linear model maximising the Likelihood function [8].

To generate model estimation data we use a comprehensive high complexity mathematical model developed by BHP-Billiton, which we shall refer to as the BHPB model. This proprietary model has of the order of 45 state-space variables and 100 tuning parameters, and operates as a black-box, in that the user is allowed to set up its inputs and has access to its outputs once the model has been run, but has no access to internal computations or structure. The BHPB model has been validated against real data from an experimental bioleaching test column. Given the typical proportions of real heap bioleaching implementations (several square kilometres in area) and time scales (transients of the order of months), the use of accurate models such as the BHPB model appear as a necessary step in devising effective feedback control strategies for these processes.

A reduced number of input variables for control is selected based on an open-loop sensitivity analysis performed on the BHPB model (Section II-B). Such sensitivity analysis quantifies the effect of small input variations around nominal set-point values on the total amount of extracted copper after a full life-cycle of the heap.

The outputs of interest in the present paper are the average temperature in the heap and the copper concentration in the leached solution. We use the Expectation Maximisation (EM) algorithm to find linear Maximum Likelihood (ML) model estimates [9], [10]. The linear model obtained describes incremental variations of the outputs of interest to the selected control inputs around nominal trajectories, which are generated with the BHPB model under nominal conditions. The perturbations on the nominal values of the inputs are generated as pseudo-random binary sequences (PRBS) [8].

In Section III we use the linear incremental model estimated following the procedures described in Section II to design a feedback controller using the Internal Model Control architecture [11], [12] to regulate small increments around the nominal values of the outputs. The closed-loop results
simulated using the BHPB model are discussed in Section III-B and show around 5% improvement in the total mass of extracted copper. These results appear very promising given the extreme simplicity of the estimated model (second order) used, and the limited (incremental) action of the control inputs applied, and indicates good potential for feedback operation of heap bioleaching processes.

Note that the proposed approach requires a priori knowledge of the nominal trajectory of interest for model estimation and control design. In a implementation on a real heap, the BHPB model could be used to predict the heap nominal trajectories with periodic parameter tuning to fit its predictions to the real heap measured trajectories. Therefore, further work will concentrate on such implementation structure by considering robustness and measurement noise issues.

II. MODEL IDENTIFICATION

A. The Process

In copper heap leaching, large heaps of up to several square kilometres by 6 to 20 metres height of crushed copper mine tailings are formed. A sulphuric acid solution, called raffinate, is sprinkled by means of an arrangement of drip lines at the top of the heap. As the solution percolates down through the heap, it becomes enriched by the copper dissolved from the heaped ore, forming the pregnant leach solution (PLS). The PLS is then collected at the base of the heap by an impervious liner and pumped to an electro-winning extraction plant, which produces cathodic copper. The residual solution is then recycled as rafinate to the top of the heap. The process is illustrated in Fig. 1.

To tackle the complexity of this process, we consider a single geometric dimension of the process, namely, the vertical direction, assuming process homogeneity in every direction on the horizontal plane. There are three fundamental inter-coupled sub-processes in bioleaching [13], [4]. We focus on the identification of a model for average temperature, which is one of these three sub-processes which directly affects bacteria population and thereby, efficiency of copper extraction [2].

B. Input Selection by Sensitivity Analysis

A sensitivity analysis of a selected set of potential control variables is carried out to determine the input variables with best control authority to affect copper extraction. The candidate manipulated variables are:

- Irrigation rate ($F_1$) and input Acid Concentration ($H_2SO_4$);
- Aeration Rate ($A_i$) and Raffinate Temperature ($T_{r}$);
- Input Ferric Sulphate Concentration ($Fe_2SO_4$) and input Ferrous Sulphate Concentration ($FeSO_4$);
- Air Humidity and Air Temperature.

The sensitivity analysis of these variables is carried out by running the BHPB model. We choose each pair of variables listed above and vary them by small values around their nominal set point values to obtain a 2-dimensional grid. Then the BHPB model is run with the set point values defined by the grid, while keeping all other variables at nominal set point values.

The effect of the set point in the open loop process is quantified using the functional defined by

$$J = \sum_{k=1}^{N} C_{Cu}[k]F_o[k]e^{-\theta k}, \quad \text{in} \quad [g/h.m^2]. \quad (1)$$

where $C_{Cu}[k]$ is the concentration of copper in $[g/L]$ and $F_o[k]$ is outflux in $[L/h.m^2]$. The functional (1) considers the total copper extracted (per hour $\times$ meter$^2$) during the bioleaching process for a given set point. Thus, for each pair of manipulated variables, and for each point in the grid of set point variations, we obtain a value $J$ of total copper extracted. The exponential factor $e^{-\theta k}$ weights more favourably runs of the BHPB model that render copper more rapidly. We choose $\theta = 4.81 \times 10^{-5}$, which corresponds to a value decrease of approximately 10% in one year; around 3000 samples when the process is sampled every 4 hrs. This exponential term is very small when time approaches the last samples in the simulation (at about 3000 samples). In this way, runs that extract copper over a long period of time are penalised over runs that extract the same amount of copper in a shorter period of time.

The results obtained are shown in the following plots. Fig. 2 (a) shows sensitivity of $J$ with respect to raffinate influx ($F_1$) and acid concentration ($H_2SO_4$); and Fig. 2 (b) shows the corresponding for aeration rate ($A_i$) and raffinate temperature ($T_{r}$). We observe in Fig. 2 (a) a monotone relationship between input variables and the functional value, which increases when variables do. On the other hand, in Fig. 2 (b), there is an interesting aspect. Namely, there is an optimal extremum value for aeration rate. The existence of such local extremum in copper extraction with respect to aeration rate is reasonable, since we should expect a less productive heap with little or no aeration [14], but also with excessive aeration, since then the heap will be cooled down.

The effect of the pair Ferric Sulphate ($Fe_2SO_4$) and Ferrous Sulphate ($FeSO_4$) is shown in Fig. 3 (a). We can observe from Fig. 3 (a) that increasing the values for each of the variables increases the copper extraction, but its effect is
less notorious than, for example, the pair *Aeration Rate* and *Raffinate Temperature*.

![Functional Value varying different possible manipulated variables.](image1)

Fig. 2. Functional Value varying different possible manipulated variables.

Finally, the last pair considered is given by *Air Humidity* and *Air Temperature*. As we can see in Fig. 3 (b), they have little effect on copper extraction. Thus, we immediately discard them as possible control variables.

![Functional Value varying different possible manipulated variables.](image2)

Fig. 3. Functional Value varying different possible manipulated variables.

We conclude from this sensitivity analysis that the most influential inputs affecting the bioleaching process in open loop are: *Influx* ($F_i$), input sulphuric acid concentration ($H_2SO_4$), raffinate temperature ($T^{rafn}$) and aeration rate ($A_i$). We expect that these variables also affect significantly the process in closed loop.

### C. Experiment design

Once we have identified the most influential variables and in order to drive the process to a better copper recovery performance, we need to design our experiment to collect the data which will be used in the routines for identification.

Notice that in this paper, the improvement on copper extraction will be done through either controlling the average temperature or directly the copper extraction. Depending on what objective variable we choose, the inputs to be selected could be different, but always within the set of the input variables already selected in Section II-B. In the case of temperature being the control objective, we discard the raffinate temperature as an input variable since it is obvious that the average temperature in the heap raises if raffinate temperature does. We want to explore with the remaining variables which seem to be easier to implement. On the other hand, in the case of copper concentration, the input variable to be discarded is the influx ($F_i$) as it is obvious that reducing the raffinate sprinkling on the top of the heap will reduce the outflux from the heap and, consequently, it will raise the copper concentration.

In the case of average temperature modelling, the selected input variables $F_i$, $H_2SO_4$ and $A_i$ excite the BHPB model as shown in Fig. 4.

![Exciting the BHPB model with PRBS around nominal values](image3)

Fig. 4. Exciting the BHPB model with PRBS around nominal values (dashed line).

The PRBS sequences generated as inputs change $\pm 30\%$ around the nominal values (dashed line). These nominal values have been proven to be efficient in copper recovery in an open loop scheme. As output, we obtain a signal varying around the nominal output (dotted line) as seen in Fig. 4.

We generate two sequences of inputs. In the case of average temperature modelling, in Fig. 5, the first sequence (continuous line) is used for estimation and the second one (dotted line) is used for a posteriori validation of the model obtained. These two sequences of inputs also generate two outputs for each of the selected variables to be controlled.

![Data for the modelling of the average temperature.](image4)

Fig. 5. Data for the modelling of the average temperature. Excited selected inputs and outputs for estimation (continuous line) and validation (dotted line).

The data obtained for the average temperature in the heap and the copper concentration in the PLS are then used to obtain discrete models, in conjunction with the selected inputs.

In a general case, with noise acting on the system, the models to be found are in the following form:

$$x_{t+1} = A_{n_x \times n_x} x_t + B_{n_x \times n_u} u_t + w_t$$

$$y_t = C_{n_y \times n_x} x_t + D_{n_y \times n_u} u_t + v_t,$$

(2)
where \( x_t \in \mathbb{R}^n \), \( y_t \in \mathbb{R}^m \), and \( u_t \in \mathbb{R}^n \) are the state, the output and the input of the system, respectively. The vector \( \eta_t = [w_t \ v_t]^T \), which combines \( w_t \) and \( v_t \), describes the process and output noise respectively. The disturbance input \( \xi_t \) is assumed to be a zero mean Gaussian process with covariance matrix given by

\[
E\{\eta_t \eta_t^T\} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}.
\]

The initial state \( x_0 \) is assumed to be Gaussian with mean \( \mu \) and covariance \( P_0 : x_0 \sim N(\mu, P_0) \).

In our case the data generated by the BHPB model, as it can be seen in Fig. 5 is noise free, that means, we have assumed ideal measurements, but the noise terms given in (2) account for the model uncertainty as we are modelling a nonlinear system using a linear approximation.

The procedure for the obtention of matrices \( A, B, C \) and \( D \) is explained in the following Section.

D. Review of the Expectation Maximisation (EM) algorithm

In order to obtain model for the average temperature and copper concentration in the bioleaching process, we use an iterative algorithm called Expectation Maximisation (EM). EM provides the Maximum-Likelihood (ML) estimate for a vector of parameters \( \theta \). We choose a ML approach because it is a popular algorithm in system identification [15], [8]. In the ML framework, the following log-likelihood function is maximised:

\[
l(\theta) = \log p(Y_1:N | \theta) \tag{4}
\]

where \( Y_N \) denotes the given data set containing the system outputs i.e. \( Y_1:N := \{y_1, y_2, \ldots, y_N\} \). For future use, we also introduce the state sequence \( X_1:N := \{x_1, x_2, \ldots, x_N\} \).

The EM algorithm may be summarised as follows [9], [16]:

1) Choose an initial estimate \( \hat{\theta}_0 \in \Omega \), where \( \Omega \) is a constraint set in the parameter space. Then, for \( i = 0, 1, \ldots \)

2) E-step: Compute the auxiliary function \( Q(\theta, \hat{\theta}_i) \) which is the expected value of the complete data log-likelihood with respect to the random variable \( X_1:N \) (usually called “hidden data” in the statistics literature) given the observed data \( Y_1:N \) and the previous estimate \( \hat{\theta}_i \):

\[
Q(\theta, \hat{\theta}_i) = \mathbb{E}_{X_1:N} \{ \log[p(X_1:N, Y_1:N | \theta)] | Y_1:N, \hat{\theta}_i \} \tag{5}
\]

3) M-step: Set \( \hat{\theta}_{i+1} = \arg \max_{\theta \in \Omega} Q(\theta, \hat{\theta}_i) \).

4) Go to step 2, and continue until convergence.

Steps 2 and 3 are usually known as the E-Step and M-Step respectively. Under quite general conditions [9], [17], [18], the EM algorithm can be proven to converge to a stationary point of the likelihood function which in many practical applications will be a local maximum of the likelihood function [19].

The advantage of using the EM (apart from the usual statistical properties of a ML estimator) to identify a linear state-space model is that the algorithm is simple and has a closed form solution for both steps.

E. EM applied to linear state space models

To apply the EM algorithm for the problem of interest in the current paper we assume that the data has been generated by the model given in (2).

The parameters to be identified are \( \theta = (\bar{A}^T, \bar{B}^T, \bar{C}^T, \bar{D}^T, \mu^T, P_0^T, Q, R, S)^T \) where the arrow (\( \rightarrow \)) denotes the vector operator which creates a vector from a matrix by stacking its columns [20]. The state vector, \( x_k \), as the hidden variable.

We describe the proposed algorithm under the two headings of E and M step.

The E-step: The E-step requires that we calculate the following

\[
-2Q(\theta, \hat{\theta}_i) = (n + Nn_y + Nn)\log [2\pi] + \log ||P_0|| + N\log||Q|| + N\log||R|| + tr[P_0^{-1}\Sigma_{1:N} + \langle \hat{x}_{1:N} - \mu \rangle - \mu \rangle^T] + tr[Q^{-1} \langle A, B \rangle^T \langle A, B \rangle^T - \Psi A, B \rangle^T - \Psi A, B \rangle^T + \Phi \rangle + tr[R^{-1} \langle C, D \rangle^T \langle C, D \rangle^T - \Lambda A^T - \Lambda A^T + \Delta \rangle] \tag{6}
\]

where \( \Delta, \Phi, \Pi, \Gamma, \Lambda \) and \( \Psi \) are matrices which can be calculated as shown in the Appendix.

The M-step: In the M-step, the estimates have the following closed form (just set the derivatives of \( Q(\theta, \hat{\theta}_i) \) equal to zero and solve for \( \theta \)):

\[
[A, B] = \Psi \Gamma^{-1} \tag{7}
\]

\[
[C, D] = \Lambda \Pi^{-1} \tag{8}
\]

\[
Q = \frac{1}{N} \langle \Phi - \Psi \Gamma^{-1} \Psi \rangle^T \tag{9}
\]

\[
R = \frac{1}{N} \langle \Delta - \Lambda \Pi^{-1} \Lambda \rangle^T \tag{10}
\]

\[
\mu = \hat{x}_{1:N} \tag{11}
\]

\[
P_0 = \Sigma_{1:N} \tag{12}
\]

F. Model Validation

We use the validation data in dot-dashed lines showed in Fig. 6 to validate our model. We generated a new sequence for the inputs, and we apply these new inputs both, to the model obtained for control (average temperature or copper concentration), and to the BHPB model. The results are shown in Fig. 6.

As we can see in Fig. 6, the models for control can reasonable predict the variations on the outputs given by the BHPB model.

III. CONTROL DESIGN

We now discuss two feedback configurations to improve the copper extraction in an aerated heap. The first configuration takes the average temperature in the heap as the control objective. We initially focus on the temperature as control objective because temperature is related to the
bacterial activity [14]. We can expect that maintaining high
temperature for longer periods of time will improve the
bacteria activity and consequently make the copper extraction
more efficient.

The second configuration considers directly the copper
concentration as the variable to be controlled, but this option
requires the constant measurement of copper concentration
in the PLS, which could be hard to implement in practice.

We implement the IMC strategy considering separately
the two cases already mentioned, that is, control of the
average temperature in the heap, and control of the copper
concentration in the PLS.

A. Brief Review of IMC

A typical Internal Model Control scheme [12], [11] is
shown in Figure 7. The following relationships can be
established

\[ \Delta u = [I + G(P - \hat{P})]^{-1}G(\Delta r - d) \]

\[ \Delta y = P[I + G(P - \hat{P})]^{-1}G(\Delta r - d) + d, \]

where \( I \) is the identity matrix; \( P \) is the plant; \( \hat{P} \) is a model
of the plant; and \( G \) is a controller. Signals \( \Delta r, \Delta y, d, \Delta u \)
are defined as reference, output, disturbance and control action
respectively. The output of the model is defined as \( \Delta y_m \).

In this case, \( \Delta \) denotes the variation of a variable with
respect to its nominal value, e.g., \( \Delta u = u - \bar{u} \) and \( \Delta y =
\bar{y} - \bar{y} \), where \( \bar{u} \) and \( \bar{y} \) are the nominal values for the inputs
and outputs, respectively.

In IMC we intend to use as \( G \) the inverse of the model
for the plant, multiplied by a function \( F_d \) used to adjust
the bandwidth of the closed loop. Due to the non-square
characteristic of the model \( \hat{P} \) (the model of the plant has
different number of inputs and outputs), we use a generalised
inverse type I of \( \hat{P} \) as defined in [21]. In particular we choose

\[ G = F_d \hat{P}^T[\hat{P} \hat{P}^T]^{-1}, \]  

where the filter \( F_d \) is a scalar bi-proper transfer function
that gives the desired bandwidth for the closed loop. Notice
that the inverse of the term \( \hat{P} \hat{P}^T \) is, in general, well defined
since it is a scalar transfer function. In the case when non-
minimum phase zeros are present, we replace them by using
their corresponding mirrored ones (with respect to the unit
circle).

B. Control of average temperature in the heap

For the control of the average temperature in the heap,
we have chosen a second order model. Using the procedure
structure explained in Section II-D, we estimate the incre-
mental model for the variations in the average temperature,
with matrices \( A, B, C, D \) given by

\[
A = \begin{bmatrix} 1.0161 & -0.7127 \\ 0.0007 & 0.9767 \end{bmatrix}, \quad B = 10^{-2} \begin{bmatrix} 1.104 & 0.44 & -0.24 \\ 0.01 & 0.03 & 0.05 \end{bmatrix}, \\
C = -\begin{bmatrix} 0.4664 & 0.6322 \end{bmatrix}, \quad D = -10^{-2} \begin{bmatrix} 0.07 & 0.17 & 0.04 \end{bmatrix}.
\]

The matrices obtained above corresponds to a linear
discrete-time model, with sampling time \( T = 4 \) [h].

The results in Fig. 8 show that the linear controller used
for the incremental gives consistent results when different \( \Delta r \)
are applied. These \( \Delta r \) corresponds to the desired increment
in the output being controlled. In this case, the output being
controlled is the temperature, then \( \Delta r \) corresponds to the
variations for this variable in [°C]. The case for \( \Delta r = 0 \) is
the open loop case using nominal inputs.

We also notice in Fig. 9 that the control efforts are rea-
sonable in terms of variations regarding the nominal values
(continuous line). Notice that the control actions are applied
once those actions can make a difference in the output of the
closed loop, that is, after the day 66-th in the process. Before
that, and as seen in Fig. 5, changes in the selected inputs do
not produce a visible change in the outputs of interest, then we have designed the control scheme to operate only after the day 66-th. The best improvement on copper extraction controlling the average temperature is approximately 5% when the maximum control effort is used ($\Delta r = 10$).

### C. Control of copper concentration in the PLS

For the control of the CuSO$_4$ in the leached solution, we have chosen a second order model. Using the procedure explained in Section II-D, we estimate our discrete-time model with sampling time $T = 4 \, [h]$. The matrices $A,B,C,D$ obtained are given by

$$A = \begin{bmatrix} 1.1357 & -0.7639 \\ 0.0257 & 0.8565 \end{bmatrix}, B = -10^{-2} \begin{bmatrix} 0.11 & 0.39 & 1.184 \\ 0.01 & 0.07 & 0.3 \end{bmatrix}, C = - \begin{bmatrix} 0.6363 & 0.6141 \end{bmatrix}, D = 10^{-2} \begin{bmatrix} 0.06 & 0.29 & 1.130 \end{bmatrix}$$

The results in Fig. 10 show that the linear controller used for the incremental gives also consistent results when different $\Delta r$ are applied. These $\Delta r$ corresponds to the desired increment in the output being controlled. In this case, the output being controlled is CuSO$_4$, then $\Delta r$ corresponds to the variations for this variable in $g/L$. The case for $\Delta r = 0$ is the open loop case using nominal inputs. In the best case, we can obtain an improvement for copper extraction of 4.5% compared to the extraction using nominal values for the inputs.

Although the improvement in copper extraction independently regulating the average temperature and the copper concentration are similar (5% and 4.5% respectively), the control efforts are less demanding in the control of the average temperature as can be seen in Fig. 9.

### IV. CONCLUSIONS AND FUTURE WORK

We have identified a model for the increments in the temperature and copper concentration. The models have been obtained solving the maximum likelihood (ML) estimators using EM algorithms in a time domain framework.

The use of incremental models for this kind of process on an IMC strategy indicates significant potential improvements on the copper extraction which is our final objective. This improvement on the copper extraction has been carried out...
controlling either the average temperature in the heap or directly the copper concentration. We have found, however, that better results in terms of copper extraction and control effort can be obtained controlling only the temperature.

The drawback of the proposed approach is the requirement of the nominal trajectory for the variable to be controlled, either the temperature or the copper concentration. This trajectory is necessary because the models to be estimated are incremental, that means, they describe the variations around a nominal trajectory of the chosen variable. This nominal trajectory is obtained using the BHPB model, but this one is another model and its predicted value could be different to the real one. Hence, as a future research topic, we will focus on a robustness analysis considering that the predicted nominal trajectory by the BHPB model could be slightly different to the real one. This can be done by adding noise to the nominal trajectory and running Montecarlo simulations.

Another topic of interest in modelling using the EM algorithm is a frequency domain approach for the estimation of the incremental models. As the EM algorithm in the frequency domain permits to focus on a specific range of frequencies [16], then more robust estimates for the models could be obtained in a desired range of frequencies. Also, it allows to consider data from different experiments having an impact on the robustness of the model obtained [16].

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VI. APPENDIX

The matrices used in Section II-E can be calculated as follows:

\[
\Delta = \sum_{t=1}^{N} y_t g_t^T \tag{16}
\]

\[
\Phi = \sum_{t=1}^{N} E \left\{ x_t x_t^T | Y_N, \theta_i \right\} = \sum_{t=1}^{N} \left[ \Sigma_{t|N} + \hat{x}_{t|N} \hat{x}_{t|N}^T \right] \tag{17}
\]

\[
\Pi = \sum_{t=1}^{N} E \left\{ \begin{bmatrix} x_t x_t^T & x_t u_t^T \\ u_t x_t^T & u_t u_t^T \end{bmatrix} \right\} | Y_N, \theta_i \tag{18}
\]

\[
\Gamma = \sum_{t=1}^{N} E \left\{ \begin{bmatrix} x_{t-1} x_{t-1}^T & x_{t-1} u_{t-1}^T \\ u_{t-1} x_{t-1}^T & u_{t-1} u_{t-1}^T \end{bmatrix} \right\} | Y_N, \theta_i \tag{19}
\]

\[
\Lambda = \sum_{t=1}^{N} E \left\{ \begin{bmatrix} y_t x_t^T & y_t u_t^T \end{bmatrix} \right\} | Y_N, \theta_i \tag{20}
\]

\[
\Psi = \sum_{t=1}^{N} E \left\{ \begin{bmatrix} x_{t-1} x_{t-1}^T & x_{t-1} u_{t-1}^T \end{bmatrix} \right\} | Y_N, \theta_i \tag{21}
\]

and

\[
\hat{x}_{t|N} = E_{X_N} \left\{ x_t | Y_N, \theta_i \right\} \tag{22}
\]

\[
\Sigma_{t|N} = E_{X_N} \left\{ (x_t - \hat{x}_{t|N})(x_t - \hat{x}_{t|N})^T | Y_N, \theta_i \right\} \tag{23}
\]

\[
\Sigma_{t-1|N} = E_{X_N} \left\{ (x_{t-1} - \hat{x}_{t-1|N})(x_{t-1} - \hat{x}_{t-1|N})^T | Y_N, \theta_i \right\} \tag{24}
\]

where \( \hat{x}_{t|N}, \Sigma_{t|N} \) and \( \Sigma_{t-1|N} \) can be easily obtained by using standard algorithms to obtain the Kalman’s smoother estimates (see e.g. [10], [22]).
REFERENCES


