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Quantized stabilization for stochastic discrete-time systems with multiplicative noises

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SUMMARY

This paper considers the problem of quadratic mean-square stabilization of a class of stochastic linear systems using quantized state feedback. Different from the previous works where the system is restricted to be deterministic, we focus on stochastic systems with multiplicative noises in both the system matrix and the control input. A static quantizer is used in the feedback channel. It is shown that the coarsest quantization density that permits stabilization of a stochastic system with multiplicative noises in the sense of quadratic mean-square stability is achieved with the use of a logarithmic quantizer, and the coarsest quantization density is determined by an algebraic Riccati equation, which is also the solution to a special stochastic linear control problem. Our work is then extended to exponential quadratic mean-square stabilization of the same class of stochastic systems. Copyright © 2011 John Wiley & Sons, Ltd.

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KEY WORDS: quantized feedback control; mean-square stability; stochastic system; multiplicative noises

1. INTRODUCTION

With the rapid advancement of digital networks, the problems of network time-delay, packet dropout, and quantization error (caused by limited data rate) arise for network-based feedback control systems. These problems can significantly deteriorate the performance of a control system, thus appropriate measures need to be taken in designing network-based control systems. Feedback control using quantized information can be traced back to 1960s (e.g., [1–5]). The development of modern network control systems has brought a resurgent interest in quantized feedback control. Recent works on quantized feedback control include [6–9].

The research on quantized feedback control can be categorized depending on whether the quantizer is static or dynamic. A static quantizer is a memoryless nonlinear function, whereas a dynamic quantizer involves memory and thus can be more complex but more powerful. Most of the research about static quantizers use either uniform or logarithmic quantization. A uniform quantizer can minimize the information loss, especially when the input signal falls into the dynamic range of the quantizer with a uniform distribution, and the number of quantization levels required for a given quantization step-size increases linearly as the dynamic range increases. A uniform quantizer is used in [10] to stabilize a linear time-invariant control system; [11] investigates the quantized H_∞ control problem for discrete-time systems with random packet losses; [12] studies the asymptotic characteristics of uniform scalar quantizers that are optimal with respect to some mean-squared error. But [13] shows that the uniform quantization approach is inappropriate when the quantization resolution is coarse or when the open-loop system is unstable.

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In [14], it has been proved that there exists a minimum data rate for a dynamic quantizer that below which, there does not exist any quantized feedback controller that can stabilize an unstable system. For a deterministic linear time-invariant system, it is revealed in [6] that the coarsest quantization density for quantized feedback stabilization of a linear time-invariant system using a static quantizer is achieved with the use of a logarithmic quantizer, and its quantization density is related to the unstable roots of the system matrix. In [7], it is shown that the quantization error of a logarithmic quantizer can be modeled as a norm-bounded multiplicative noise, and the problem of quantized feedback control can be transformed into a classical robust control problem. As for the stochastic systems, [14] considers the system with additive noises under the assumption that the noises are bounded. A logarithmic quantizer with finite levels to guarantee the practical stability for the closed-loop system has been designed. Quantized feedback with packet dropouts is considered in [8], and it is shown that the coarsest quantization density for a system with quantized feedback control subject to the Bernoulli packet dropout model is related to the packet dropout rate and the unstable roots of the system matrix. Reference [9] considers the minimum data rate for mean-square stability of linear systems over a lossy channel that is modeled as a time-homogenous binary Markov process. The minimum data rate for a scalar system is given in terms of the magnitude of the unstable roots and the transition probabilities of the Markov process, and necessary and sufficient conditions are provided for scalar systems. For stabilization of stochastic systems with multiplicative noises, [15] uses a generalized Lyapunov equation approach to give some testable criteria for stochastic stabilization. Under a different class of criteria, [16] uses a linear matrix inequality approach to illustrate how to design an almost surely stabilizing controller for a stochastic system that is otherwise unable to be stabilized in the mean-square sense. As for the stochastic systems with additive noises, [17] gives an explicit expression for the minimum data rate by using the entropy power inequality of information theory and a new quantization error bound.

In this paper, we consider the problem of finding the coarsest quantization density required for quadratic mean-square stabilization of a class of stochastic systems with multiplicative noises. The coarsest quantizer is proved to be logarithmic with countable levels. By finding the optimal quadratic Lyapunov function that allows for the coarsest logarithmic quantizer, it is shown that the solvability of the coarsest quantization density is related to a special stochastic linear control problem. The exact solution to the coarsest quantization density is given in terms of a special Riccati equation, and an approximate numerical solution is given in terms of a linear matrix inequality. The coarsest quantizer for exponential quadratic mean-square stability is discussed in this paper.

This paper is organized as follows: Section 2 formulates the quantized state feedback control problem; Section 3 presents the solution of the problem; Section 4 generalizes the previously mentioned results to exponential mean-square stabilization; and Section 5 draws conclusions.

2. PROBLEM FORMULATION

Consider the following discrete-time stochastic system with multiplicative noise:

$$x(t+1) = Ax(t) + Bu(t) + [A_0 x(t) + B_0 u(t)]\, w(t), x(0) = x_0, \tag{1}$$

where $x(t) \in R^n$ and $u(t) \in R$ are, respectively, the system state and control input, $w(t) \in R$ is a white noise with zero mean and variance σ^2 and is uncorrelated with initial state $x(0)$. It is assumed that (A, A_0, B, B_0) is stabilizable and (A, A_0) is observable. The stabilization assumption means that there exists a constant matrix K such that the control law $u(t) = Kx(t)$ renders the following closed-loop system

$$x(t+1) = (A + BK)x(t) + (A_0 + B_0 K)x(t)w(t),\ x(0) = x_0, \tag{2}$$

asymptotically mean-square stable, that is, $\lim_{t\to\infty} E[||x(t)||^2] = 0$ for any initial state x_0.

We will denote

$$V_P(x(t)) \triangleq E\left[x^T(t)Px(t)\right],\ P = P^T > 0, \tag{3}$$

$$\nabla V_P(x(t)) \triangleq V_P(x(t+1)) - V_P(x(t)). \tag{4}$$

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The dependence on time t will be dropped when there is no confusion. We will call $V_P(x)$ a control Lyapunov function.

Definition 1

The discrete-time stochastic system (1) is said to be quadratically mean-square stabilizable if there exists a state feedback control law $u(t) = U(x(t))$ such that for any initial state $x_0 \in R^n$, the closed-loop system

$$x(t+1) = Ax(t) + BU(x(t)) + [A_0 x(t) + B_0 U(x(t))]w(t),\ x(0) = x_0 \tag{5}$$

satisfies

$$\nabla V_P(x(t)) < 0,\ \ \forall\ t \geqslant 0. \tag{6}$$

The system (1) is said to be exponentially quadratically mean-square stabilizable with convergence rate $0 < \alpha < 1$ if (6) is replaced with

$$V_P(x(t+1)) < \alpha V_P(x(t)),\ \ \forall\ t \geqslant 0. \tag{7}$$

We first adopt some notation as defined in [6]: $\mathcal{Q}(V_P)$ denotes the set of all the symmetric quantizers that solve Problem 1 for a given control Lyapunov function $V_P(x)$. For $f \in \mathcal{Q}(V_P)$ and $0 < \varepsilon < 1$, denote $\sharp f[\varepsilon]$ the number of quantization levels in the interval $[\varepsilon, 1/\varepsilon]$. Then, the quantization density of f is defined as

$$\eta_f = \limsup_{\varepsilon \to 0} \frac{\sharp f[\varepsilon]}{\ln(1/\varepsilon)}. \tag{8}$$

A quantizer $f^\star \in \mathcal{Q}(V_P)$ is said to be coarsest for a given $V_P(x)$ if it has the least quantization density, that is,

$$f^\star = \arg \inf_{f \in \mathcal{Q}(V_P)} \eta_f. \tag{9}$$

Note that, because of the infimum action mentioned earlier, the coarsest quantization density may not be reachable, that is, $f^\star$ may belong to the closure of $\mathcal{Q}(V_P)$ instead of being in the interior of $\mathcal{Q}(V_P)$.

A logarithmic quantizer is expressed as follows:

$$Q(u) = \begin{cases} u_i, & if\ \frac{1}{1+\Delta}u_i < u \leqslant \frac{1}{1-\Delta}u_i,\ u > 0 \\ 0, & if\ u = 0 \\ -Q(-u), & if\ u < 0 \end{cases} \tag{10}$$

with quantization levels as

$$U = \left\{\pm u_i : u_i = \rho^i u_0, i = 1, 2, \cdots\right\} \bigcup \{\pm u_0\} \bigcup \{0\}, with\ 0 < \rho < 1, u_0 > 0, \tag{11}$$

where ρ is called quantized density of the quantizer and u is the signal that has to be quantized. It is easy to show that $\eta_f = \frac{2}{\ln(1/\rho)}$. Because η_f is a monotonic function of ρ (i.e., a smaller ρ corresponding to a coarser quantizer density), we will abuse the notation a bit and also refer ρ as the quantization density for a logarithmic quantizer.

Problem 1: The first (main) quantized stabilization problem to be considered in the paper aims to find a control law $U(x)$ called quantized state feedback control law to achieve quadratic mean-square stabilization. In particular, the control signal involves a linear or nonlinear state feedback control law u and a quantized version of it with a symmetric quantizer $f(u)$ (i.e., $f(-u) = -f(u)$) and a finite or countable number of quantization levels. Our goal is to find a coarsest quantization density that permits quadratic mean-square stabilization.

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3. SOLUTION TO QUADRATIC MEAN-SQUARE STABILIZATION

In order to find the coarsest quantization density for system (1), we will first derive all the quantized feedback controllers that render the closed-loop system of (1) quadratically mean-square stable for a given P. The following lemma characterizes all such controllers.

Lemma 1
Let $V_P(x) = E[x^T P x]$ for some $P > 0$ be a control Lyapunov function for the system (1). A necessary condition for the system (1) to be quadratically mean-square stabilizable via the given control Lyapunov function is that

$$Q = P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0} > 0. \quad (12)$$

Under the assumption that (12) holds, then for any $x \neq 0$, the set of all the u that yield $\nabla V_P(x) < 0$ along the trajectory of (1) is characterized by

$$U(x) = \{u \in R \mid u_1(x) < u < u_2(x)\}, \quad (13)$$

where

$$u_1(x) = K_{GD}x(t) - \sqrt{\frac{x^T(t)Qx(t)}{B^T PB + \sigma^2 B_0^T PB_0}}, u_2(x) = K_{GD}x(t) + \sqrt{\frac{x^T(t)Qx(t)}{B^T PB + \sigma^2 B_0^T PB_0}} \quad (14)$$

with

$$K_{GD} = -\frac{B^T PA + \sigma^2 B_0^T PA_0}{B^T PB + \sigma^2 B_0^T PB_0}. \quad (15)$$

Proof
It is direct to verify that

$$\nabla V_P(x) = u(t)\left(B^T PB + \sigma^2 B_0^T PB_0\right)u(t) + 2x^T(t)\left(A^T PB + \sigma^2 A_0^T PB_0\right)u(t) + x^T(t)\left(A^T PA + \sigma^2 A_0^T PA_0 - P\right)x(t). \quad (16)$$

Note that $\Pi = B^T PB + \sigma^2 B_0^T PB_0 > 0$. By completing the squares, (16) can be rewritten as

$$\nabla V_P(x) = \left(u - \Pi^{-1}\left(A^T PB + \sigma^2 A_0^T PB_0\right)\right)^T \Pi \left(u - \Pi^{-1}\left(A^T PB + \sigma^2 A_0^T PB_0\right)\right) - x^T Qx, \quad (17)$$

where Q is given in (12). Thus, it is necessary that $Q > 0$ in order to achieve quadratic mean-square stabilization using the given control Lyapunov function. Now, assuming $Q > 0$, setting $\nabla V_P(x) = 0$ in (17) leads to the two boundary points $u_1(x)$ and $u_2(x)$ as in (14) and the admissible control set as in (13). □

It is easy to check that the admissible control set (13) has the following properties:

- P1: $U(\alpha x) = \alpha U(x)$ for $\alpha > 0$;
- P2: $u_1(x) = -u_2(x)$ if $K_{GD}x = 0$.

From P2, we know that when $x \perp K_{GD}^T$, $u = 0$ can be used to ensure that the Lyapunov function decreases along the state trajectory. Thus, it suffices to quantize the state x in the direction of K_{GD}. In fact, consider the subspace spanned by

$$Y_{GD} = \left\{x \in R^n : x = y\frac{K_{GD}^T}{K_{GD}K_{GD}^T}, y \in R\right\}. \quad (18)$$

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Without sacrificing the quantization density, we can restrict the quantizer to be on Y_{GD} by taking

$$f^{GD}(x) = h(K_{GD}x),$$

where $h(\cdot)$ is a scalar quantizer. In view of P2, it follows that $f^{GD} \in \mathcal{Q}(V_P)$. Thus, we have

$$\inf_{f \in \mathcal{Q}(V_P)} \eta_f = \inf_{f^{GD} \in \mathcal{Q}(V_P)} \eta_{f^{GD}}. \tag{19}$$

The aforementioned analysis leads to the following result.

Theorem 1
Given a control Lyapunov function $V_P(x)$ and suppose (12) holds, then the coarsest quantizer for quadratic mean-square stabilization of the system (1) using static quantizer and the given control Lyapunov function are logarithmic, and the corresponding quantization density is determined by

$$\rho = \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1}, \tag{20}$$

where

$$\Delta = \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T Q^{-1} \left(A^T PB + \sigma^2 A_0^T PB_0\right)}{B^T PB + \sigma^2 B_0^T PB_0}. \tag{21}$$

Proof
Let

$$z(t) = Q^{\frac{1}{2}} x(t).$$

Then the boundary points of the control set as given in Lemma 1 are changed to

$$u_1(x) = K_{GD} Q^{-\frac{1}{2}} z(t) - \sqrt{\frac{z^T(t) z(t)}{B^T PB + \sigma^2 B_0^T PB_0}},$$

$$u_2(x) = K_{GD} Q^{-\frac{1}{2}} z(t) + \sqrt{\frac{z^T(t) z(t)}{B^T PB + \sigma^2 B_0^T PB_0}}.$$

Decompose $z(t)$ into the following form:

$$z(t) = Q^{-\frac{1}{2}} \left(A^T PB + \sigma^2 A_0^T PB_0\right) \alpha + w\beta,$$

for some $\alpha \in R$ and $\beta \in R^{n-1}$, where $w \in R^{n \times n-1}$, $w \perp Q^{-\frac{1}{2}} \left(A^T PB + \sigma^2 A_0^T PB_0\right)$ with $w^T w = \mathrm{I}$. With some abuse of notation, we can rewrite $u_1(x)$ and $u_2(x)$ as $u_1(\alpha, \beta)$ and $u_2(\alpha, \beta)$, respectively, which are given by

$$u_1(\alpha, \beta) = \Delta\alpha - \sqrt{\alpha^2 \Delta + \frac{\|\beta\|^2}{B^T PB + \sigma^2 B_0^T PB_0}}, \tag{22}$$

$$u_2(\alpha, \beta) = \Delta\alpha + \sqrt{\alpha^2 \Delta + \frac{\|\beta\|^2}{B^T PB + \sigma^2 B_0^T PB_0}}. \tag{23}$$

Denoting the set $U(x) = U(\alpha, \beta) = \{u_1(\alpha, \beta) < u < u_2(\alpha, \beta)\}$, then $U(\alpha, \beta)$ is a minimal set when $\beta = 0$, which means that the worst direction of quantization that requires highest quantization density is along the direction that parallels to $Q^{-\frac{1}{2}} K_{GD}^T$.

In the next part, we show the coarsest covering in the direction of $Q^{-\frac{1}{2}} K_{GD}^T$ that follows a logarithmic law. To this end, we set $\beta = 0$. It is easy to verify that $K_{GD} x = \Delta\alpha$ in this case. Let

$$u^{(0)} = \Delta - \sqrt{\Delta}. \tag{24}$$

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Figure 1. The optimal quantized control of $K_{GD}x(t)$.

Then the set of $K_{GD}x$ (or $\Delta\alpha$) such that $\nabla V_P(x) \leqslant 0$ with $u = u_0$ is given by

$$U^{(0)} = \left\{\Delta\alpha : \left(\Delta - \sqrt{\Delta}\right)\alpha \leqslant u^{(0)} \leqslant \left(\Delta + \sqrt{\Delta}\right)\alpha\right\}.$$

Changing the previously mentioned inequalities to equalities and solving the corresponding values for α lead to $U^{(0)} = \{\rho\Delta, \Delta\}$. For $K_{GD}x < \rho\Delta$, a smaller value of u needs to be used. By Property P1, it is clear that $u^{(1)} = \rho u^{(0)}$ guarantees the non-increasing of $V_P(x)$ for all the $K_{GD}x \in U^{(1)}$, where $U^{(1)} = \left\{\rho^2\Delta, \rho\Delta\right\}$. The same argument applies for all $u^{(i)} = \rho^i u^{(0)}$ and $U^{(i)} = \left\{\rho^{i+1}\Delta, \rho^i\Delta\right\}$ $i = \pm 1, \pm 2, \ldots$. It is clear from the aforementioned analysis that this partition of $KGD(x)$ leads to a logarithmic quantizer and this quantizer gives the coarsest quantization density. The following diagram gives a visual description of the quantizer (see Figure 1). □

Remark 1

When the covariance σ^2 of the noise $w(t)$ is 0, which means that the noise $w(t) \equiv 0$, the quantization density is the same as in the deterministic case in [6]. When considering the system with only multiplicative noises in the input channel (i.e., $A_0 = 0$) for the same P, it can be shown that Δ is larger in the stochastic system than that in the deterministic case. Thus, the coarsest quantization density for a stochastic system with multiplicative noises is larger than that for a deterministic system. This implies that a finer quantization density is needed to tradeoff the destabilizing effect caused by the noises.

In what follows, we want to characterize the coarsest quantizer by searching over all quadratic stochastic control Lyapunov functions and derive the optimal $P > 0$ such that $V_P(x) = Ex^T Px$ is a Lyapunov function and the corresponding quantization density ρ is minimized. We show that the solvability of the optimal quantizer density is related to solving a special stochastic linear quadratic regulator (LQR) problem.

Theorem 2

The optimal P corresponding to the coarsest quantization density is given by the unique semi-positive-definite solution of the following algebraic Riccati equation:

$$P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0 + 1} = 0, \quad (25)$$

the coarsest quantization density is given by

$$\rho^* = \frac{\gamma^* - 1}{\gamma^* + 1} \quad (26)$$

with $(\gamma^*)^2 = B^T PB + \sigma^2 B_0^T PB_0 + 1$ and P solving (25). Furthermore, The algebraic Riccati equation (25) is also the solution to the stochastic LQR problem of min $J(x_0, u)$ with

$$J(x_0, u) = \sum_{t=0}^{\infty} E\left[u^2(t)\right], \quad (27)$$

which corresponds to the minimum energy control that quadratically mean-square stabilizes the system (1) (without quantization).

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Proof
Denote $\gamma = \sqrt{\Delta}$. Note that $\gamma > 1$ is required from (20). From Theorem 1, it is clear that the coarsest quantization density can be achieved by varying $P > 0$ such that γ is minimized, that is, we need to solve

$$\gamma^* = \inf_{P>0} \gamma \tag{28}$$

subject to $Q > 0$ and $\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T Q^{-1} \left(A^T PB + \sigma^2 A_0^T PB_0\right) \leqslant \gamma^2$. The following implications follow immediately:

$$\begin{aligned}
&\frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T Q^{-1} \left(A^T PB + \sigma^2 A_0^T PB_0\right)}{B^T PB + \sigma^2 B_0^T PB_0} \leqslant \gamma^2 \\
&\Longleftrightarrow \text{trace}\left\{ \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T Q^{-1} \left(A^T PB + \sigma^2 A_0^T PB_0\right)}{B^T PB + \sigma^2 B_0^T PB_0} \right\} \\
&\quad = \text{trace}\left\{ Q^{-1/2} \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0} Q^{-1/2} \right\} \leqslant \gamma^2 \\
&\Longleftrightarrow \lambda_{max}\left(Q^{-1/2} \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0} Q^{-1/2} \right) \leqslant \gamma^2 \\
&\Longleftrightarrow Q^{-1/2} \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0} Q^{-1/2} \leqslant \gamma^2 I \\
&\Longleftrightarrow \left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(B^T PB + \sigma^2 B_0^T PB_0\right)^{-1}\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T \leqslant \gamma^2 Q \\
&\Longleftrightarrow \left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(B^T PB + \sigma^2 B_0^T PB_0\right)^{-1}\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T \\
&\quad \leqslant \gamma^2 \left(P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0} \right) \\
&\Longleftrightarrow P - A^T PA - \sigma^2 A_0^T PA_0 + \left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(\frac{\gamma^2\left(B^T PB + \sigma^2 B_0^T PB_0\right)}{\gamma^2 - 1} \right)^{-1} \\
&\quad \times \left(A^T PB + \sigma^2 A_0^T PB_0\right)^T \geqslant 0.
\end{aligned} \tag{29}$$

Define

$$\beta = \frac{B^T PB + \sigma^2 B_0^T PB_0}{\gamma^2 - 1}. \tag{30}$$

Note that $\beta > 0$ because $\gamma > 1$. Equation (29) becomes

$$P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0 + \beta} \geqslant 0. \tag{31}$$

Because the previously mentioned inequality is not affected by positive scaling of P, we can assume that $\beta = 1$ without loss of generality. It is known from [18] that the minimum P satisfying (31) is obtained by replacing the inequality sign with equality, which is (25), and that the corresponding solution P becomes semi-positive-definite. It is clear from (29) that when equality is achieved, γ is minimized. The minimum γ can be recovered by using (30) with $\beta = 1$, that is,

$$(\gamma^*)^2 = 1 + B^T PB + \sigma^2 B_0^T PB_0.$$

The connection between (25) and the stochastic LQR problem (27) comes from [19] which shows that the solution to the stochastic LQR optimal control problem with the following cost function

$$J(x_0, u) = \sum_{t=0}^{\infty} E\left[x^T(t)\Omega x(t) + u^2(t)\right], \ \Omega > 0 \tag{32}$$

is given by algebraic Riccati equation as follows:

$$\Omega = P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{1 + B^T PB + \sigma^2 B_0^T PB_0}, \tag{33}$$

under the assumptions that (A, A_0, B, B_0) is stabilizable and $(A, A_0/\Omega^{1/2})$ is exactly observable. More specifically, (33) is guaranteed to have a unique solution $P > 0$. The optimal controller is uniquely determined by

$$u^*(t) = Kx(t) = -\left(1 + B^T PB + \sigma^2 B_0^T PB_0\right)^{-1}\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T x(t),$$

and the optimal cost is given by

$$\min J(x_0, u) = E\left[x^T(0)Px(0)\right].$$

Note that the stabilizability and observability assumptions are satisfied if we take $\Omega = \varepsilon I$ for $\varepsilon > 0$. The system is mean-square stabilized, so $Ex^T(t)x(t) \to 0$ as $t \to 0$, and so $Ex^T(t)x(t)$ is finite over all the time t. Now, taking the limit that $\varepsilon \to 0$, (33) becomes (25) and (32) becomes (27). □

Remark 2
Equation (25) is not a standard algebraic Riccati equation (because of the nonzero σ). We thus need to comment on its solution. Indeed, (25) can be transferred into solving

$$\max \text{ trace(P)}$$

subject to

$$\begin{bmatrix} P - A^T PA - \sigma^2 A_0^T PA_0 - \varepsilon I & A^T PB + \sigma^2 A_0^T PB_0 \\ \left(A^T PB + \sigma^2 A_0^T PB_0\right)^T & -\left(B^T PB + \sigma^2 B_0 PB_0 + 1\right) \end{bmatrix} < 0, \tag{34}$$

for some sufficiently small $\varepsilon > 0$. The inequality (34) is motivated by the fact that (25) can be rewritten as

$$P - A^T PA - \sigma^2 A_0^T PA_0 - \varepsilon I + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{1 + B^T PB + \sigma^2 B_0^T PB_0} = -\varepsilon I < 0$$

which is converted into the inequality (34) by using the well-known Schur complement method. The slack introduced by ε is negligible when ε is sufficiently small. Note that the computation of a similar algebraic Riccati equation for the continuous-time case is considered in [20]. The proof in [20] can be easily extended to prove that the solution to (34) overbounds that of (25), but the two solutions' coverage to each other was $\varepsilon \to 0$. As the quantization density is monotone, increasing with the trace of P [21], the quantization density deduced from the solution of (34) is larger than the coarsest quantization density. ■

4. SOLUTION TO EXPONENTIAL QUADRATIC MEAN-SQUARE STABILIZATION

This section serves as an extension to the results in the previous section by considering the exponential quadratic mean-square stabilization problem. The problem setting is the same as before, except that a convergence rate α is required for some $\alpha > 0$. The main result is given as follows:

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Theorem 3

The system (1) is exponentially quadratically mean-square stabilizable with convergence rate $0 < \alpha < 1$ by using a quantized state feedback control law if and only if the following algebraic Riccati inequality

$$\alpha P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0 + 1} > 0 \tag{35}$$

has a positive solution $P > 0$. The corresponding quantizer is a logarithmic one with quantization density given by

$$\rho = \frac{\sqrt{\Delta} - 1}{\sqrt{\Delta} + 1} \tag{36}$$

where Δ is defined in (21) with

$$Q = \alpha P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0}. \tag{37}$$

The coarsest quantization density is achieved by solving the positive definite P for the following algebraic Riccati equation:

$$\alpha P - A^T PA - \sigma^2 A_0^T PA_0 + \frac{\left(A^T PB + \sigma^2 A_0^T PB_0\right)\left(A^T PB + \sigma^2 A_0^T PB_0\right)^T}{B^T PB + \sigma^2 B_0^T PB_0 + 1} = 0. \tag{38}$$

Moreover, the algebraic Riccati equation (38) is also the solution to the special linear quadratic regulator problem

$$\min \sum_{i=0}^{\infty} E\left[u_i^2\right] \tag{39}$$

corresponding to the minimum energy control that exponentially mean-square stabilizes the system:

$$x(t+1) = \frac{A}{\sqrt{\alpha}} x(t) + \frac{A_0}{\sqrt{\alpha}} x(t)w(t) + Bu(t) + B_0 u(t)w(t), x(0) = x_0. \tag{40}$$

Proof

The proof is similar to the proofs of the results in the previous section, so only a sketch of proof is provided here. The requirement that (35) must hold with some $P > 0$ for quadratic mean-square stabilization of (1) is obtained similarly to Lemma 1, and (36)–(37) are obtained similarly to Theorem 1. The coarsest quantization density result to (38)–(39), and the connection to the minimum energy control problem is obtained similarly to Theorem 2. □

It is interesting to consider the special case that $A_0 = A$ and $B_0 = B$. In this case, the system (1) becomes

$$x(t+1) = Ax(t)(1 + w(t)) + Bu(t)(1 + w(t)), x(0) = x_0. \tag{41}$$

We have the following interesting observation: define $P_1 = (1 + \sigma^2)P$, then (21) becomes

$$\Delta = \frac{B^T P_1 A Q^{-1} A^T P_1 B}{B^T P_1 B}, \tag{42}$$

and (31) becomes

$$\frac{\alpha}{1 + \sigma^2} P_1 - A^T P_1 A + \frac{A^T P_1 B (A^T P_1 B)^T}{B^T P_1 B + 1} = 0. \tag{43}$$

After comparing this with (38), we come to the conclusion that the previously mentioned solution is equivalent to the exponential stabilization with quantized feedback without multiplicative noise, but the exponential convergence rate is altered to be $\frac{\alpha}{1+\sigma^2}$. That is, the multiplicative noise worsens the convergence rate.

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5. ILLUSTRATIVE EXAMPLE

In this section, we give a numerical example to demonstrate the main results of this paper. Consider the system (1) with $A = a = 1$, $A_0 = a_0 = 1$, $B = b = 1$, $B_0 = b_0 = 0.5$, $E\xi(t) = 0$, and $E\xi^2(t) = 1 = \sigma^2$. It is easy to check that (a, a_0, b, b_0) is mean-square stabilizable and that (a, a_0) is observable.

By taking $\varepsilon = 0.1$, it is easy to see that $\left(a, a_0/\varepsilon^{\frac{1}{2}}\right)$ is observable. Solving the inequality (34) by hand, we get $P = 1.2078$. The corresponding quantization density is characterized by (26), which yields

$$\rho = 1 - \frac{2}{\sqrt{b^2 P + \sigma^2 P b_0^2 + 1} + 1} \approx 0.2261.$$

The control gain K_{GD} is computed by using (15). By using the quantized controller of $K_{GD}x(t)$ with logarithmic quantization density ρ as indicated earlier, the closed-loop system's response is illustrated in Figure 2, which shows the effectiveness of the quantized control.

Figure 3 compares the convergence rate of the quantized feedback system subject to multiplicative noise with those not subject to multiplicative noise. It is clear that the presence of the multiplicative noise slows down the convergence rate under the same quantization density. Figure 4 shows the convergence rates for different quantization densities. It is clear from this figure that the convergence rate is faster when ρ is closer to 1 (which means without quantization). We note that the coarsest quantization density for mean-square stabilization can be computed by hand using (25) and (26), which is $\rho^* = 0.2$. It is interesting to see from Figure 4 that the state of the closed-loop system still converges to zero when $\rho = 0.2$.

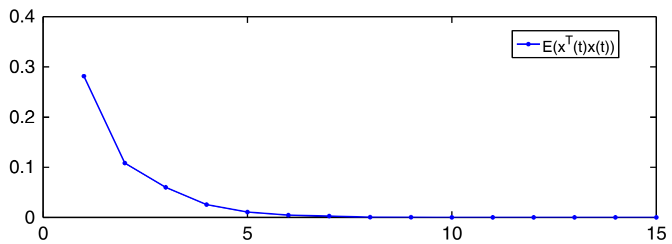

Figure 2. The response of $Ex^T(t)x(t)$.

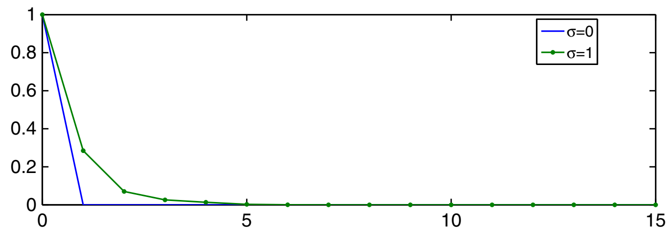

Figure 3. The convergence rate of $Ex^T(t)x(t)$ with different σ but the same quantization density.

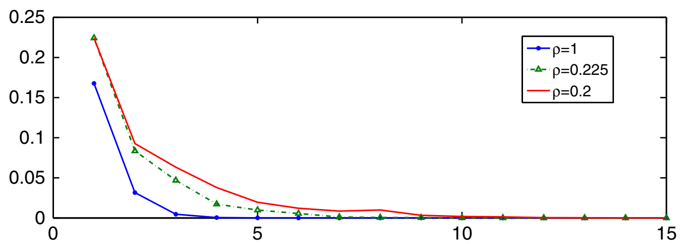

Figure 4. The convergence rate of $Ex^T(t)x(t)$ with different ρ.

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6. CONCLUSION

In this paper, we have investigated quantized state feedback stabilization problems for stochastic systems with multiplicative noises. Results are given for determining the coarsest quantization density required to quadratically mean-square stabilize such a system. The optimal quantization structure is shown to be logarithmic. The solution is expressed in terms of an algebraic Riccati equation. This solution also corresponds to the minimum energy control problem for the given system without quantization. We have also extended the aforementioned results to the exponential quadratic mean-square stabilization of such a system.

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