# $L_2$-induced Norms and Frequency-gains of Sampled-data Sensitivity Operators.*

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**Abstract**

This paper develops exact, computable formulas for the frequency-gain and $L_2$-induced norm of the sensitivity operator in a sampled-data control system. With sampled-data, we refer to a system that combines both continuous-time and discrete-time signals, and which is studied in continuous-time. The expressions are obtained using lifting techniques in the frequency-domain, and have application in performance and stability robustness analysis taking in account full intersample information.

**Keywords:** Sampled-data systems, $L_2$-induced norms, Frequency response, Sensitivity analysis, Generalized sampled-data holds.

## 1 Introduction.

This paper studies the computation of the $L_2$-induced norm of the sensitivity operator in a sampled-data (SD) control system. The term SD indicates that we approach the system in continuous-time, i.e., considering full intersample information. The $L_2$-induced norm is the operator norm when inputs and outputs belong to the space of square-integrable signals $L_2$, and it is linked to important control problems. Indeed, for linear time-invariant (LTI) systems, the $L_2$-induced norm of a system operator is the $H_\infty$-norm of its transfer matrix, which represents a very useful measure in many applications of modern control theory (e.g., Francis 1991).

The concepts and methods associated with LTI $H_\infty$ control bear no immediate extension to SD systems since, when intersample behavior is taken in account, the operators are time-varying and no transfer functions are associated with them. In view of this, considerable research during the last years has focused on the study of $L_2$-norms and $H_\infty$ related problems for SD systems.

Early works addressing the computation of $L_2$-induced norms of SD systems include Thompson, Stein & Athans (1983),Thompson, Daily & Doyle (1986), Chen & Francis (1990), and Leung, Perry & Francis (1991). Thompson et al. (1983), Thompson et al. (1986), and Chen & Francis (1990) considered simple open-loop connections involving a sampler and a zero order hold (ZOH). A formula for the computation of the $L_2$-induced norm in a feedback setup was given in Leung

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et al. (1991) under the assumptions of band-limited input signals. In particular, the numerical implementation of our results relies on the same tools used in the latter paper for the case of ZOH.

More general results on the computation of $L_2$-induced norms and the first solutions to SD $H_\infty$ optimal control problems were obtained using different time-domain frameworks. One of these frameworks is the so-called *lifting technique* (Yamamoto 1990, Bamieh, Pearson, Francis & Tannenbaum 1991, Bamieh & Pearson 1992, Toivonen 1992, Yamamoto 1993, Dullerud & Glover 1993, Yamamoto 1994). Time-domain approaches that dispense with the use of lifting techniques include Tadmor (1992), Kabamba & Hara (1993), Sun, Nagpal & Khargonekar (1993), and Sivashankar & Khargonekar (1994). In particular, Sun et al. (1993) applied techniques based on *linear systems with jumps* to the synthesis of $H_\infty$ controllers for SD systems. An interesting conclusion from (Sun et al. 1993) is that, if one does not assume a ZOH, then the optimal solution involves the use of a generalized SD hold function (GSHF) *à la* Kabamba (1987).

A recently introduced concept, which is closely related to $L_2$-induced norms, is that of the *frequency-gain* of a SD operator. This concept extends the LTI notion of frequency response to SD systems, in the sense that the maximum magnitude of the frequency-gain of a SD operator is its $L_2$-induced norm (Yamamoto & Khargonekar 1993, Hagiwara, Ito & Araki 1995, Yamamoto & Araki 1994, Chen & Francis 1995). Yamamoto & Khargonekar (1993) used lifting techniques to compute the frequency-gain of a general SD system, while Hagiwara et al. (1995) obtained similar results for the class of SD *compact* operators based on the frequency-domain framework developed in Araki, Ito & Hagiwara (1993). Relations between both approaches have been discussed in Yamamoto & Araki (1994). Although more general, the procedures proposed in Yamamoto & Khargonekar (1993) do not seem to have an easy numerical implementation. On the other hand, the formulas provided by Hagiwara et al. (1995) are readily numerically implementable in a reliable fashion. An iterative procedure was also suggested in Hagiwara et al. (1995) to compute the frequency-gain of operators such as the sensitivity operator, which, as it turns out, is *not* compact.

In this note we apply a frequency-domain lifting technique to obtain exact formulas for the computation of the frequency-gain of the SD sensitivity operator. The $L_2$-induced norm is then obtained from the frequency-gain by performing a simple search of maximum over a finite interval of frequencies. These expressions have direct application in performance and stability robustness analyses of SD systems. In particular, our results are formulated in terms of GSHFs, thus allowing the analysis of design techniques using D/A devices other than the ZOH (e.g., Kabamba 1987, Sun et al. 1993, Yang & Kabamba 1994).

The paper is organized as follows. §2 defines notation and introduces the system under study. §3 reviews the frequency-domain lifting formalism that will be used to state and prove our results, which appear in §4. Expressions for the numerical implementation of the results and an illustrative example are given in §5. §6 has some concluding remarks.

## 2 Preliminaries.

We consider the multivariable SD feedback system shown in Figure 1, where $P$ and $F$ are the transfer functions of the plant and anti-aliasing filter, $C_d$ is the digital controller, and $r$, $d$ and $n$ are the command, disturbance and noise inputs to the system. The system output is $y$, and $u$ and $\{u_k\}$ are the analog and discrete control inputs. The plant and controller are assumed to be proper, and the filter strictly proper and stable,[1] and they are all free of unstable hidden modes.

We denote the sampling period by $T$, and the sampling frequency by $\omega_s \triangleq 2\pi/T$. The *Nyquist frequency range* is defined as the interval $\Omega_N \triangleq [-\omega_s/2, \omega_s/2]$. If $\nu$ is a continuous-time signal, we define the sampling operation with period $T$ by

$$\mathcal{S}_T\, \nu = \{\nu_k\}_{k=-\infty}^{\infty},$$

[1] The assumption that the filter is strictly proper is standard and guarantees that the sampling operation is well-defined (e.g., Sivashankar & Khargonekar 1993, Dullerud & Glover 1993). The assumption of stability is only made for simplicity of exposition and may be removed.

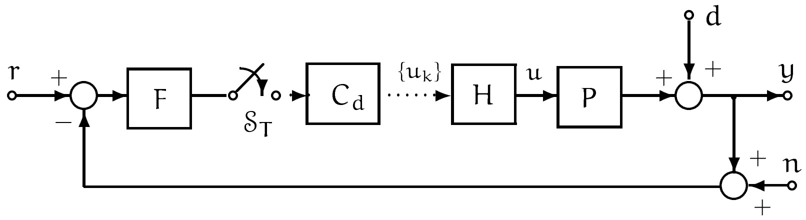


Figure 1: SD control system.

where the sequence $\{v_k\}_{k=-\infty}^{\infty}$ represents the sampled signal, with $v_k = v(kT)$ for any integer k. The z-transform operator is denoted by $\mathcal{Z}$, i.e., $\mathcal{Z}\{u_k\} \triangleq \sum_{k=-\infty}^{\infty} u_k z^{-k}$, and the Laplace transform operator is denoted by $\mathcal{L}$, $\mathcal{L}u = U$.

The hold device H is a GSHF (Kabamba 1987), defined by

$$u(t) = h(t - kT)u_k, \quad kT \le t < (k+1)T, \quad k \in \mathbb{Z}.$$

The function h, which characterizes the hold, is defined over the interval $[0, T]$ and satisfies some mild technical conditions described in Middleton & Freudenberg (1995). Associated with this hold we define its *frequency response function* by $H = \mathcal{L}h$. Since h is supported on a finite interval, it follows that H has no singularities at any finite s in $\mathbb{C}$; e.g., for the ZOH, $H(s) = (1 - e^{-sT})/s$. Frequency responses for other types of hold functions are derived in Middleton & Freudenberg (1995).

We denote by $(FPH)_d$ the *discretized plant*, defined as

$$(FPH)_d \triangleq \mathcal{Z}\mathcal{S}_T\mathcal{L}^{-1}FPH,$$

where $\mathcal{L}^{-1}FPH$ denotes the inverse Laplace transform of FPH. In connection with $(FPH)_d$, we also introduce the *discrete input sensitivity function* and the *discrete output complementary sensitivity function*, respectively as

$$S_d \triangleq [I + C_d(FPH)_d)]^{-1}, \quad \text{and} \quad T_d \triangleq (FPH)_d S_d C_d. \tag{1}$$

In Middleton & Freudenberg (1995), the well-known non-pathological sampling condition for plants discretized with a ZOH is generalized to the case of a GSHF. This result was also extended to the multivariable case in Middleton & Xie (1995). In particular, since, for GSHFs, H may have zeros in the right half plane, it is necessary to require that none of these coincides with an unstable pole of the analog plant. Under the non-pathological sampling hypothesis, it is straightforward to extend the exponential and $L_2$ input-output stability results of Chen & Francis (1991) and Francis & Georgiou (1988) to the case of a GSHF. We shall assume throughout that the system considered in Figure 1 is $L_2$ input-output stable.

The assumptions on P, H and F stated above suffice to guarantee (e.g., Freudenberg, Middleton & Braslavsky 1994) that $(FPH)_d$ satisfies the well-known formula

$$(FPH)_d(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_k(j\omega)P_k(j\omega)F_k(j\omega), \tag{2}$$

where the notation $Y_k(s)$ represents $Y(s + jk\omega_s)$, with k an entire number. This notation will be frequently used in the sequel.

We shall also use the function G, defined as

$$G(j\omega) \triangleq \frac{1}{T} P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T}). \tag{3}$$

Associated with G and F, we introduce now two discrete transfer matrices that will be required to formulate our results,

$$G_d(e^{j\omega T}) \triangleq \sum_{k=-\infty}^{\infty} G_k^*(j\omega)G_k(j\omega), \quad \text{and} \tag{4}$$

$$F_d(e^{j\omega T}) \triangleq \sum_{k=-\infty}^{\infty} F_k(j\omega)F_k^*(j\omega), \tag{5}$$

where $F^*$ denotes the conjugated transpose of F. Note that if $y$, $n$, and $d$ are valued in $\mathbb{R}^m$, then $G_d(e^{j\omega T})$ and $F_d(e^{j\omega T})$ are $m \times m$ discrete transfer matrices.

A characteristic feature of SD systems is evident from (2), namely, the response of the discretized plant at a frequency $\omega \in \Omega_N$ depends upon the response of the analog plant, prefilter, and hold function at an infinite number of frequencies. Indeed, it is well-known that the steady-state response of a stable SD system to a sinusoidal input consists of a fundamental component and infinitely many aliases shifted by multiples of the sampling frequency. Analogous expressions are obtained for the response to more general inputs, such as noise $n$ and output disturbance $d$ (cf. Goodwin & Salgado 1994, Araki, Ito & Hagiwara 1993, Freudenberg et al. 1994). In particular, if $n$ is in $L_2(0,\infty)$ and N is its Laplace transform, then we have that the system response is given by

$$Y(j\omega) = -P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)N_k(j\omega). \tag{6}$$

Similarly, for the response to a disturbance $d$ in $L_2(0,\infty)$, we have that

$$Y(j\omega) = D(j\omega) - P(j\omega)H(j\omega)S_d(e^{j\omega T})C_d(e^{j\omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)D_k(j\omega). \tag{7}$$

Although a SD system is time-varying, its intrinsic periodic nature allows the use of model transformation techniques that yield time-invariant characterizations. The following section deals with one of them.

## 3 Frequency-domain lifting.

Several frameworks are available for the treatment of SD systems embodying intersample information in the model. Among them, we have time-domain approaches, as the lifting technique of, e.g., Yamamoto (1990), Bamieh & Pearson (1992), Yamamoto & Khargonekar (1993); and frequency-domain approaches, as the *FR-operators* introduced in Araki, Ito & Hagiwara (1993). We shall use here a frequency domain setting that we refer to as a — frequency-domain — lifting technique. The transformation involved in this approach may be viewed as a generalization of the Fourier transform in the FR-operators framework of Araki, Ito & Hagiwara (1993). The idea of lifting in frequency-domain is not new; it was developed in the signal processing literature for linear discrete-time periodic systems. See for example Shenoy, Burnside & Parks (1994).

Let $y$ be a signal in the space $L_2(0,\infty)$. Then, it is a fact that its Fourier transform $Y(j\omega)$ belongs to $L_2(-\infty,\infty)$. Now, from $Y(j\omega)$ construct the sequence of functions

$$\{Y_k(j\omega)\}_k = \{Y(j(\omega + k\omega_s))\}_k, \tag{8}$$

for $\omega$ in the Nyquist range $\Omega_N$ and $k$ integer. Arrange this sequence in an infinite vector, which we denote by

$$y(\omega) \triangleq \begin{bmatrix} \vdots \\ Y_1(j\omega) \\ Y_0(j\omega) \\ Y_{-1}(j\omega) \\ \vdots \end{bmatrix}. \tag{9}$$

We say that the infinite-dimensional vector $y$ is the *lifting* of $Y$, and we denote the lifting operation as $y = \mathcal{F}Y$. As a function, $y$ is defined at almost every[2] $\omega$ in $\Omega_N$, and takes values in $\ell_2$. Moreover, these $\ell_2$-valued functions form a Hilbert space (Balakrishnan 1981) under the norm and inner product

$$\|y\| \triangleq \left( \int_{\Omega_N} \|y(\omega)\|_{\ell_2}^2 \, d\omega \right)^{1/2}, \quad \langle y, x \rangle \triangleq \int_{\Omega_N} \langle y(\omega), x(\omega) \rangle_{\ell_2} \, d\omega \,.$$

We denote this space by $L_2(\Omega_N;\ell_2)$. Since the elements of $L_2(\Omega_N;\ell_2)$ are essentially rearrangements of those of $L_2(-\infty,\infty)$, it is not difficult to see that both spaces are in fact isomorphic with preservation of norm, as stated in the following lemma (See also Yamamoto & Araki 1994).

**Lemma 3.1** *The space* $L_2(\Omega_N;\ell_2)$ *is isometrically isomorphic to* $L_2(-\infty,\infty)$.

**Proof:** See Appendix. □

A key point of the lifting is that, in the new space, operators of the SD system are represented as *multiplication operators* described by infinite-dimensional "transfer matrices". In other words, if $\mathcal{M}$ is a bounded operator in $L_2$, and $\mathcal{F}\mathcal{M}\mathcal{F}^{-1}$ is the corresponding operator in $L_2(\Omega_N;\ell_2)$, then its action can be described as $(\mathcal{F}\mathcal{M}\mathcal{F}^{-1}y)(\omega) = M(\omega)y(\omega)$. An important consequence of these facts is that the $L_2$-induced norm of the operator can then be computed as (Bamieh et al. 1991, Yamamoto & Araki 1994)

$$\|\mathcal{M}\| = \sup_{\omega \in \Omega_N} \|M(\omega)\|, \tag{10}$$

where the supremum is understood as the essential supremum in $\Omega_N$, and $\|M(\omega)\|$ is the $\ell_2$-induced operator norm of $M(\omega)$. The scalar-valued function $\|M\| : \Omega_N \to \mathbb{R}_0^+$ is the so-called *frequency-gain* of the SD operator $\mathcal{M}$ (Yamamoto & Khargonekar 1993, Hagiwara et al. 1995).

Notice the similarity of (10) to the familiar expression of the $L_2$-induced norm of an operator in a LTI system, i.e., the $H_\infty$-norm of a transfer matrix.

# 4 $L_2$-induced norms of Sensitivity Operators.

We concentrate on the sensitivity and complementary sensitivity operators in the SD system of Figure 1. These operators are defined as the mappings relating output disturbance $d$ and noise $n$ to the output $y$, and are respectively denoted by

$$\begin{aligned} \mathcal{S} : L_2 &\to L_2 \\ \mathcal{S}d &\mapsto y \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{T} : L_2 &\to L_2 \\ \mathcal{T}n &\mapsto y. \end{aligned}$$

Under the assumptions of closed-loop $L_2$-stability, $\mathcal{S}$ and $\mathcal{T}$ are bounded operators on $L_2$.

The actions of the complementary sensitivity and sensitivity operators are respectively defined in frequency-domain by the steady-state responses (6) and (7). Using the frequency-domain lifting of §3, it is straightforward to alternatively write (6) and (7) as

$$y = -Tn \qquad \text{and} \qquad y = Sd\,. \tag{11}$$

[2] With respect to the standard Lebesgue measure.

Here, $T(\omega)$ is the following infinite-dimensional transfer matrix defined on $\Omega_N$ (cf. Araki, Hagiwara & Ito 1993, Yamamoto & Araki 1994)

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \\ \cdots & G_k F_k & G_k F_{k-1} & \cdots \\ \cdots & G_{k-1} F_k & G_{k-1} F_{k-1} & \cdots \\ & \vdots & \vdots & \ddots \end{bmatrix}, \tag{12}$$

where $F$ is the transfer matrix of the prefilter, and $G$ is the function introduced in (3). Obviously, the corresponding matrix for $\mathcal{S}$ is given by $S = I - T$, where $I$ represents the identity operator on $\ell_2$.

Operators $T$ and $S$ are infinite-dimensional transfer matrix representations of the SD complementary sensitivity and sensitivity operators $\mathcal{T}$ and $\mathcal{S}$ (Araki & Ito 1993, Araki, Hagiwara & Ito 1993). We are interested in the computation of their frequency-gains $\|T(\omega)\|$ and $\|S(\omega)\|$. The corresponding $L_2$-induced norms are then obtained, from (10), by searching for suprema over the finite interval $\Omega_N$.

An important fact about the complementary sensitivity operator $\mathcal{T}$ is that it has finite rank (and therefore is compact). We show this in the following lemma.

**Lemma 4.1** *If the inputs to the system in Figure 1 are valued in $\mathbb{R}^m$, then $\mathcal{T}$ has at most rank $m$.*

**Proof:** Partition $F(j\omega)$ by rows, and $G(j\omega)$ by columns, i.e., $F = [f_1^*, f_2^*, \dots, f_m^*]^*$, and $G = [g_1, g_2, \dots, g_m]$. Introduce the liftings for $F^*$ and $G$, denoted $f = \mathcal{F}F^*$, and $g = \mathcal{F}G$, which can then be written $f = [f_1, f_2, \dots, f_m]$, and $g = [g_1, g_2, \dots, g_m]$. Each column $f_i = \mathcal{F}f_i^*$ in $f$, and $g_i = \mathcal{F}g_i$ in $g$, certainly is a vector in $L_2(\Omega_N;\ell_2)$, since $F$ and $G$ are both stable and strictly proper from our assumptions in § 2. Using this notation, the action of $T$ can be alternatively written as

$$Tn = \sum_{i=1}^{m} g_i \langle n, f_i \rangle_{\ell_2}, \tag{13}$$

where, $\langle n, f_i \rangle_{\ell_2}$ is a scalar-valued function defined a.e. on $\Omega_N$. Equation (13) shows that $T$ is the sum of $m$ rank-one operators on $L_2(\Omega_N;\ell_2)$; hence it has at most rank $m$, and so does $\mathcal{T}$. □

Since $\mathcal{T}$ is compact, the numerical computation of the norm of $\mathcal{T}$ may be approximated by truncating $T$ between harmonics $-n$ and $n$, say, and evaluating the maximum singular value of the finite dimensional transfer matrix so obtained, as suggested in Araki, Hagiwara & Ito (1993). The convergence of this sequence of computations could be slow, though, since in general $G(j\omega)$ and $F(j\omega)$ decay as $1/\omega^p$, for some integer $p$ depending on the relative degrees of the transfer matrices involved.

In fact, since $\mathcal{T}$ is finite-rank, more efficient ways of computing $\|T(\omega)\|$ are possible. In Hagiwara et al. (1995), the authors have shown for this case that the frequency-gain is given as the maximum eigenvalue of an associated finite-dimensional discrete transfer matrix evaluated on the unit circle. For the case of ZOH, they show how to implement their procedures in a numerically reliable fashion.

The following theorem is analogous to the result of Hagiwara et al. (1995). The pattern of our proof is quite different though, and we include it here because the same pattern will be used for the case of the SD sensitivity operator. Denote by $\lambda_{max}[M]$ the maximum eigenvalue of a square matrix $M$. Recalling the functions $G_d$ and $F_d$ defined by (4) and (5), we then have the following result.

**Theorem 4.2 ($L_2$-induced Norm of $\mathcal{T}$)** *If the SD system of Figure 1 is $L_2$-input-output stable, then*

$$\|\mathcal{T}\| = \sup_{\omega \in \Omega_N} \|T(\omega)\|, \tag{14}$$

*where*

$$\|T(\omega)\|^2 = \lambda_{\max}\left[G_d(e^{j\omega T})F_d(e^{j\omega T})\right] \; . \tag{15}$$

**Proof:** From (10) we have that $\|\mathcal{T}\| = \sup_{\omega\in\Omega_N}\|T(\omega)\|$. Fix $\omega$ in $\Omega_N$. Since $T$ is a finite-rank operator in $\ell_2$, from (13) we can write as a dyadic product $T = gf^*$. We can then decompose $\ell_2$ into $\ell_2 = \wp_F \oplus \wp_F^\perp$, where $\wp_F$ is the subspace of $\ell_2$ spanned by the range of $f$, and $\wp_F^\perp$ its orthogonal complement. Hence, if $v$ is a vector in $\wp_F^\perp$ then $Tv = 0$, and we can write

$$\begin{aligned}\|T\| &= \sup_{\substack{v\in\ell_2\\ v\neq 0}} \frac{\|Tv\|_{\ell_2}}{\|v\|_{\ell_2}}\\ &= \sup_{\substack{v\in\wp_F\\ v\neq 0}} \frac{\|Tv\|_{\ell_2}}{\|v\|_{\ell_2}}.\end{aligned}$$

Since vectors of $\ell_2$ in $\wp_F$ can be finitely parameterized as $v = f\alpha$, where $\alpha$ belongs to $\mathbb{C}^m$, with $m$ the number of inputs of $F$, we then have that

$$\begin{aligned}\|T\|^2 &= \sup_{\substack{\alpha\\ f\alpha\neq 0}} \frac{\alpha^* f^* f g^* g f^* f\alpha}{\alpha^* f^* f\alpha}\\ &= \lambda_{\max}\left[(f^*f)^{1/2}(g^*g)(f^*f)^{1/2}\right]. \end{aligned} \tag{16}$$

Notice that $(g^*g)(\omega) = G_d(e^{j\omega T})$ and $(f^*f)(\omega) = F_d(e^{j\omega T})$ are the discrete transfer matrices defined in (4) and (5). Moreover, they are finite $m\times m$ matrices and, particularly, $f^*f$ is non-singular since $F$ was assumed full column rank.

Finally, since eigenvalues are invariant under similarity transformations, (15) follows from (16), completing the proof. □

The case of $\mathcal{S}$ has to be considered more carefully, since this is a non-compact operator, and as such, it may not be in principle approximable by sequences of finite-rank operators, meaning that the norms of progressive truncations of $S$ may not necessarily converge to the norm of the operator.

Frequency-gains of possibly non-compact SD operators have been discussed in Yamamoto & Khargonekar (1993), but, as pointed out in Hagiwara et al. (1995), the proposed procedure seems in general hard to implement numerically. In Hagiwara et al. (1995), a numerically reliable method is suggested for the case of operators like the sensitivity, i.e., the sum of a compact and a constant operator. Yet, to compute the frequency gain $\|S\|$, this last method still requires a $\gamma$-iteration *at each frequency* $\omega\in\Omega_N$.

The following theorem gives an exact, computable formula for the frequency-gain and $L_2$-induced norm of the SD sensitivity operator $\mathcal{S}$. Our result relies on the fact that $\mathcal{S}$ verifies the complementarity relation $\mathcal{S} = \mathcal{I} - \mathcal{T}$, and since $\mathcal{T}$ is of finite rank, it turns out that the computation of the frequency-gain of $\mathcal{S}$ also reduces to a finite-dimensional eigenvalue problem. As for Theorem 4.2, these results admit a simple and reliable numerical implementation.

**Theorem 4.3** ($L_2$**-induced Norm of** $\mathcal{S}$) *If the SD system of Figure 1 is $L_2$-input-output stable, then*

$$\|\mathcal{S}\| = \sup_{\omega\in\Omega_N}\|S(\omega)\| \; , \tag{17}$$

*where*

$$\|S(\omega)\|^2 = 1 + \lambda_{\max}\begin{bmatrix} F_d(e^{j\omega T})G_d(e^{j\omega T}) - T_d(e^{j\omega T}) & -F_d(e^{j\omega T}) \\ T_d^*(e^{j\omega T})G_d(e^{j\omega T}) - G_d(e^{j\omega T}) & -T_d^*(e^{j\omega T}) \end{bmatrix},$$

$G_d$, $F_d$ *are the functions given by* (4) *and* (5), *and* $T_d$ *is the discrete complementary sensitivity function defined in* (1).

**Proof:** The same idea for the proof of Theorem 4.2 works here. Again, for a fixed $\omega$ in $\Omega_N$, decompose $\ell_2$ into $\ell_2 = \wp_{(F,G)} \oplus \wp^{\perp}_{(F,G)}$, where $\wp_{(F,G)}$ denotes the subspace spanned by both f and g, and $\wp^{\perp}_{(F,G)}$ its orthogonal complement. Since S is block diagonal in these spaces, we have ,that

$$\begin{aligned} \|S\| &= \max\left\{ \sup_{\substack{v\in\wp_{(F,G)}\\ v\neq 0}} \frac{\|Sv\|_{\ell_2}}{\|v\|_{\ell_2}} \quad , \sup_{\substack{v\in\wp^{\perp}_{(F,G)}\\ v\neq 0}} \frac{\|Sv\|_{\ell_2}}{\|v\|_{\ell_2}} \right\} \\ &= \max\left\{ \sup_{\substack{v\in\wp_{(F,G)}\\ v\neq 0}} \frac{\|Sv\|_{\ell_2}}{\|v\|_{\ell_2}} \quad , \quad 1 \right\}. \end{aligned} \tag{18}$$

Now, any vector v in $\wp_{(F,G)}$ can be finitely parameterized as

$$\begin{aligned} v &= f\alpha + g\beta \\ &= [f, g]\,\gamma, \end{aligned} \tag{19}$$

with $\gamma$ in $C^{2m}$. Denote $h \triangleq [f, g]$, and $M \triangleq h^*h$. Notice that $M$ is a finite-dimensional, positive semi-definite Hermitian matrix. Using the notation introduced in (4) and (5), and noticing that the discrete complementary sensitivity in (1) may be expressed as $T_d = f^*g$, we can write $M$ as

$$M = \begin{bmatrix} F_d & T_d \\ T_d^* & G_d \end{bmatrix}.$$

Introducing the matrix $N$,

$$N \triangleq \begin{bmatrix} G_d & -I \\ -I & 0 \end{bmatrix},$$

it then follows that $h^*S^*Sh = h^*(I - fg^*)(I - gf^*)h = (I + MN)M$, and hence, we obtain from (19) that

$$\begin{aligned} \sup_{\substack{v\in\wp_{(F,G)}\\ v\neq 0}} \frac{\|Sv\|^2_{\ell_2}}{\|v\|^2_{\ell_2}} &= \sup_{\gamma\in\mathbb{C}^{2m}} \frac{\gamma^*M\gamma + \gamma^*MNM\gamma}{\gamma^*M\gamma} \\ &= 1 + \lambda_{\max}\left[M^{1/2}NM^{1/2}\right] \qquad (20) \\ &= 1 + \lambda_{\max}\left[MN\right] \, . \qquad (21) \end{aligned}$$

Since in (21) the product $MN$ is

$$MN = \begin{bmatrix} F_dG_d - T_d & -F_d \\ T_d^*G_d - G_d & -T_d^* \end{bmatrix},$$

from (18) and (21) we see that it only remains to show that $\lambda_{\max}[MN]$ is nonnegative to complete the proof. But this follows easily from the fact that $M \geq 0$. Indeed, suppose first that $M$ is positive definite, i.e., $M > 0$. Then,

$$\delta = \begin{bmatrix} F_d & T_d \\ T_d^* & G_d \end{bmatrix}^{-1/2} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

gives $\delta^*M^{1/2}NM^{1/2}\delta = G_d \geq 0$, and thus $\lambda_{\max}$ in (20) is nonnegative. Otherwise, if $M$ is not positive definite, it is then necessarily singular, which means that 0 must be in the spectrum of $M^{1/2}NM^{1/2}$, and therefore shows that $\lambda_{\max}[MN] \geq 0$. The proof is now complete. □

The closed-form expressions given by Theorems 4.2 and 17 can be used for performance and stability robustness analysis of SD systems (cf. Sivashankar & Khargonekar 1993, Dullerud & Glover 1993, Hagiwara & Araki 1995).

In the particular case of SISO systems, these formulas simplify and show some interesting connections. In this case, the operator $\mathcal{T}$ is then of rank one, and so the frequency-gains are given by the magnitude of the frequency response of scalar discrete transfer functions.

**Corollary 4.4** *If the SD system of Figure 1 is SISO, then*

$$\|T\| = \Phi_d |T_d|, \quad \textit{and} \tag{22}$$

$$\|S\| = \frac{1}{2}\left(\sqrt{(\Phi_d^2 - 1)|T_d|^2 + (|S_d| + 1)^2} + \sqrt{(\Phi_d^2 - 1)|T_d|^2 + (|S_d| - 1)^2}\right), \tag{23}$$

*where $S_d$ and $T_d$ are the discrete sensitivity and complementary sensitivity functions, evaluated at $z = e^{j\omega T}$, and*

$$\Phi_d^2(e^{j\omega T}) = \frac{\left(\sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2\right)\left(\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2\right)}{|(FPH)_d(e^{j\omega T})|^2}. \tag{24}$$

**Proof:** The proof of (22) follows immediately from Theorem 4.2. Formula (23) may be checked by computing $\lambda_{max}$ in (17) and after some straightforward but tedious algebraic manipulation. □

The expressions given in Corollary 4.4 show a direct connection to the discrete sensitivity functions $S_d$ and $T_d$. In fact, the magnitudes of their frequency responses are correspondingly lower bounds on the frequency-gains of $\mathcal{S}$ and $\mathcal{T}$, as we see in the following corollary.

**Corollary 4.5** *Under the assumptions of Corollary 4.4,*

$$\|T(\omega)\| \geq |T_d(e^{j\omega T})|, \quad \textit{and} \tag{25}$$

$$\|S(\omega)\| \geq \max\{|S_d(e^{j\omega T})|, 1\}, \tag{26}$$

*at all frequencies $\omega$ in $\Omega_N$.*

**Proof:** First notice from (24) that $\Phi_d$ is greater than or equal to 1 at any $\omega$ in $\Omega_N$, since by Cauchy-Schwarz

$$\begin{aligned} |(FPH)_d(e^{j\omega T})|^2 &= \left|\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)P_k(j\omega)H_k(j\omega)\right|^2 \\ &\leq \left(\sum_{k=-\infty}^{\infty} |F_k(j\omega)|^2\right) \times \left(\frac{1}{T^2}\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2\right). \end{aligned}$$

Hence, (25) follows immediately. For (26), we have from (23) that

$$\|S\| \geq \frac{||S_d| - 1| + |S_d| + 1}{2}, \tag{27}$$

since $\Phi_d \geq 1$. Therefore, from (27), if $|S_d| > 1$ then $\|S\| \geq |S_d|$, and otherwise $\|S\| \geq 1$, which completes the proof. □

Unsurprisingly, it then follows from Corollary 4.5 that the $L_2$-induced norms of the discretized system also give lower bounds for the $L_2$-induced norms of the SD system. The gap in norms, then, is due to the intersample information missing in the discrete description of the system. Note that, in this sense, $\sup_{\omega \in \Omega_N} \Phi_d$ may be interpreted as a "fidelity index", *independent of the controller*, that quantifies how close we can expect to be the discrete and SD performances.

# 5 Numerical implementation.

The expressions for the frequency-gains and $L_2$-induced norms obtained in the last section can be readily numerically implemented by computing $G_d$ and $F_d$ from (4) and (5). These computations can be approached as "special discretizations" by considering relations similar to (2). In this way, the arguments of $\sup_{\omega\in\Omega_N}$ in (14) and (17) are expressed by two rational transfer functions in $z = e^{j\omega T}$ — the frequency-gains of the SD sensitivity operators. The induced norms can then be computed by a straightforward search of maxima over the finite interval $\Omega_N$. Similar formulas have been derived for the case of ZOH in (Leung et al. 1991, Theorem 3).

## 5.1 Computation of $F_d(e^{j\omega T})$

Consider $F_d = T\mathcal{Z}\mathcal{S}_T\mathcal{L}^{-1}(F\tilde{F})$, where $\tilde{F}(s) \triangleq F(-s)^t$, and the superscript "t" denotes transpose. Since $F$ is strictly proper, then the sampling of the output of $F\tilde{F}$ is well-defined. If $\{a, b, c, 0\}$ is a minimal state-space realization for $F$, then, a minimal realization for $F\tilde{F}$ is given by

$$A = \begin{bmatrix} a & bb^t \\ 0 & -a^t \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -c^t \end{bmatrix}, \quad C = \begin{bmatrix} c & 0 \end{bmatrix},$$

and it is straightforward to see that the function $F_d(e^{j\omega T})$ is then computed as $F_d(e^{j\omega T}) = TC(e^{j\omega T}I - e^{AT})^{-1}B$.

## 5.2 Computation of $G_d(e^{j\omega T})$

The case of $G_d$ is slightly more complicated than the previous one, but can be approached in a similar fashion. From (4) we can write $G_d = \frac{1}{T}C_d^* S_d^* E_d S_d C_d$, where

$$E_d \triangleq \frac{1}{T}\sum_{k=-\infty}^{\infty} H_k^* P_k^* P_k H_k. \tag{28}$$

Hence, to compute $G_d$ we need to evaluate $E_d(e^{j\omega T})$. This is done by discretizing the cascade of the hold $\tilde{H}$, the system $P\tilde{P}$, and the hold $H$. Since $H$ is proper by definition, so is the cascade, and therefore the sampling operation is again well-defined. If the plant $P$ has a minimal realization $\{a, b, c, d\}$, then a minimal realization for $\tilde{P}P$ is given by

$$A = \begin{bmatrix} a & 0 \\ c^t c & -a^t \end{bmatrix}, \; B = \begin{bmatrix} b \\ c^t d \end{bmatrix}, \; C = \begin{bmatrix} d^t c & -b^t \end{bmatrix}, \; D = \begin{bmatrix} d^t d \end{bmatrix}.$$

Suppose that the hold is as defined in §2. Then, following Middleton & Freudenberg (1995), its pulse response can be described as

$$h(t) = \begin{cases} Ke^{L(T-t)}M & \text{if } t \in [0, T) \\ 0 & \text{otherwise} \end{cases}, \tag{29}$$

for matrices $K$, $L$, and $M$ of appropriate dimensions. Using these data, it may be checked that the function $E_d(e^{j\omega T})$ in (28) is given by $E_d(e^{j\omega T}) = C_d(e^{j\omega T}I - A_d)B_d + D_d$, where $A_d = e^{AT}$, $B_d = \int_0^T e^{A\tau}BKe^{L\tau}M\, d\tau$, and

$$\begin{aligned} C_d &= \int_0^T M^t e^{L^t(T-\tau)}K^t C e^{A\tau}\, d\tau \\ D_d &= \int_0^T M^t e^{L^t\tau}K^t D K e^{L\tau}M\, d\tau + \int_0^T M^t e^{L^t(T-\tau)}K^t C \int_0^\tau e^{A(\tau-\sigma)}BKe^{L(T-\sigma)}M\, d\sigma d\tau \end{aligned}$$

Matrices $B_d, C_d$ and $D_d$ in the above expressions can be easily numerically evaluated using matrix exponential formulas suggested by Van Loan (1978), which yield

$$B_d = [e^{AT} \quad 0] \exp\left\{ \begin{bmatrix} -A & BK \\ 0 & L \end{bmatrix} T \right\} \begin{bmatrix} 0 \\ M \end{bmatrix},$$
$$C_d = [M^t \quad 0] \exp\left\{ \begin{bmatrix} L^t & K^t C \\ 0 & A \end{bmatrix} T \right\} \begin{bmatrix} 0 \\ I \end{bmatrix},$$
$$D_d = [M^t e^{L^t T} \quad 0] \exp\left\{ \begin{bmatrix} -L^t & K^t D K \\ 0 & L \end{bmatrix} T \right\} \begin{bmatrix} 0 \\ M \end{bmatrix} + [M^t \quad 0] \exp\left\{ \begin{bmatrix} L^t & K^t C & 0 \\ 0 & A & BK \\ 0 & 0 & -L \end{bmatrix} T \right\} \begin{bmatrix} 0 \\ e^{LT} M \end{bmatrix}.$$

### 5.3 Example: Sensitivity of GSHF Control

These formulas may be readily used to analyze sensitivity and robustness properties of SD systems. As an illustration, we compute the frequency-gain of a system taken from Kabamba (1987, Example 1). In this example, the author considered a harmonic oscillator that cannot be made asymptotically stable by LTI control. Yet, as shown in Kabamba (1987), the system can be stabilized with a GSHF in a feedback configuration as displayed in Figure 2.

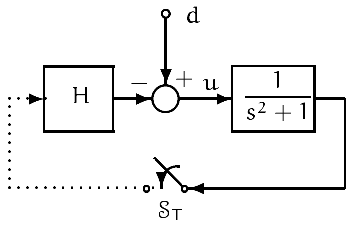


Figure 2: System with plant input disturbance.

The GSHF given by matrices

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} -13.1682 \\ 7.0898 \end{bmatrix},$$

and sampling period of $T = 1$ renders the closed-loop system stable with a response that is deadbeat in two sampling periods at most. As noted in Kabamba (1987), this system can also be asymptotically stabilized with a ZOH and a constant gain $p$, but then the discrete closed-loop eigenvalues cannot be arbitrarily assigned, as is the case with the GSHF.

However, the GSHF solution is more sensitive and less robust than the ZOH solution. Indeed, consider the SD operator $\mathcal{S}$ mapping input disturbance $d$ to $u$ in Figure 2. Figure 3 shows the frequency-gains of $\mathcal{S}$ for the GSHF solution (left) and the ZOH with $p = 0.75$ (right). For comparison, we also plotted the magnitude of the frequency responses of the corresponding discrete sensitivity functions $S_d$.

We see from Figure 3 that, in the GSHF solution, $\|S\|$ has peaks that more than double those in the ZOH solution. This indicates higher sensitivity to input disturbances and poorer robustness properties to plant perturbations.

## 6 Conclusions

This paper has considered the worst-case disturbance/noise performance ($L_2$-induced norm) of a SD system with full intersample information. Using a frequency-domain lifting technique, we have derived exact expressions for the computation of the frequency-gain and $L_2$-induced norm of

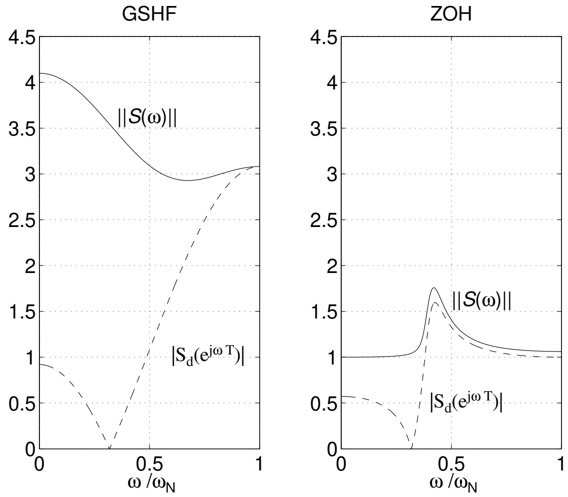


Figure 3: Sampled-Data frequency gains.

the SD sensitivity operator. The formulas involve a finite-dimensional eigenvalue problem that is readily numerically implemented.

These formulas have immediate application in the analysis of stability robustness for LTV unstructured perturbations, and $H_\infty$ control synthesis problems. Particularly, since our expressions allow the use of GSHFs, they provide a reliable computational tool for the evaluation of performance of a general class of SD designs.

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## Appendix: Proof of Lemma 3.1

Let $Y(j\omega)$ be in $L_2$. Then we have that

$$\begin{aligned} \|Y\|^2 &= \int_{-\infty}^{\infty} |Y(j\omega)|^2\, d\omega \qquad (30) \\ &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\omega_N}^{(2k+1)\omega_N} |Y(j\omega)|^2\, d\omega \\ &= \sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y(j(\omega + k\omega_s))|^2 d\omega. \qquad (31) \end{aligned}$$

As $\|Y\|^2$ is finite by assumption, the series $\sum \int |Y_k(j\omega)|^2\, d\omega$ is convergent. Then, by Levi's Theorem (Riesz & Sz.-Nagy 1955), we can interchange summation and integration in (31), and using (8) we have that

$$\begin{aligned} \sum_{-\infty}^{\infty} \int_{-\omega_N}^{\omega_N} |Y_k(j\omega)|^2 d\omega &= \int_{-\omega_N}^{\omega_N} \sum_{-\infty}^{\infty} |Y_k(j\omega)|^2 d\omega \qquad (32) \\ &= \|y\|^2 \qquad (33) \end{aligned}$$

From (30)-(33) it follows that there is an isometry between $L_2(-\infty, \infty)$ and $L_2(\Omega_N; \ell_2)$. To see that the isometry is isomorphic, we have to show that it is *onto*, that is, *each* function in $L_2(\Omega_N; \ell_2)$ is the image of a function in $L_2(-\infty, \infty)$. It will suffice to show that this is the case for each element in a basis for $L_2(\Omega_N; \ell_2)$.

Let $\{\phi_k\}_{k=-\infty}^{\infty}$ be an orthonormal basis for $\ell_2$, and $\{\psi_k(\omega)\}_{k=-\infty}^{\infty}$ an orthonormal basis for $L_2(\Omega_N)$. It can be proved that the sequence

$$\{\psi_n(\omega)\,\phi_m\}_{n,m=-\infty}^{\infty}$$

is an orthonormal base on $L_2(\Omega_N;\ell_2)$. Now, take for example $\psi_n(\omega)\,\phi_m$, for fixed integers $n$, $m$. This element of $L_2(\Omega_N;\ell_2)$ is precisely

$$\begin{array}{cc} \begin{bmatrix} \vdots \\ 0 \\ \psi_n(\omega) \\ 0 \\ \vdots \end{bmatrix} & \begin{array}{c} \vdots \\ {\scriptstyle m+1} \\ {\scriptstyle m} \\ {\scriptstyle m-1} \\ \vdots \end{array} \end{array}, \tag{34}$$

which corresponds to the function

$$\psi(\omega) = \begin{cases} \psi_n(\omega) & \text{if } \omega \in [-\omega_N + m\omega_s, \omega_N + m\omega_s] \\ 0 & \text{otherwise} \end{cases} . \tag{35}$$

But $\psi(\omega)$ is in $L_2(-\infty,\infty)$, as it is a function of finite support and integrable there. Therefore, every element in $L_2(\Omega_N;\ell_2)$ is the image of an element in $L_2(-\infty,\infty)$ and the proof is completed.

$\square$