Towards Quantitative Time Domain Design Tradeoffs in Nonlinear Control*

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Abstract

This paper analyzes the feasibility of quantifying design tradeoffs on the transient step response of a class of nonlinear systems. This feasibility analysis builds on available tools for the characterization of performance limitations in the optimal quadratic response of the class of strict feedback nonlinear systems. We present results that show that, as in linear systems, for certain classes of nonminimum phase systems, the closed loop transient step response must display undershoot. A lower bound on this undershoot can be computed based on the settling time of the system, and this bound increases as the settling time is decreased.

Keywords: Time-domain analysis, Nonlinear systems, Performance analysis, Cheap control, Quadratic optimal regulators, Non-minimum phase systems.

1 Introduction

Nonlinear control systems theory has in recent years significantly evolved towards the generation of systematic design procedures of effective application in engineering practice. As exposed in the recent survey paper [11], this evolution, during the 1990s, was marked by the transformation of the predominantly *descriptive* and analysis-oriented concepts and tools of the 1980s into the *constructive* concepts and tools that are nowadays applied to the control of special classes of nonlinear systems that include ships, jet engines, turbo-diesel engines and electric induction motor drives [11, 7, 13].

As pointed out in [11], the industrial application of these systematic design procedures for nonlinear systems under acceptable margins of safety requires the availability of clear design rules indicating how to select a satisfactory design approach. For linear systems such design rules have been developed through the precise quantification of the system attainable performance, which aides the generation of appropriate design specifications [6, 18]. Well-known examples of these developments are the traditional Bode frequency-response methods and the modern H_∞ design techniques. These tools for performance quantification, however, are generally applied in the transformed domain (frequency response) and do not extend to nonlinear systems.

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The quantification of performance limitations in nonlinear systems has been recently studied in [19, 3] for the L_2 optimal regulation problem, and in [9] for the L_2 optimal disturbance attenuation problem. In particular, [19, 3] approach the problem of quantification of performance limitations in the time domain by considering the best attainable optimal quadratic regulation performance. One of the main conclusions in [19] is that for the class of strict-feedback nonlinear systems it is impossible to arbitrarily reduce the L_2 norm of the regulation error if the system is non-minimum phase; a result that matches the linear case [14]. Recent works applying optimal control tools to characterize performance limitations include [2, 1] for nonlinear systems, and [20, 4, 16, 5] for linear systems.

Nevertheless, the development of design tradeoffs and practical rules for general classes of nonlinear systems largely remains as an open area to research. One possible approach to this problem is to study the quantification of tradeoffs in the transient step response of the system by considering specifications such as rise time, over and undershoot, and settling time. This kind of tradeoffs and a set of practical design rules have already been developed for linear systems in [15]. This paper analyzes the feasibility of developing similar design tradeoffs for a class of nonlinear systems based on the characterization of performance limitations introduced in [19, 3]. We present results that show that, as in linear systems, for certain classes of nonminimum phase systems, the closed loop transient step response must display undershoot. A lower bound on this undershoot can be computed based on the settling time of the system, and this bound increases as the settling time is decreased, as we illustrate by considering a simple second order example. Alhough these results are preliminary, they indicate that generalizations are indeed possible.

2 Performance Limitations in Nonlinear Control Systems

2.1 Cheap Nonlinear Control

The tools developed in [19, 3] are based on the application of optimal *cheap* control techniques and singular perturbations theory [12, 17]. The idea behind optimal cheap control is that the control effort may be scaled by means of a single scalar parameter $\varepsilon > 0$ in the cost function

$$\mathscr{J} = \tfrac{1}{2}\int_0^\infty \left(\|y(t)\|^2 + \varepsilon^2\|u(t)\|^2\right) dt \tag{1}$$

to be minimized. As $\varepsilon \to 0$, the control effort is less penalized, becoming *free* in the limit when $\varepsilon = 0$. Hence, the "ideal" limiting performance attained by the optimal cheap control represents a bound that cannot be improved by any other control.

The works in [19, 3, 2, 1] show that for nonlinear systems with the *strict feedback* structure

$$\begin{aligned}
\dot{x}_0 &= f_0(x_0) + g_0(x_0)x_1\,, && x_0 \in \mathbb{R}^p\,,\\
\dot{x}_1 &= f_1(x_0,x_1) + g_1(x_0,x_1)x_2\,, && \\
&\ \vdots && x_i \in \mathbb{R}^m, i = 1,\dots,r\,,\\
\dot{x}_r &= f_r(x_0,x_1,\dots,x_r) + g_r(x_0,x_1,\dots,x_r)u\,, && u \in \mathbb{R}^m\,,
\end{aligned} \tag{2}$$

as it happens for linear systems [14, Thm. 3.14], the optimal value of the cost (1) must be positive even when the control effort is free ($\varepsilon = 0$) if the plant lacks a stable inverse. Namely, there will be a fundamental obstacle to L_2 performance if, on taking y as a performance output of (2), the system is

(i) non-minimum phase (e.g., if $y = x_1$ and the corresponding zero dynamics $\dot{x}_0 = f_0(x_0)$ is unstable), or

(ii) non-right invertible (there are more independent outputs than control inputs, e.g., $y = [x_0, x_1]^T$).

To be more precise, consider $y = x_1$ in (1), (2), and suppose that the corresponding zero dynamics $\dot{x}_0 = f_0(x_0)$ is anti-stable (the origin of $\dot{x} = -f_0(x_0)$ is globally asymptotically stable). We assume, as in [1], that for $i = 1,\dots,r$, the functions g_i are nonsingular and the functions f_i have linear growth in a sufficiently small neighborhood of the origin. It then follows from [19, 1] that the minimum attainable value for $\mathscr{J}$

$$\lim_{\varepsilon\to 0}\min_{u}\mathscr{J} = V_0(x_0), \tag{3}$$

where V_0 (whenever it exists) is the $\mathscr{C}^{r+1}$ proper and positive semidefinite solution of the (reduced order) Hamilton-Jacobi equation (HJE)

$$\frac{\partial V_0}{\partial x_0} f_0(x_0) - \frac{1}{2}\left\| g_0^T(x_0)\frac{\partial^T V_0}{\partial x_0}\right\|^2 = 0\,, \tag{4}$$

with

$$\alpha(x_0) \triangleq -g_0^T(x_0)\frac{\partial^T V_0}{\partial x_0} \tag{5}$$

such that the equilibrium of $\dot{x}_0 = f_0(x_0) + g_0(x_0)\alpha(x_0)$ is globally asymptotically stable.

Although they could be treated numerically, HJEs are in general very difficult to solve. Nevertheless, even without solving an HJE, the characterization of the limiting cheap control problem may reveal important qualitative interpretations. For example, the minimum attainable value of the regulation cost $\mathscr{J}$ (3) is the minimum energy required to stabilize the unstable zero dynamics of the system, that is, the optimal value

$$\min_y \mathscr{J}_0 = \tfrac{1}{2}\int_0^\infty \|y(t)\|\,dt \tag{6}$$

such that y, promoted to control input for the zero dynamics equation

$$\dot{x}_0 = f_0(x_0) + g_0(x_0)y,$$

achieves global asymptotic stability of the origin [19].

2.2 Time Domain Interpretation of Cheap Nonlinear Control Results

In special cases, however, it may be possible to derive quantitative time domain interpretations of the results reviewed in Section 2.1 in terms of step response parameters. Consider further the strict feedback system (2) with output $y = x_1$. Suppose for example that we wish to take the system from the origin, $x_i(0) = 0, i = 0\dots r$ to an equilibrium wherein $y(t) = \bar{y}$. Further, let us assume that there exist unique equilibrium states and inputs associated with this new equilibrium, that is, there exist unique $\bar{x}_i, i = 0\dots r$ and $\bar{u}$ such that

$$\begin{aligned}
0 &= f\left(\bar{x}_0\right) + g_0\left(\bar{x}_0\right)\bar{x}_1\\
0 &= f_1\left(\bar{x}_0,\bar{x}_1\right) + g_1\left(\bar{x}_0,\bar{x}_1\right)\bar{x}_2\\
&\vdots\\
0 &= f_r\left(\bar{x}_0,\bar{x}_1,\dots\bar{x}_r\right) + g_r\left(\bar{x}_0,\bar{x}_1,\dots\bar{x}_r\right)\bar{u}\\
\bar{y} &= \bar{x}_1
\end{aligned}$$

Then define "error coordinates" $e(t) = y(t) - \bar{y}$, $v(t) = u(t) - \bar{u}$, and $\eta_i(t) = x_i(t) - \bar{x}_i$. In these new error coordinates, the dynamic equations (2) become[1]

$$\begin{aligned}
\dot{\eta}_0 &= \bar{f}\left(\eta_0\right) + \bar{g}_0\left(\eta_0\right)\eta_1\\
\dot{\eta}_1 &= \bar{f}_1\left(\eta_0,\eta_1\right) + \bar{g}_1\left(\eta_0,\eta_1\right)\eta_2\\
&\vdots\\
\dot{\eta}_r &= \bar{f}_r\left(\bar{x}_0,\bar{x}_1,\dots\bar{x}_r\right) + \bar{g}_r\left(\bar{x}_0,\bar{x}_1,\dots\bar{x}_r\right)\bar{u}\\
e &= \eta_1
\end{aligned} \tag{7}$$

[1] Note that we have suppressed the dependence of $\bar{f}_i$ and $\bar{g}_i$ on $\bar{y}$ in these equations. For example, $\bar{f}_0\left(\eta_0\right) = f_0\left(\eta_0 + \bar{x}_0\left(\bar{y}\right)\right) - f_0\left(\bar{x}_0\left(\bar{y}\right)\right) + \left[g_0\left(\eta_0 + \bar{x}_0\left(\bar{y}\right)\right) - g_0\left(\bar{x}_0\left(\bar{y}\right)\right)\right]\bar{x}_1\left(\bar{y}\right)$ and $\bar{g}_0\left(\eta_0\right) = g_0\left(\eta_0 + \bar{x}_0\left(\bar{y}\right)\right)$.

with initial conditions $\eta_i(0) = -\bar{x}_i, \quad i = 0, 1 \ldots r$. We then have the following results, which follow as simple extensions of the results in [19, 1].

Proposition 1. *For a given $\bar{y}$, suppose that there exists a suitably continuous, proper and positive definite solution, $\bar{V}_0(\eta_0)$ to the error coordinates HJE*

$$\frac{\partial \bar{V}_0}{\partial \eta_0}\bar{f}_0(\eta_0) - \frac{1}{2}\left\| \frac{\partial \bar{V}_0}{\partial \eta_0}\bar{g}_0(\eta_0)\right\|^2 = 0 \tag{8}$$

such that the origin of $\dot{\eta}_0 = \bar{f}_0(\eta_0) - \bar{g}_0(\eta_0)g_0^T(\eta_0)\frac{\partial \bar{V}_0}{\partial \eta_0}$ is globally asymptotically stable. Then any stabilizing control which takes the system (2) from rest to the equilibrium $y = \bar{y}$ satisfies

$$\int_0^\infty \|y(t) - \bar{y}\|^2 dt \geq 2\bar{V}_0(-\bar{x}_0). \tag{9}$$

Corollary 1. *Under the conditions of Proposition 1, and the additional constraint that equilibrium is attained at $t \leq T$, that is for all $t > T$, $y(t) = \bar{y}$ then*

$$\sup_{t\in(0,T)} \|y(t) - \bar{y}\| \geq \sqrt{\frac{2\bar{V}_0(-\bar{x}_0)}{T}} \tag{10}$$

Proof. Follows immediately from (9) since $\int_T^\infty \|y(t) - \bar{y}\|^2 dt = 0$. □

Note that (10) gives a lower bound on the peak error for a given reference step, $\bar{y}$ and a given "settling" time, T. This lower bound is dictated by the unstable zero dynamics in the error equations (7). Note further that in general, this lower bound will not be tight. In the next subsection, we show how for certain classes of zero dynamics, other bounds on the transient response may be obtained.

2.3 Scalar Nonminimum Phase Zero Sub-Dynamics

Suppose the zero dynamics equation of (2) can be transformed to the equations

$$\begin{aligned} \dot{z}_0 &= F_0(z_0) + G_0(z_0)y \\ \dot{\xi}_0 &= h_0(\xi_0, z_0, y), \end{aligned} \tag{11}$$

where both z_0 and y are scalar functions of time; with zero initial conditions. We refer to the $\dot{z}_0$ equation in (11) as *sub-dynamics*, since they represent a partially decoupled set of dynamics of the overall nonlinear zero dynamics. We make use of the following assumption.

Assumption 1.

(i) The set point for tracking, $\bar{y}$ is positive.

(ii) The zero sub-dynamics, $F_0(z_0)$ lies entirely in the first and third quadrants.

(iii) The zero sub-dynamics gain matrix, $G_0(\cdot)$, is positive for all values of its argument.

(iv) There exists at least one solution, $\bar{z}_0$ to the steady state equation $F_0(\bar{z}_0) + G_0(\bar{z}_0)\bar{y} = 0$.

We then have the following results.

Proposition 2. *Suppose that for all t, $y(t) \geq -y_{us}$ and the conditions of Assumption 1 are satisfied. Let $z_-(t)$ denote the solution to*

$$\dot{z}_- = F_0(z_-) - G_0(z_-)y_{us} \tag{12}$$

with initial condition $z_-(0) = 0$. Then for all t, $z_0(t) \geq z_-(t)$.

Proof. Note that at $t = 0$ we have $z_0(0) = z_-(0) = 0$. Also note from the formulation, that for any t such that $z_0(t) = z_-(t)$ it follows from the comparison principle [10, §2.5] that $\dot{z}_0(t) \geq \dot{z}_-(t)$. ☐

Corollary 2. *Under the conditions of Assumption 1, for any suitable $\bar{y}$ and any $\alpha_{us} > 0$ define*

$$T_s(\alpha_{us}) \triangleq \int_{\bar{z}_0}^{0} \frac{dz}{\alpha_{us}\bar{y}G_0(z) - F_0(z)}. \tag{13}$$

Then at least one of the following two statements is false:

(i) For all $t \geq 0$, it holds $y(t) > -y_{us} = -\alpha_{us}\bar{y}$.

(ii) There exists $t_1 < T_s(\alpha_{us})$ such that $y(t_1) = \bar{y}$ and $z_0(t_1) = \bar{z}_0$.

Proof. Note firstly that under the conditions of Assumption 1, the denominator of the integrand in (13) is always positive, and so T_s in (13) is well defined and positive. Then consider the solution to (12) with $y_{us} = \alpha_{us}\bar{y}$. This equation defines a scalar differential equation with a negative right-hand side and therefore $z_-(t)$ is monotonically decreasing. The solution to this differential equation can be expressed by

$$t = \int_{z_-(t)}^{0} \frac{dz}{\alpha_{us}\bar{y}G_0(z) - F_0(z)}.$$

Therefore, T_s in (13) defines the first time at which z_- reaches $\bar{z}_0$ and the contradiction follows from Proposition 2. ☐

Corollary 2 therefore gives a nonlinear generalization of the time domain constraints introduced in [15]. In particular, it follows that for systems with scalar, anti-stable zero sub-dynamics which satisfy Assumption 1, any stable step response must undershoot. Furthermore, for a given step change $\bar{y}$ and permitted level of undershoot α_{us}, there is a non-trivial lower bound on the settling time permitted. This lower bound decreases with an increase in α_{us} (the permitted level of undershoot).

In the linear case, where without loss of generality we may take $G_0(z) = \frac{1}{\tau_z}$ and $F_0(z) = \frac{z}{\tau_z}$, we obtain $\bar{z}_0 = -\bar{y}$ and

$$\begin{aligned} T_s(\alpha_{us}) &= \int_{\bar{z}_0}^{0} \frac{dz}{\alpha_{us}\bar{y}G_0(z) - F_0(z)} \\ &= \tau_z \int_{-\bar{y}}^{0} \frac{dz}{\alpha_{us}\bar{y} - z} \\ &= \tau_z \ln\left(\frac{\alpha_{us}+1}{\alpha_{us}}\right) \end{aligned}$$

which is in accord with the equivalent interpretation of the results in [15].

3 Tradeoffs in the Step Response: A Simple Example

Let us consider the simple second order nonlinear system

$$\begin{aligned} \dot{x}_0 &= x_0^3 + y \\ \dot{y} &= 6x_0 + u, \end{aligned} \tag{14}$$

where $y \in \mathbb{R}$ is the output, $x_0 \in \mathbb{R}$ is the zero dynamics state, and $u \in \mathbb{R}$ is the system control input. When the output is identically zero, $y(t) \equiv 0$, it is seen that the zero dynamics $\dot{x}_0 = x_0^3$ is *anti-stable* (that is, $\dot{x}_0 = -x_0^3$ is asymptotically stable), which means that the system is (strictly) non-minimum phase in the sense of Isidori [8].

Suppose that the output $y(t)$ is specified to asymptotically track a constant reference of amplitude $\bar{y} > 0$ starting from zero initial states $y(0) = 0 = x_0(0)$. Then, from (14), this specification requires that

$$\bar{y} \triangleq \lim_{t\to\infty} y(t), \quad \bar{x}_0 \triangleq \lim_{t\to\infty} x_0(t) = -\sqrt[3]{\bar{y}}, \quad \bar{u} \triangleq \lim_{t\to\infty} u(t) = 6\sqrt[3]{\bar{y}}.$$

Qualitative properties of the transient response of system (14) can be obtained from a steady-state analysis. For example, it can be concluded that, because the zero dynamics is anti-stable, the response in $y(t)$ *must display undershoot*, as is the case for non-minimum phase linear systems [15]. This fact is easily seen by analyzing the zero dynamics equation $\dot{x}_0 = x_0^3 + y$: the assumption that $y(t) \geq 0$ for all $t \geq 0$ implies that $y(t)$ cannot drive $x_0(t)$ from $x_0(0) = 0$ to its required asymptotic value $\bar{x}_0 = -\sqrt[3]{\bar{y}} < 0$. Thus, $y(t)$ must be negative at some t, which implies that $y(t)$ must undershoot before reaching its asymptotic value $\bar{y}$.

Quantitative information about the system transient response, such as bounds on the undershoot or settling time, can be obtained by making a change of variables as in Section 2.2, that is,

$$e(t) \triangleq y(t) - \bar{y}, \quad \eta(t) \triangleq x_0(t) - \bar{x}_0, \quad \nu(t) \triangleq u(t) - \bar{u},$$

which takes the system (14) to

$$\begin{aligned} \dot{\eta} &= \eta^3 - 3\sqrt[3]{\bar{y}}\,\eta^2 + 3\sqrt[3]{\bar{y}^2}\,\eta + e \\ \dot{e} &= 6\eta + \nu. \end{aligned} \tag{15}$$

Now the original tracking problem is set in (15) as the asymptotic regulation problem in which the variables $e(t)$ (the tracking error) and $\eta(t)$ must be driven to zero from the initial states $e(0) = -\bar{y}$ and $\eta(0) = \sqrt[3]{\bar{y}}$.

System (15) has a strict-feedback structure and is in Isidori's *normal form* [8]. Because its zero dynamics is anti-stable, system (15) is also non-minimum phase. Then, it follows from the results in [19] and [3] that there exists a positive lower bound on the L_2 norm of $e(t)$ *independently of the control applied to achieve the regulation*. This lower bound can be computed as the solution to (4), (5), which from the $\dot{\eta}$ equation in (15) gives the proper and positive definite function

$$\begin{aligned} V_0(\eta) &= \begin{bmatrix} \eta^2 \\ \eta \end{bmatrix}^T \begin{bmatrix} 1/2 & -\sqrt[3]{\bar{y}} \\ -\sqrt[3]{\bar{y}} & 3\sqrt[3]{\bar{y}^2} \end{bmatrix} \begin{bmatrix} \eta^2 \\ \eta \end{bmatrix} \\ &= \tfrac{1}{2}\eta^4 - 2\sqrt[3]{\bar{y}}\,\eta^3 + 3\sqrt[3]{\bar{y}^2}\,\eta^2. \end{aligned}$$

Thus, from (3) we obtain that the regulation error must satisfy

$$\tfrac{1}{2}\int_0^\infty |e(t)|^2 dt \geq V_0(\sqrt[3]{\bar{y}}) = \tfrac{3}{2}\sqrt[3]{\bar{y}^4}. \tag{16}$$

The bound (16) on the L_2 norm of the tracking error holds for any control that achieves the system specified regulation. In particular, the lowest value $V_0(\sqrt[3]{\bar{y}})$ can be arbitrarily closely approximated by using the *near optimal cheap* control law [19] (in the original x_0, y coordinates)

$$u(t) = 6\sqrt[3]{\bar{y}} - \frac{y(t) + 2x_0^3(t) + \bar{y}}{\varepsilon} \quad \text{as } \varepsilon \to 0, \text{ with } \varepsilon > 0. \tag{17}$$

Alternatively, faster tracking may be achieved at the expense of a larger L_2 norm in the tracking error by using the control law

$$u(t) = 6\sqrt[3]{\bar{y}} - \frac{y(t) + 2kx_0^3(t) + (2k-1)\bar{y}}{\varepsilon} \quad \text{for any } k > 1. \tag{18}$$

Figure 1 shows the closed-loop responses of the system (14) to a unitary step change in the reference, $\bar{y} = 1$, using the control law (18) with $\varepsilon = 10^{-3}$ and $k = 1, 1.5$ and 2. The response for $k = 1$, which yields the near optimal cheap control (17), has the lowest L_2 norm of the tracking error, but is also the slowest, with a settling time $T \approx 2.5$. Note that, as predicted, there is undershoot in all three responses. Note also the tradeoff

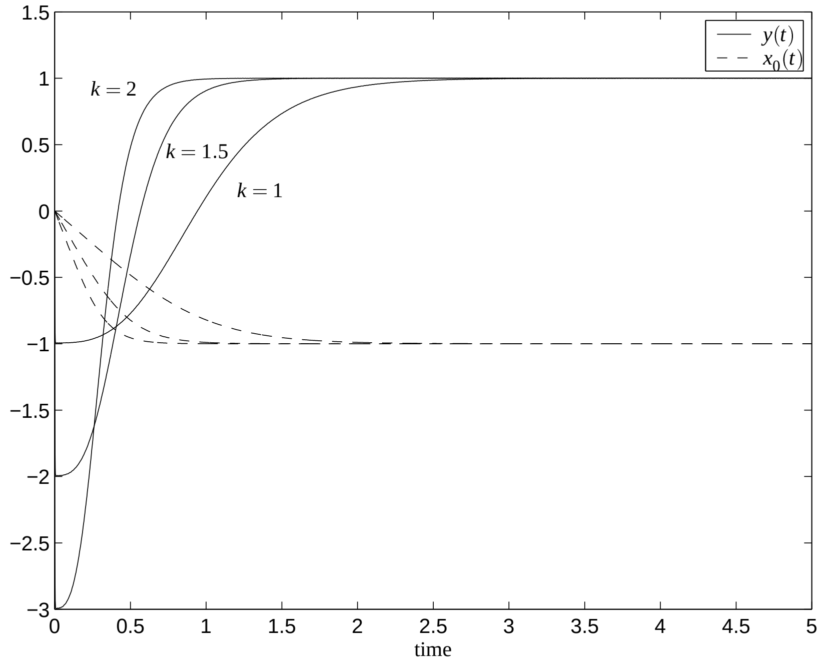

Figure 1: Tracking responses of the system (14) using $u(t)$ from (18) with $k = 1, 1.5$ and 2.

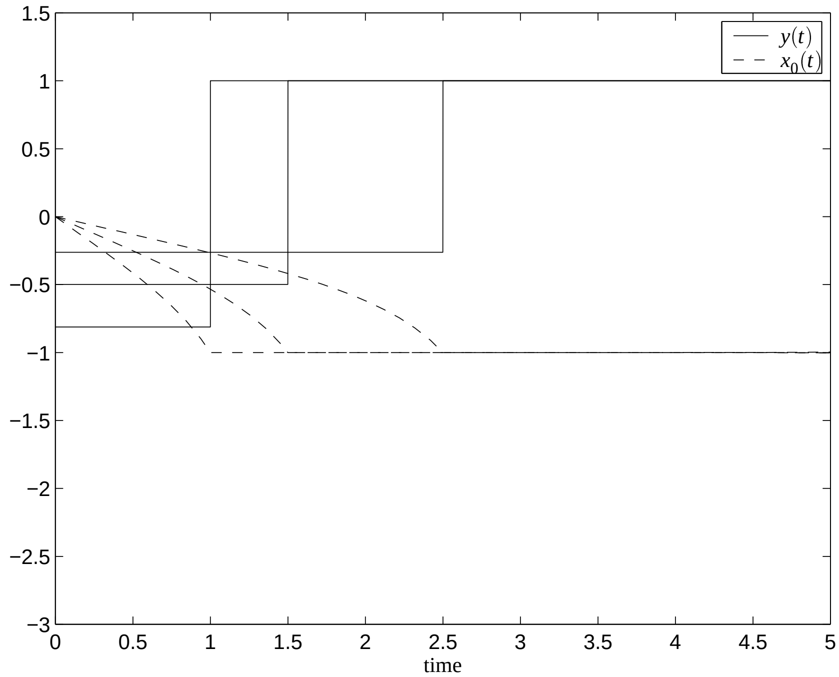

Figure 2: Responses of $\dot{x}_0 = x_0^3 + y$ with minimum undershoot in $y(t)$.

between settling time and undershoot, the magnitude of which increases as the response is made faster. This design tradeoff seems inescapable, as the following argument indicates.

Suppose that $y(t)$ settles at the finite time $T > 0$. Then it follows from Corollary 1 that

$$3\sqrt[3]{\bar{y}^4} \leq \int_0^T |e(t)|^2 dt \leq \sup_{t\in[0,T)} |e(t)|^2 T \quad \Rightarrow \quad \sup_{t\in[0,T)} |e(t)| \geq \frac{\sqrt{3}}{\sqrt{T}} \sqrt[3]{\bar{y}^2}. \tag{19}$$

The inequality on the right of (19) shows that the peaks in the tracking error must increase as the settling time is made shorter, which is confirmed by the magnitude of the undershoot in the responses of Figure 1. Strictly, the bound (19) is on the *peak deviation* in the error signal rather than on the undershoot. It may be shown, however, that the control law (18), based on the near optimal cheap control law, cannot produce overshoot ($e(t) > 0$) in this system. Indeed, the high gain action of the near optimal cheap control initially drives $e(t)$ in a fast motion towards the *slow invariant manifold*

$$e = \alpha(\eta) \triangleq -2\eta^3 + 6\sqrt[3]{\bar{y}}\eta^2 - 6\sqrt[3]{\bar{y}^2}\eta,$$

to then make it slide along it in a slow motion towards the origin [19]. Thus, the initial fast transition of $e(t)$ from $e(0) = -\bar{y}$ to $\alpha(\eta(0)) = -2\bar{y}$ produces the undershoot in $y(t)$; since the slow invariant manifold is monotonic in η, the error $e(t)$ cannot subsequently change sign as it drifts towards the origin along the manifold.

On dividing both sides of the inequality on the right of (19) by $\bar{y}$,

$$\frac{\sup_t |e(t)|}{\bar{y}} \geq \frac{\sqrt{3}}{\sqrt{T}\sqrt[3]{\bar{y}}},$$

we see that for larger values of $\bar{y}$ the relative peak in $e(t)$ gets smaller, and hence, the penalty imposed by the unstable zero dynamics relaxes. This observation is consistent with the known relaxation of constraints in linear systems, for which fast non-minimum phase zeros impose less constraints than slow ones [6]. Indeed, on writing the zero dynamics equation as

$$\dot{x}_0 = (x_0^2)x_0 + y, \tag{20}$$

system (14) may be assimilated to a "linear" one with a state-dependent non-minimum phase "zero". A larger step change $\bar{y}$ implies relatively larger values of $x_0(t)$, that is, "faster" zero dynamics in (20).

Lower values of undershoots could be obtained using a control law that specifically accounts for an L_∞ optimization criterion. For example, the responses in Figure 2 show minimum undershoot solutions, which have been numerically computed from $\dot{x}_0 = x_0^3 + y$ to satisfy the tracking specifications. Table 1 compares the magnitudes of undershoot in these responses with those in Figure 1 and the lower bounds on the peak tracking error computed from (19). Note that the bounds given by (19) are conservative ("optimistic") for the undershoots actually achieved by the L_2 norm-based solutions of Figure 1, but are, as should be expected, much more tight with respect to the L_∞ norm-based solutions of Figure 2.

T	Lower bound from (19) $\sqrt{3/T}$	Actual undershoot $\sup -e(t)$: Using $u(t)$ from (18)	Actual undershoot $\sup -e(t)$: With minimum undershoot in $y(t)$
2.5	1.0954	2	1.2617
1.5	1.4142	3	1.4985
1	1.7320	4	1.8112

Table 1: Peak tracking error versus settling time for system (14).

Since this system naturally falls in the class described in Section 2.3, we may also quantify the undershoot/settling-time tradeoff using Corollary 2. Figure 3 shows the relation between settling-time T_s and undershoot permitted

α_{us} for different values of the reference $\bar{y}$, obtained from the explicit solution of (13). We see how the settling time increases as we reduce the permitted level of undershoot. These bounds hold for *any* controller achieving the regulation specifications, and are "optimistic", in the sense that they are *lower bounds* on the actual settling time that would be obtained with a specific controller.

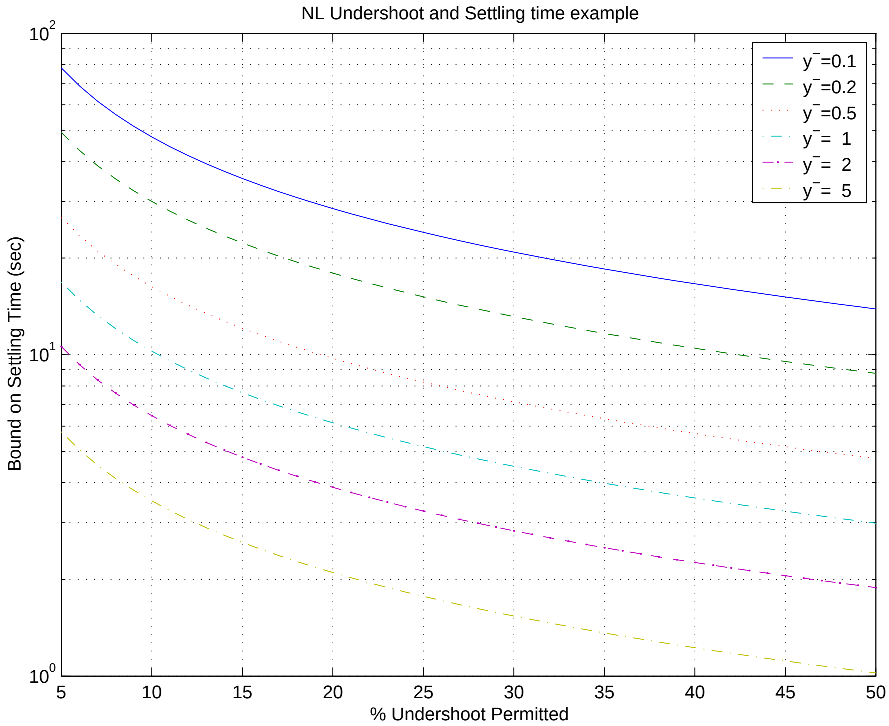

Figure 3: Undershoot and settling time

4 Conclusions

This paper studied the feasibility of quantifying design tradeoffs in the step response of nonlinear systems. Building on the recent characterizations of bounds on the achievable L_2 optimal performance for strict feedback nonlinear systems, we have presented results that show that, as in linear systems, for certain classes of nonminimum phase systems, the closed loop transient step response must display undershoot. The results and the simple example considered show how a bound on attainable L_2 optimal performance can be translated into qualitative and quantitative properties of the transient response of the system. Since these properties hold irrespective of the particular control design applied, they may be used to compare different approaches or to make an appropriate choice of parameters in a given design. We expect to extend these tools to more general classes of nonlinear control systems in follow-up studies.

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