Level Crossing Sampling in Feedback Stabilization under Data-Rate Constraints*

Ernesto Kofman[†] and Julio H. Braslavsky[‡]

ARC Centre for Complex Dynamic Systems and Control
The University of Newcastle
Callaghan, NSW 2308, Australia

Technical Report EE06010
March 7 2006

Abstract

This paper introduces a novel event-driven sampled-data feedback scheme based on hysteretic quantization. In the proposed sampling scheme, the plant output samples are triggered by the crossings—with hysteresis—of the signal through its quantization levels. The plant and controller communicate over binary channels that operate asynchronously and are assumed to be error and delay–free. The paper proposes two systematic output feedback control design strategies. The first strategy consists in the digital emulation of a previously designed analog controller. If such analog controller achieves closed-loop asymptotic stability, the proposed emulation design guarantees closed-loop practical stability of the resulting asynchronous sampled-data system. The second design strategy is a simple direct design that drives the plant state to the origin in finite time after a total transmission of $2n + 2$ bits, where n is the order of the plant. These results exhibit the potential of the proposed scheme for the development of general tools for the analysis and design of practical event-driven sampled-data control systems.

1 Introduction

The study of networked control systems has attracted increasing research interest in recent years (for example, see the special issue [1] and the references therein). While standard control theory assumes the instantaneous availability of sensor and actuator information for control computations, networked control system formulations generally assume that such information is transmitted over communication channels, which pose additional constraints to system analysis and design. Such constraints include limitations in communication bandwidth, transmission power, data rates, and effects of signal quantization. A general theory for networked control systems is as yet unavailable, and therefore research efforts have concentrated on the study of partial aspects of the problem.

A number of works have considered the problem of feedback stabilization over a communication channel subject to data-rate constrains; see the upcoming paper [2] for an overview. A

*Submitted to CDC 2006 on March 7 2006.

†E. Kofman is with the Escuela de Ingeniería Electrónica, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Rosario 2000, Argentina, kofman@fceia.unr.edu.ar

‡J.H. Braslavsky is with the ARC Centre for Complex Dynamic Systems and Control, The University of Newcastle, Callaghan NSW 2308, Australia jhb@ieee.org

fundamental result in stabilization under data-rate constraints states that there exist a positive lower bound on the data-rate [bits per second] required to achieve closed loop stability of a linear time invariant, continuous-time plant with poles in the open right-half plane [3, 4].

An underlying assumption for this fundamental result is that the output of the plant is sampled in an essentially *time-driven* fashion; be it uniform and periodic, as in [3], or nonuniform, as in [5, 6]. Namely, the samples are obtained at instants that are largely independent of the plant current dynamic evolution.

Recent works have discussed *event-driven* alternatives to conventional time-driven sampling schemes [7, 8, 9]. One of such alternatives is *Lebesgue* sampling [7], in which samples are triggered by the crossing of the signal through quantization levels. Such sampling scheme presents a number of potential advantages for networked control, such as clock-free operation, low data-rate requirements [7], and higher energy efficiency in digital-to-analog conversion [10]. On the other hand, a main disadvantage is that control analysis and design are more difficult. Thus, little theory seems available, in contrast with the well-developed theory for time-driven periodic sampled-data systems [11].

Motivated by the recent developments in networked control systems and the growing interest in event-driven sampling, the present paper explores the use of a level-crossing sampling (LCS) scheme based on *hysteretic quantization* for feedback stabilization under data-rate constraints. The proposed LCS scheme may be viewed as a Lebesgue sampling scheme in which the quantizer includes hysteresis. Hysteresis allows us to implement 1-bit coding feedback communication in the proposed control strategies, which has the potential to achieve the most efficient data-rates [6]. In addition, under noisy scenarios, hysteresis would also minimize spurious sampling, further contributing to low data-rate feedback communication. Such hysteretic quantized LCS scheme has been used for discrete-event control and simulation in [12, 13].

The present paper aims to contribute to the development of a sampled-data control theory based on event-driven sampling schemes by proposing two systematic control strategies based on hysteretic LCS:

1. a control design strategy by event-driven “emulation” of a previously designed analog controller

2. a control design strategy by “direct” design using a combined deadbeat estimation and control scheme.

If the original analog control design achieves closed-loop asymptotic stability, we show that the proposed “emulation” design based on such controller guarantees closed-loop practical stability. The emulated controller is implemented as a *quantized state system* [12] that approximates the original analog controller. We give a bound for the approximation error between the closed-loop trajectories obtained by emulation with respect to those that would be obtained with the original analog controller. We believe that the proposed LCS emulation design is a systematic approach to obtain a practical event-based sampled-data controller with guaranteed stability and error properties.

Assuming no disturbances in the plant states or outputs, the proposed “direct” design control strategy drives the plant state to the origin in *finite time* under the constraint of 1-bit (delay and error-free) transmissions, both between sensor and controller, and between controller and actuator. The control strategy consists in two separate open-loop procedures carried out sequentially: 1. deadbeat estimation of the unknown initial condition, 2. deadbeat bang-bang control to the origin. Under the idealized conditions assumed, the proposed strategy achieves stability with effectively *zero* average data-rate, thus appearing to circumvent the data-rate limitations known for feedback based on conventional time-driven sampling. However, it can be anticipated that noisy measurements, plant uncertainties and time delays will certainly impose a non-zero average data-rate, as we discuss in Section 4.4.

The paper is organized as follows: In Section 2 we present our assumptions and describe the scheme under study: a feedback loop with binary communication channels between plant and controller. We describe the proposed hysteretic quantized LCS scheme in detail, as well as the proposed 1-bit coding and decoding associated strategies. Section 3 presents the proposed asynchronous emulation design, and gives its main properties. A numerical simulation example is given at the beginning of the section and is revisited at the end. Section 4 presents the proposed direct design for asynchronous deadbeat stabilization based on LCS. At the end of this section we discuss in more detail the issue of stabilization under data-rate constraints with the conventional smapling, and with the proposed LCS design strategy. Section 5 summarizes the paper conclusions.

Contents

2 Level crossing sampling scheme

2.1 Feedback system

The general feedback scheme considered is shown in Figure 1. We model two communication links in the feedback loop: a sensor communication link, between the measured plant output and the controller input, and an actuator communication link, between controller output and plant input. These communication links are assumed to be error-free and have no noise disturbances nor transmission delays.

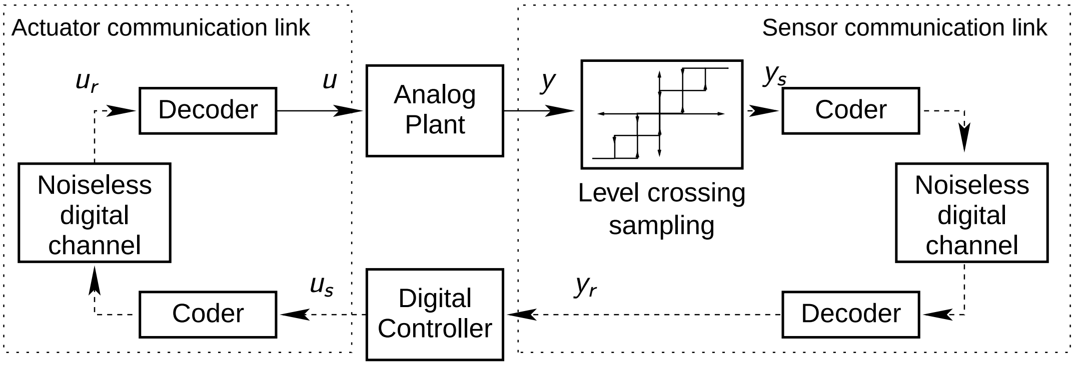

Figure 1: Event-driven sampled-data scheme for feedback stabilization over digital channels

2.2 Plant.

The plant is a continuous-time, finite dimensional LTI system given by the minimal realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

The matrices A, B, C are assumed known, but the system initial condition $x(0)$ is unknown.

2.3 Actuator.

We consider an actuator that can only produce values in the set $\mathcal{U} \triangleq \{-pU, \cdots, -U, 0, U, \cdots, pU\}$ where U is a positive real constant and p is a positive integer. We assume that the switching of the actuator is instantaneous and the values are held constant (in a zero-order-hold fashion) until a different value of $u(t)$ is generated.

2.4 Level crossing sampling with hysteresis

The proposed event-driven sampling scheme generates a new sample of this signal whenever $y(t)$ differs from the previous sample in a fixed quantity $h > 0$, which we call the quantization interval.

Let $y(t)$ be a continuous-time real valued scalar signal. Then the LCS device produces the quantized (but exact) samples

$$y_s(t_k) = y(t_k) \tag{2}$$

at the sampling instants t_k, $k = 0, 1, 2, \ldots$ defined by

$$t_k = \inf \{\tau \in (t_{k-1}, \tau] : |y(\tau) - y_s(t_{k-1})| > h\}. \tag{3}$$

The quantization interval $h > 0$ is assumed regular for simplicity; a quantization interval dependent on y could be implemented similarly. Figure 2 illustrates an output signal produced

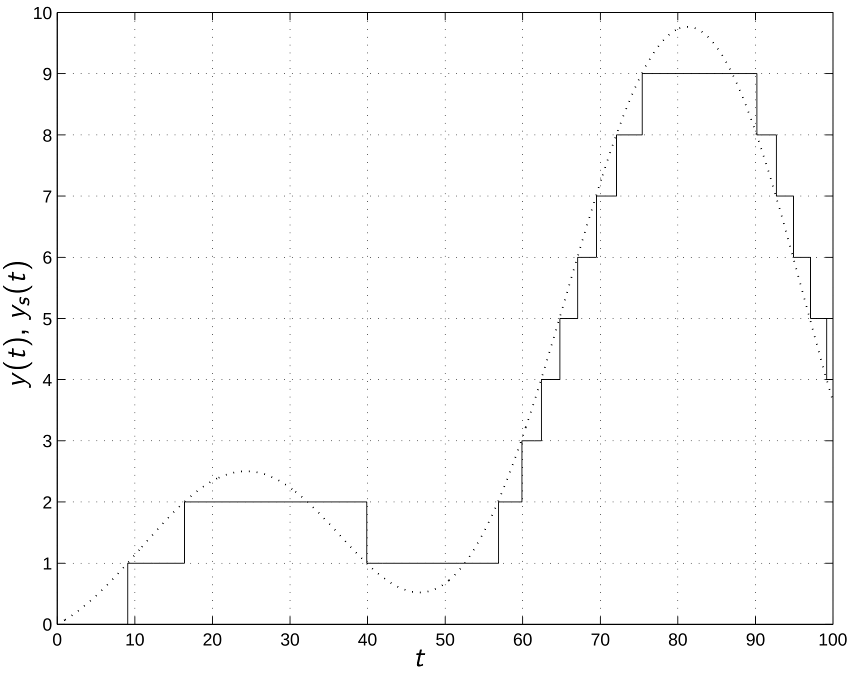

Figure 2: Event-driven sampling/quantization

by such LCS device, plotted together with the input signal that generates it, for a quantization interval $h = 1$.

Notice that the proposed LCS scheme (3), (2) has *hysteresis*, which in general reduces the number of samples generated. Moreover, if the derivative of $y(t)$ is bounded, the hysteresis guarantees that a nonzero time interval exists between successive samples. Figure 3 depicts the nonlinear characteristic of the proposed hysteretic quantizer.

We denote by $Q_h\{y\}$ the hysteretic quantized signal obtained by applying the hysteretic quantization operator with interval h to the signal y. Note that this is a nonlinearity *with memory*, in that $Q_h\{y\}(t_0)$ depends not only on the value $y(t)$ at $t = t_0$, but also on the history of $y(t)$ for $t \leq t_0$.

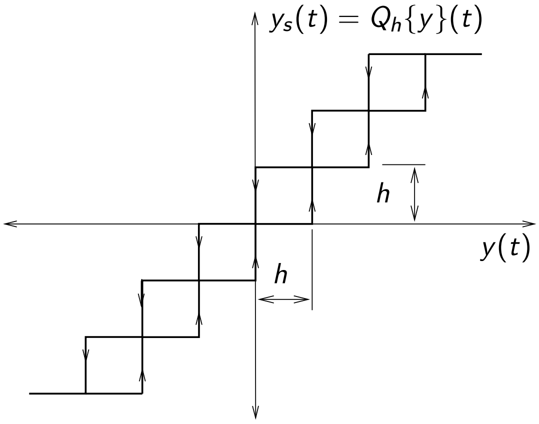

Figure 3: Quantizer with hysteresis, $y_s(t) = Q_h\{y\}(t)$

A fundamental property of this LCS scheme is that successive samples always differ in $\pm h$. Consequently, each sample can be coded using only one bit, thus reducing the amount of information that needs to be transmitted.

2.5 Coder/digital channel. Sensor link

We assume that the digital channel is memory-less and error-free, and can only transmit one bit, 0 or 1, per asynchronous sample produced by the LCS device. The coding strategy is the following: When a sample is produced, at say $t = t_k$, the channel transmits

$$\begin{cases} 1, & \text{if } y_s(t_k) > y_s(t_{k-1}), \\ 0, & \text{if } y_s(t_k) < y_s(t_{k-1}). \end{cases} \tag{4}$$

Thus, a 1 is transmitted if the current sample of the output has increased with respect to its previous sample, or a 0 if it has decreased. If no samples are produced, no transmission takes place. However, notice that in the proposed scheme there is also information when no samples are produced; namely, the output $y(t)$ remains within its quantization band.

2.6 Decoder. Sensor link.

The decoder receives the sequence of bits indicating the changes in y_s and calculates

$$y_r(t_k) = \begin{cases} h, & \text{when 1 is received,} \\ -h, & \text{when 0 is received.} \end{cases} \tag{5}$$

Notice that

$$y_r(t_k) = y_s(t_k) - y_s(t_{k-1}). \tag{6}$$

2.7 Digital controller

The controller receives an asynchronous sequence of values $y_r(t_k)$ and produces a sequence of control actions $u_s(\tau_j)$, where—in general—the time sequences t_k and τ_j are different.

We consider that $u_s(\tau_j) \in \mathcal{U}$, and $u_s(\tau_0) = 0$. We shall also restrict $|u_s(\tau_j) - u_s(\tau_{j-1})| \in \{U, 2pU\}$, that is, successive control actions differ in one quantization level or they can jump between the saturation limits. This restriction permits coding the successive control values using only one bit.

In the next sections we shall propose two different strategies to calculate the sequence $u_s(\tau_j)$.

2.8 Coder/digital channel. Actuator link

The coding strategy in the actuator link is similar to that of the sensor link, transmitting only when the controller produces a control signal that differs from its value at the previous sampling instant. Then, when a sample is produced at $t = \tau_j$, the channel transmits

$$\begin{cases} 1, & \text{if } u_s(\tau_j) - u_s(\tau_{j-1}) \in \{-2pU, U\} \\ 0, & \text{if } u_s(\tau_j) - u_s(\tau_{j-1}) \in \{-U, 2pU\} \end{cases} \tag{7}$$

2.9 Decoder. Actuator link

The actuator decoder receives the sequence of bits informing the changes in u_s, and it builds the signal $u_r(\tau_j)$ according to following logic

$$\begin{cases} u_r(\tau_{j-1}) + U, & \text{if } u_r(\tau_{j-1}) < pU \text{ and 1 is received} \\ u_r(\tau_{j-1}) - U, & \text{if } u_r(\tau_{j-1}) > -pU \text{ and 0 is received} \\ -pU, & \text{if } u_r(\tau_{j-1}) = pU \text{ and 1 is received} \\ pU, & \text{if } u_r(\tau_{j-1}) = -pU \text{ and 0 is received} \end{cases}$$

Notice that, provided that $u_r(\tau_0) = u_s(\tau_0) = 0$, it is always true that $u_r(\tau_j) = u_s(\tau_j)$.

3 Controller design by emulation

In this section, we introduce a method for designing a digital control law as an approximation of a previously designed continuous-time controller. In our asynchronous level crossing sampling setting, such digital approximation is carried out by implementing the continuous-time controller as a *quantized state controller* [12].

We first introduce the basic idea of the method on a simple example.

3.1 An introductory example

The unstable plant

$$\dot{x}(t) = x(t) + u(t); \quad y(t) = x(t) \tag{8}$$

is asymptotically stabilized by the PI controller

$$\begin{aligned} \dot{z}(t) &= y(t) \\ u(t) &= -2y(t) - z(t). \end{aligned} \tag{9}$$

Let us replace the continuous controller law (9) by the *quantized state* approximation

$$\begin{aligned} \dot{z}(t) &= y_s(t) \\ y_c(t) &= -2y_s(t) - \zeta(t), \end{aligned} \tag{10}$$

where

$$y_s(t) = Q_h\{y\}(t) \quad \text{and} \quad \zeta(t) = Q_g\{z\}(t) \tag{11}$$

are variables quantized with intervals h and g using the nonlinear hysteretic characteristic shown in Figure 3.

Let us also assume that $\mathcal{U} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$; that is, $U = 2$ in $\mathcal{U}$ as defined in Section 2.3. Then, $u_s(t)$ is calculated as the nearest element of $\mathcal{U}$ from $y_c(t)$.

We select the quantization intervals in (11) $h = 0.5$ and $g = 0.1$, and considering the initial conditions $x(0) = 10$ and $z(0) = \zeta(0) = 0$ the system will evolve as follows:

- At $t = 0$ the controller receives the value $y_s(t) = 10$. According to (10) we have $y_c(t) = -20$ and $\dot{z}(t) = 10$. Thus, $u_s(t) = -20$. These values remain constant until a new sample y_s is received or when $\zeta(t)$ changes. The next change in $\zeta(t)$ is produced when it differs from $z(t)$ in 0.1. Since $z(t)$ grows with a slope of 10, that will happen when $t = 0.1/10$. Thus, the controller schedules the next change in $\zeta(t)$ for $t = 0.01$.

- Then, at $t = 0.01$ we have $\zeta(t) = z(t) = 0.1$, $y_c(t) = -20.1$ and $u(t)$ remains equal to -20. The slope $\dot{z}(t)$ also remains constant and next change in $\zeta(t)$ is then scheduled for $t = 0.02$.
- At $t = 0.02$ it turns out that $\zeta(t) = z(t) = 0.2$, $y_c(t) = -20.2$ and the remaining variables do not change. After two identical steps, we have $t = 0.04$ and $\zeta(t) = z(t) = 0.4$, $y_c = -20.4$ and the next change is scheduled for $t = 0.05$.
- However, when $t = 0.0488$, it turns out that $y(t) = 9.5$ and a new sample $y_s = y = 9.5$ is then sent to the controller. Thus, $y_c(t) = -19.4$ and then $u_s(t)$ remains equal to -20. The controller state can be calculated as $z(t) = 0.4 + 0.0088 \times 10 = 0.488$, while its new slope becomes 9.5. Thus, the next change in $\zeta(t)$ will occur after $(0.05 - 0.0488)/9.5 = 0.000126$ units of time and the calculations continue in the same way.

The quantized controller (10) works as a discrete event system: it receives a sequence of values $y_s(t)$ and it schedules *internal* changes (when $\zeta(t)$ changes) as well as generating *output* events (when $u_s(t)$ changes).

As we can see from the initial sequence of events in the evolution of the closed-loop system (8), (10), the calculations inside the controller are very simple and can be implemented in a digital system. As we shall show, the emulated controller (10) behaves similarly to the continuous controller (9), in the sense that the closed-loop trajectories generated will be close to those that would be generated with the original analog controller.

In the next section we shall generalize and more formally define the idea behind the controller approximation in this example.

3.2 Quantized state controller (QSC)

Let the continuous-time controller previously designed for the plant (1) be given by the state realization

$$\begin{aligned} \dot{z}(t) &= A_c z(t) + B_c\, y(t) \\ u(t) &= C_c z(t) + D_c\, y(t), \end{aligned} \tag{12}$$

and assume that the continuous-time closed loop system (1), (12) is asymptotically stable. For simplicity, we also assume that the state realization (12) is in the observer canonical form [14].

The QSC approximation of the continuous-time controller (12) is a continuous-time system defined by the state equations

$$\begin{aligned} \dot{z}(t) &= A_c \zeta(t) + B_c\, u_c(t) \\ y_c(t) &= C_c \zeta(t) + D_c\, u_c(t), \end{aligned} \tag{13}$$

where $z \in \mathbb{R}^{n_c}$, and $\zeta \in \mathbb{R}^{n_c}$ is the component-wise quantized version of the state vector z, that is,

$$\zeta = \begin{bmatrix} Q_{g_1}\{z_1\} \\ Q_{g_2}\{z_2\} \\ \vdots \\ Q_{g_{n_c}}\{z_{n_c}\} \end{bmatrix}, \tag{14}$$

where each $Q_{g_i}\{\cdot\}$ is a quantization function with hysteresis, as depicted in Figure 3, with quantization intervals g_i, $i = 1, \ldots, n_c$.

Since the plant output is only available from the received samples $y_r(t_k) = y_s(t_k) - y_s(t_{k-1})$, we can calculate

$$u_c(t) = u_c(t_{k-1}) + y_r(t_k), \quad \text{for } t_k \le t < t_{k+1}. \tag{15}$$

Then, provided that the sequence $y_r(t_k)$ includes the information about the initial output $y(t_0)$,[1] it will be true that $u_c(t_k) = y(t_k)$.

Since $u_c(t)$ follows a piecewise constant trajectory, it follows that the components of $\zeta(t)$ and $y_c(t)$ are also piecewise constant and the components of $z(t)$ are piecewise linear and continuous.[2]

We remark that the calculation of the resulting output sequence $y_c(\tau_j)$ does not require numerical integration as (13) is already a discrete system.

Notice that each component $\zeta_i(t)$ of $\zeta(t)$ only changes when $|\zeta_i(t_j) - z_i(t_j)| = g_i$. Since $x_i(t)$ is continuous, this implies that $|\zeta_i(t_j) - \zeta_i(t_{j-1})| = g_i$.

By taking into account that System (12) is in the observer canonical form, it follows that $y_c(t) = \zeta_1(t) + Du_c(t)$ and then $y_c(t)$ changes with jumps of either $\pm g_1$ (when ζ_1 changes) or $\pm D_c h$ (when u_c changes).

We have assumed that the controller output $u_s(t) \in \mathcal{U}$. To accomplish this condition, we define $u_s(t)$ as the nearest element of $\mathcal{U}$ from $y_c(t)$. Then, provided that $g_1 \le U$ and $D_c h \le U$, it will be true that $u_s(\tau_j) - u_s(\tau_{j-1}) = \pm U$, and we can apply the proposed coding and decoding strategies described in Sections 2.8 and 2.9.

3.3 Properties of the QSC scheme

Defining $\Delta z(t) \triangleq \zeta(t) - z(t)$, $\Delta u(t) \triangleq u(t) - y_c(t)$ and $\Delta y(t) \triangleq u_c(t) - y(t)$, we can rewrite Equation (13) as

$$\begin{aligned} \dot{z}(t) &= A_c(z + \Delta z) + B_c(y + \Delta y) \\ u(t) &= C_c(z + \Delta z) + D_c(y + \Delta y) + \Delta u(t). \end{aligned} \tag{16}$$

Notice that $|\Delta y(t)| \le h$ and $|\Delta z_i| \le g_i$ for $i = 1, \cdots, n_c$.

If we assume also that the controller output $y_c(t)$ cannot reach the saturation bounds $\pm p\,U$ (because p is sufficiently large), then it is also true that $|\Delta u(t)| \le U$.

The closed loop system (1), (16) can be written as

$$\dot{w}(t) = \tilde{A}(w(t) + \Delta w(t)) + F\Delta v(t) \tag{17}$$

where

$$w(t) \triangleq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}; \Delta w(t) \triangleq \begin{bmatrix} 0_{n\times 1} \\ \Delta z(t) \end{bmatrix}; \Delta v(t) \triangleq \begin{bmatrix} y(t) \\ u(t) \end{bmatrix},$$

and

$$\tilde{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}; \quad F = \begin{bmatrix} BD_c & B \\ B_c & 0 \end{bmatrix}.$$

[1] To this end, we may for example allow the coder to transmit a sequence of m bits during the initialization informing that the initial output is $\pm mh$.

[2] Consequently, System (13) is equivalent to a *discrete event system* in terms of the DEVS formalism [15]. The DEVS model equivalent to this system can be found in [13].

Notice that the ideal closed loop system (1), (12) is

$$\dot{w}(t) = \tilde{A}w(t) \tag{18}$$

Thus, (17) is just a perturbed version of the plant with the original continuous-time controller (12).

In the perturbed system (17), the absolute value of each component of the perturbation terms $\Delta w(t)$ and $\Delta v(t)$ is bounded by either h, g_i or U. More formally, we can write

$$|\Delta w(t)| \leq g \triangleq \begin{bmatrix} 0_{n\times 1} \\ g_1 \\ \vdots \\ g_{n_c} \end{bmatrix}; \quad |\Delta v(t)| \leq \gamma \triangleq \begin{bmatrix} h \\ U \end{bmatrix} \tag{19}$$

where $|\cdot|$ denotes the component-wise module and the inequality sign $\leq$ denotes a component-wise inequality.

Since the plant with the quantized state controller is a perturbed version of the plant with the original continuous-time controller, the following theorem can be derived

Theorem 1. *Suppose that the closed loop system* (18) *is asymptotically stable and let $w(t)$ be its solution from the initial condition $w(0) = w_0$. Let $\tilde{w}(t)$ be the solution of* (17) *from the same initial condition w_0 and let $\Lambda = V^{-1}\tilde{A}V$ be a modal decomposition of $\tilde{A}$.*

Then, for all $t \geq 0$,

$$|\tilde{w}(t) - w(t)| \leq |V| \times (|\mathbb{R}e(\Lambda)^{-1}\Lambda||V^{-1}|g + |\mathbb{R}e(\Lambda)^{-1}V^{-1}F|\gamma). \tag{20}$$

Proof. Define $\varepsilon(t) \triangleq \tilde{w}(t) - w(t)$. Then, subtracting (18) from (17) we obtain

$$\dot{\varepsilon}(t) = \tilde{A}(\varepsilon(t) + \Delta w(t)) + F\Delta v(t)$$

with $\varepsilon(0) = 0$. The perturbation terms verify (19).

When $\tilde{A}$ is diagonalizable, Theorem 3 of [16] concludes that

$$|\varepsilon(t)| \leq |V| \times (|\mathbb{R}e(\Lambda)^{-1}\Lambda||V^{-1}|g + |\mathbb{R}e(\Lambda)^{-1}V^{-1}F|\gamma)$$

for all $t \geq 0$.

For the non-diagonalizable case, the same result can be derived following the procedure developed in Theorem 3.3 of [17]. □

This theorem says that the state trajectory of the plant with the quantized state controller remains close to the state trajectory of the plant with the original continuous controller. Moreover, since the original closed loop system is asymptotically stable, its solutions go to zero. Thus, the state trajectories of the plant with the QSC go to a bounded region around the origin. The size of that bounded region is calculated by the right hand side of Equation (20).

Therefore, the closed loop system with QSC is always practically stable, irrespective of the size of the discretization parameters h, U, and g_i, $i = 1, \cdots, n_c$.

3.4 Example revisited

Figures 4–6 show the simulation results of the System (8),(10).

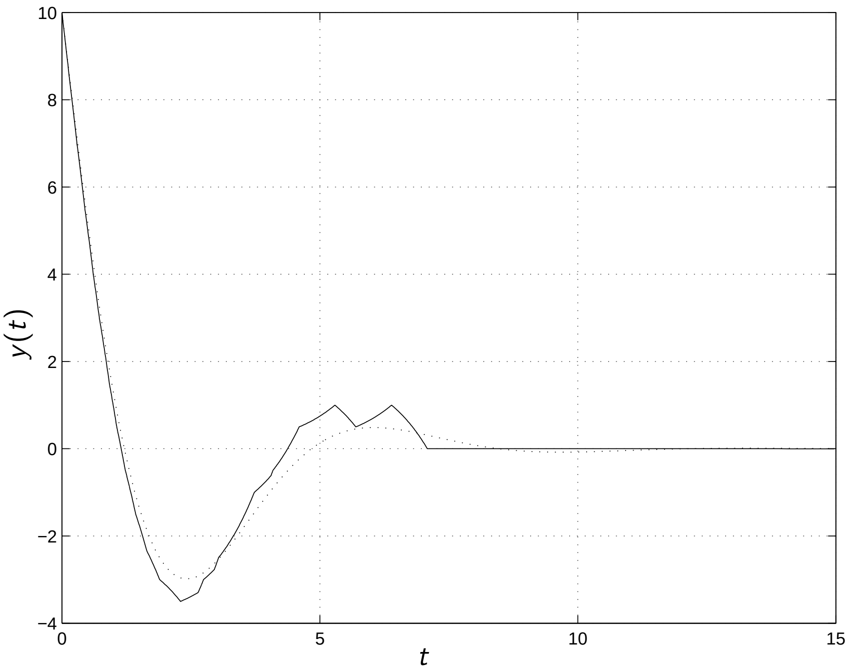

Figure 4: Plant state $x(t)$ (ideal trajectory in dotted line).

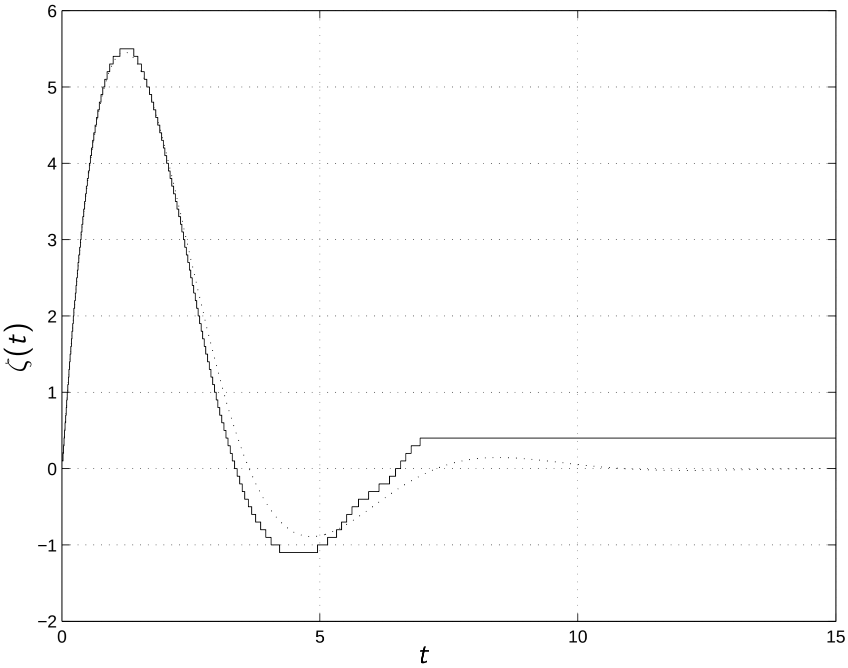

Figure 5: Quantized variable $\zeta(t)$ (ideal $z(t)$ in dotted line).

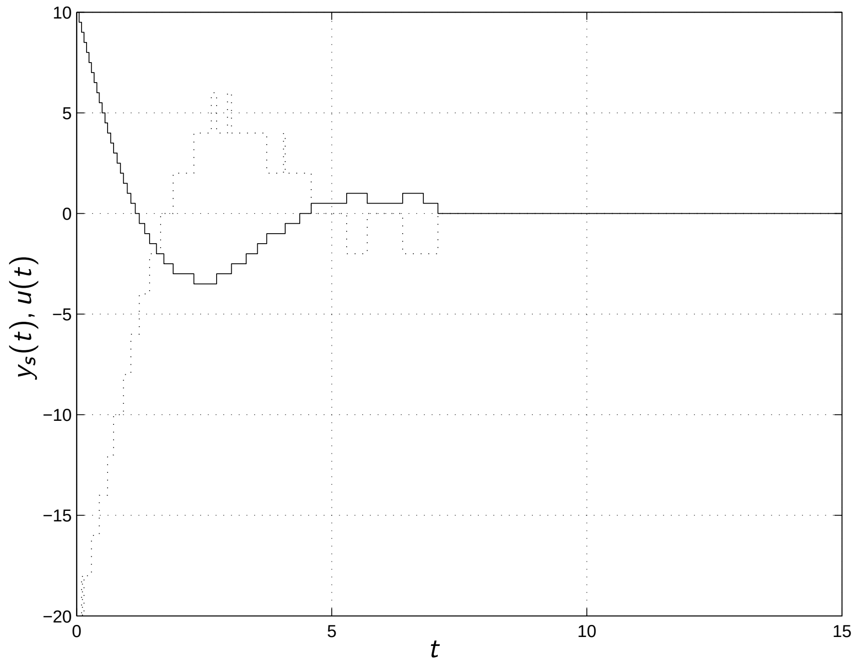

Figure 6: Controller ($y_s(t)$) and Plant ($u(t)$, dotted) input values.

After $t = 7.0823$ the plant reaches the origin. In that moment the controller produces the input $u = 0$ and then the system remains in that state without producing more samples or calculations at the controller.

In this example, the total number of samples y_s received by the controller were 41, while the number of samples u sent by the controller were 28. Since each change can be coded with only one bit, the number of transmitted bits was 61 in the sensor link (we added here 20 bits to transmit the initial value of y_s) and 28 in the actuator link.

Using Theorem 1 we can prove that the trajectories of the plant with the continuous controller and those of the system with the discrete controller never differ in more than 1.8856 (in both variables, x and z). The dotted lines in Figs 4–5 corroborate this bound.

In this case, due to the particular choice of the controller and quantization intervals h, g, U, the plant reached the origin. However, in general the QSC implementation will yield final oscillations (bounded according to Theorem 1) requiring a nonzero average data-rate.

In the next section, we introduce a different idea to design the control law so that the state of the system can be driven to the origin with a minimum number of of bits transmitted.

4 Direct deadbeat control design

In this section we present a simple stabilization strategy based on LCS by hysteretic quantization that drives the state of the system (1) to the origin in finite time from an unknown initial condition $x(0) = x_0$. Thus, we will show that with an asynchronous LCS scheme, it is possible to achieve stability by transmitting a finite number of bits.

We consider again the feedback scheme of Figure 1, with components as described in Section 2, and we assume that the plant (1) is unstable.

The proposed stabilization strategy consists in two open-loop procedures performed sequentially:

1. Finite-time initial state estimation
2. Finite-time control to the origin.

4.1 Finite-time initial state estimation

The controller receives one bit per sample indicating whether the system output has increased (a 1 received) or decreased (a 0 received) in a value h with respect to the previous sample. The sampling times $\{t_0, t_1, \dots\}$ are also recorded. Notice that the sampling times coincide with the samples *arrival* times, since transmission is assumed delay-free.

Let $x(0) = x_0$ be the plant initial state and let $\{t_0, t_1, \dots, t_n\}$, be the sequence of sampling times corresponding to the first $n+1$ samples, where n, recall, is the order of the plant (1). By setting the control input to $u(t) = 0$ during the state estimation cycle, we have

$$y(t_0) = Ce^{At_0}x_0, \quad y(t_1) = Ce^{At_1}x_0, \quad \cdots \quad y(t_n) = Ce^{At_n}x_0.$$

Since $y(0)$ is unknown, from (6) we write

$$Y_r \triangleq \begin{bmatrix} y_r(t_1) \\ y_r(t_2) \\ \vdots \\ y_r(t_n) \end{bmatrix} = \begin{bmatrix} C(e^{At_1} - e^{At_0}) \\ C(e^{At_2} - e^{At_1}) \\ \vdots \\ C(e^{At_n} - e^{At_{n-1}}) \end{bmatrix} x_0 \triangleq E\,x_0. \tag{21}$$

Then, since the sampling instants $\{t_0, t_1, \dots, t_n\}$ are available, and the vector Y_r is known independently of the value $y(0)$ (due to the property (6)), we can obtain $x_0 = E^{-1}Y_r$ after receiving $n+1$ bits, provided E is nonsingular.

4.2 Non pathological sampling

The nonsingularity of the matrix E can be seen as analogous to the well-known non pathological sampling condition in periodic sampled-data systems [18, p. 40–42].

Theorem 2. *Assume that the matrix A is unstable and it has only nonzero real eigenvalues. Then, provided that the pair (A, C) is observable, the matrix E defined in Equation* (21) *is non singular.*

Proof. Suppose that the matrix E is singular. Then, the equation $E\,x_0 = 0$ has non trivial solutions. Let $\tilde{x}_0 \neq 0$ be one of these solutions and define $\tilde{y}(t) \triangleq Ce^{At}\tilde{x}_0$. It follows that $\tilde{y}(t_k) - \tilde{y}(t_{k-1}) = E_k\tilde{x}_0 = 0$ for $k = 1, 2, \dots, n$, where $E_k \triangleq C(e^{At_k} - e^{At_{k-1}})$. Then, $\tilde{y}(t_0) = \tilde{y}(t_1) = \cdots = \tilde{y}(t_{n+1})$, which implies that the system output crosses $n+1$ times the value $\tilde{y}(t_0)$. This, however, is impossible for a system of order n in absense of null or complex eigenvalues. □

If the plant had complex or null eigenvalues, the matrix E might be singular. However, the pathological sampling condition can only occur if the sampling instants $t_1, \dots, t_{n+1}$ (the instants of crossing of the output $y(t)$ by levels separated by $\pm h$) coincide with the instants in which $\tilde{y}$ (the output from a different initial condition) crosses a unique level. Such situation is indeed possible if the plant has null eigenvalues, because the output might then remain constant from some initial condition. For complex eigenvalues, we conjecture that pathological sampling can be made impossible by choosing h sufficiently small with respect to $y(t_0)$. In that case, the sampling instants $t_0, t_1, \cdots$ will turn out to be such that the elapsed time $t_{n+1} - t_0$ results shorter than the shortest period of any natural oscillation mode in the plant.

4.3 Finite-time control to the origin

Now, at any time $\tau_0 > t_n$ we have the complete knowledge of $x(0)$, and thus we can compute

$$x(\tau_0) = e^{A\tau_0} x_0,$$

and from $x(\tau_0)$ we can drive the system to the origin in finite time by an open loop bang-bang strategy in a maximum of n switches of the input $u(t)$, which for simplicity we can assume to take values in the set $\{-U, 0, U\}$, for some sufficiently large $U > 0$.

The switching times $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ and the time of arrival to the origin τ_n can be computed by different methods by solving the minimum time control problem; see for example [19].

Thus, stability with the proposed LCS scheme is achieved in *finite-time*; namely, with the transmission of $2n + 2$ bits.

4.4 Remarks on stabilization under data-rate constraints

The above two-cycle stabilization strategy based on an LCS scheme appears to, at least in some simple cases, circumvent limitations in the problem of feedback stabilization under data-rate constraints. This problem has received significant attention in recent years [1], and a number of exciting new results have been obtained. In this context, it is interesting to observe that if the digital feedback stabilization problem is approached by periodic sampling of the plant output $y(t)$ at a rate $1/T$ Hz and a synchronous zero-order-hold is used to generate the control input $u(t)$, then the plant (1) can be stabilized if and only if the average data-rate [in bits/s] satisfies [20, 4]

$$\frac{R}{T} > (\log_2 e) \sum_{i=1}^{m} \operatorname{Re} p_i > 0 \quad \text{bits per second.}$$

On the other hand, we have seen above that a strategy based on an LCS scheme can be argued to achieve feedback stability after the transmission of $2n + 2$ bits; namely, with *zero* average data-rate.

We believe this is an interesting property of the proposed asynchronous digital control strategy, which we will further analyze in future work. However, it should be noted that the proposed LCS strategy relies on:

- instantaneous availability of information about the sampling instants $\{t_0, t_1, \ldots, t_n\}$
- knowledge of the sampling instants $\{t_0, t_1, \ldots, t_n\}$ to infinite precision.

In practice, there will be delays in transmission, which, if unknown, will prevent stability with the proposed ideal strategy. Also, note that although the information about the measured output is quantized, it is exact. However, a necessary approximation would arise in recording the sampling times $\{t_0, t_1, \ldots\}$ digitally, which would then need to be quantized to finite precision.

Effectively, this stabilization strategy makes use of an inherent coding of the output amplitudes into the sequence of sampling instants. If these sampling instants were not exactly known (due to unknown delays or quantization), the plant will not in general reach the origin in finite time at the end of the control cycle. Since the plant is assumed to be unstable, the output will then start crossing new quantization levels, and a new set of estimation/control cycles will have to be repeated. Following this idea, we would then necessarily arrive at a nonzero average data-rate and practical closed-loop stability.

Let us now consider an alternative time-driven sampling stabilization approach. In the ideal situation we assumed that the channel can transmit one bit at any instant of time. Thus, it could be argued to alternatively take a sample of the plant outut periodically and then send two bits, together with an interval of time proportional to that output value, in a pulse-width modulated fashion. In other words, we could then code (and also decode) with infinite accuracy the plant output. Thus, after receiving n samples (corresponding in this case to $2n$ bits) it would be possible to estimate the plant initial state and apply a control strategy to drive the state to the origin, again achieving stability with a finite number of transmitted bits.

From the perspective of information transmitted, the above reasoning shows that by allowing the channel to transmit at any instant of time is equivalent to the transmission of an infinite amount of information per sample. Under these conditions, a time-driven sampling strategy could also achieve stability in a finite number of transmissions.

It would thus seem that the only advantage of the proposed event-driven sampling scheme is that it uses simpler devices than those required by the above pulse-width modulation coding strategy.

However, in a non-ideal case (that is, in the presence of unknown delays, noise, etc), such time-driven sampling scheme must continue to sample using a *worst-case* sampling period, so that the output does not escape too far away from the origin.

On the other hand, the proposed event-driven sampling scheme will only generate samples (and act on the actuator communication link) only when the plant output reaches the value $\pm h$. Thus, if the estimation of the initial state was sufficiently good—for example, because during such period we did not have large perturbations—the plant will reach a state very close to the origin and it will take a relatively long time until the output reaches $\pm h$.

5 Conclusions

We have presented a digital control formulation based on an asynchronous level-crossing sampling scheme by hysteretic quantization. Such formulation appears attractive in control scenarios in which feedback is implemented over data-rate limited communication channels, which we have illustrated by presenting two approaches for control design: an asynchronous digital emulation design, and a direct digital design strategy.

In asynchronous emulation, a previously designed analog controller is implemented as a quantized state controller, which is shown to guarantee practical stability. In some cases—as in the example introduced—it also achieves stability after a finite number of transmissions.

The direct digital design strategy consists in an open loop deadbeat estimation cycle, followed by a bang-bang control cycle to drive the plant state to the origin. Stability is achieved with a finite number of transmissions.

The effect of measurement disturbances and model uncertainty will in general prevent finite-time stability with these schemes. Nevertheless, it would be of interest to quantify and compare their overall performance with that of approaches based on conventional periodic time-driven sampling and quantized measurements. This will be the topic of future work.

We believe that the proposed level crossing sampling scheme and control strategies exhibit a clear potential for the development of general systematic asynchronous control design methods for event-driven sampled-data systems.

References

[1] "Special issue on networked control systems", *IEEE Transactions on Automatic Control*, vol. 49, no. 9, September 2004.

[2] G.N. Nair, F. Fagnani, S. Zampieri, and R.J. Evans, "Feedback control under data rate cosntraints: an overview", *IEEE Proceedings. Special Issue on Technology of Networked Control Systems*, December 2006, To appear.

[3] J. Baillieul, "Feedback designs in information-based control", in *Stochastic Theory and Control: Proceedings of a Workshop held in Lawrence, Kansas*, 2002, vol. 280 of *Lecture Notes in Control and Information Sciences*, pp. 35–58.

[4] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilisation over signal-to-noise ratio constrained channels", in *Proc. 2004 American Control Conference*, Boston, USA, 2004, vol. 6, pp. 4903–4908.

[5] W.S. Wong and R.W. Brockett, "Systems with finite communication bandwidth constraints. ii. stabilization with limited information feedback", *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049–1053, May 1999.

[6] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (dfcb) control", *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1573–1584, 2004.

[7] K.J. Åström and B.M. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems", in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, USA, December 2002, pp. 2011–2016.

[8] M. Miśkowicz, "The event-triggered sampling optimization criterion for distributed networked monitoring and control systems", in *Proc. of the IEEE International Conference on Industrial Technology*, 2003, vol. 2, pp. 1083–1088.

[9] R. McCann, A.K. Gunda, and S.R. Damugatla, "Improved operation of networked control systems using Lebesgue sampling", in *Conference Record of the 2004 IEEE Industry Applications Conference, 2004. 39th IAS Annual Meeting*, 2004, vol. 2, pp. 1211–1216.

[10] E. Allier, G. Sicard, L. Fesquet, and M. Renaudin, "Asynchronous level crossing analog to digital converters", *Measurement*, vol. 37, pp. 296–309, 2005.

[11] K.J. Åström, "Event based sampling", Presented at the Pravin Varaiya Symposium on Systems, Control and Networks, June 2005.

[12] E. Kofman, "Quantized state control: a method for discrete event control of continuous systems", *Latin American Applied Research*, vol. 33, no. 4, pp. 399–406, 2003.

[13] E. Kofman, "Discrete Event Simulation of Hybrid Systems", *SIAM Journal on Scientific Computing*, vol. 25, no. 5, pp. 1771–1797, 2004.

[14] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

[15] B. Zeigler, T.G. Kim, and H. Praehofer, *Theory of Modeling and Simulation. Second edition*, Academic Press, New York, 2000.

[16] E. Kofman, "Non conservative ultimate bound estimation in LTI perturbed systems", *Automatica*, vol. 41, no. 10, pp. 1835–1838, 2005.

[17] Ernesto Kofman, Hernan Haimovich, and Maria Seron, “A systematic method to obtain ultimate bounds for perturbed systems”, *International Journal of Control*, 2006, in press.

[18] T. Chen and B.A. Francis, *Optimal sampled-data control systems*, Springer-Verlag, 1995.

[19] F. Grognard and R. Sepulchre, “Computation of time-optimal switchings for linear systems with complex poles”, in *Proceedings of the European Control Conference*, Cambridge, United Kingdom, 2003, CDROM.

[20] G.N. Nair and R.J. Evans, “Exponential stabilisability of finite-dimensional linear systems with limited data rates”, *Automatica*, vol. 39, no. 4, pp. 585–593, April 2003.