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State Estimation Subject to Random Communication Delays

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Abstract: The state estimation problem is studied for networked control systems (NCSs) subject to random communication delays and the measurements without time stamps. With the random delay bounded by one step only, a new measurement model is proposed for possible out-of-sequence measurements. For unstable systems, to guarantee linearly unbiased estimator and uniformly bounded estimation error variance, that the estimator structure is based on the average of all received measurements at each time. The estimator gains can be derived by solving a set of recursive discrete-time Riccati equations. The estimator is guaranteed to be optimal in the sense that it is unbiased with uniformly bounded estimation error covariance. A simulation example shows the effectiveness of the proposed algorithm.

Keywords: State estimation, networked control systems (NCSs), random time delay, Riccati equations

In the past decade or so, networked control systems (NCSs) have gained a lot of attention in communication networks, control and state estimation[1−3]. In an NCS, data typically travel through a communication network from sensors to controller and from controller to actuators. As a direct consequence of the finite bandwidth for data transmission over networks, random communication delays, out-of-sequence measurements, and packet losses are inevitable in networked systems where a common medium is shared among different users for data transfers. These problems should be properly handled in order to achieve satisfactory estimation and control performance[4−7].

The estimation problem for NCSs with random delays has gained many results in the past years[8−21]. In the networked system, the sensor measures the output of the system at every sampling instant time and transmits the measurement to the estimator, and time delay is unavoidable due to network congestion. The standard Kalman filtering cannot be directly applied to systems with random output delays. For random time delays, there are two approaches with, i.e., either using time stamps or not using time stamps. Time stamps are often used to reorder the packets when the measurements arrive out of order. In [17], Schenato considered that the measurements with time stamps, encapsulated into packets, and then transmitted through a digital communication network (DCN), thus the estimator was presented by re-ordering the measurements at the estimator site. Zhang et al.[8] studied the optimal estimation problem for discrete-time systems with time-varying delay in the measurement channel, and the measurements were time-stamped which could only take one value at each time instant. With using no time stamps, Sun[20] investigated the estimation problem for systems with bounded random measurement delays and packet dropouts, which were described by some binary distributed random variables whose probabilities are known. Sun[21] also studied the optimal estimation with one-step random delays and packet dropouts.

For the case without time stamps, the commonly used model does not use time stamps, but assumes that at each time k, one and only one randomly delayed measurement is received, i.e., $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}_{k-\tau_k}$, where $\tilde{\boldsymbol{y}}_k$ is the measurement of the system at time k, $\boldsymbol{y}_{k-\tau_k}$ is the received measurement at time k, $\tau_k \in \{0, 1, \cdots, N\}$ is the random time delay with N being the maximum time delay. This model has been widely used, e.g., [19−22]. But this model does not represent practical communication systems, because it allows the same measurement to be received multiple times and can generate too much packet loss. To illustrate this, we suppose the case where $N = 1$ and $\rho_0 = \rho_1 = 0.5$. Then $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}_k$ with probability of 0.5 and $\tilde{\boldsymbol{y}}_{k+1} = \boldsymbol{y}_k$ with probability of 0.5 as well. Since $\boldsymbol{y}_k$ can be received only at k or $k + 1$, it is clear that the probability that $\boldsymbol{y}_k$ gets lost equals the probability that $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}_{k-1}$ and $\tilde{\boldsymbol{y}}_{k+1} = \boldsymbol{y}_k$, which equals 0.25. It is not possible for any network protocol to be designed to produce such a high inherent packet loss probability or to allow duplicated reception of the same measurement.

Since the network transmission has a limited capability, one measurement should not be re-received. Furthermore, the packet delay is random, thus it is possible that between two consecutive sampling periods no packet or multiple packets are delivered. This means packets will arrive in burst or even out of order at the receiver side. So far, estimation problems with such a communication model are seldom reported without using time stamps.

In this paper, we assume that the sequence of received measurement at each sampling time does not have any time stamp. We first provide a time delay model that removes the shortcomings of previous models, i.e., it avoids re-receiving packets and any packet loss. So the presented model is more appropriate for the actual communication protocols. Then, we want that the estimator is unbiased and the estimation error covariance is uniformly bounded. For unstable systems, we provide a novel state estimator using the average of all the received measurements at each sampling time. The optimal estimator is designed. For technical simplicity but without altering the core difficulty,

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we consider the case where the maximum time delay is one time step.

This paper is mainly organized as follows. Problem formulation is given in Section 1; Section 2 firstly considers the estimation problem with received multi-measurements; the whole estimation solution with the random delay is given in Section 3. In Section 4, a simulation example is given. Finally, Section 5 draws some conclusions.

1 Problem formulation

Consider the following discrete-time linear stochastic system:

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + \boldsymbol{v}_k, \tag{1}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{\omega}_k, \tag{2}$$

where $\boldsymbol{x}_k \in \mathbf{R}^n$ is the system state, $\boldsymbol{y}_k \in \mathbf{R}^l$ is the measured output, $\boldsymbol{v}_k \in \mathbf{R}^n$ and $\boldsymbol{\omega}_k \in \mathbf{R}^l$ are system noise and measure noise respectively. A, C are matrices of appropriate dimensions. The initial state $\boldsymbol{x}_0$ and $\boldsymbol{v}_k, \boldsymbol{\omega}_k$ are Gaussian, uncorrelated, white, with mean $(\bar{\boldsymbol{x}}_0, 0, 0)$ and covariance (P_0, Q_k, R_k), respectively. We also assume that the pair (A, C) is observable, and $R > 0$.

In this paper, we will consider the case that there is no time stamps for the measurement, i.e., we do not know the correct order of received measurements because of random delays. We only consider the case where the maximum random delay is $N = 1$, i.e., the random delay is either 0 or 1. Assuming that there is no packet loss and the packets cannot be received repeatedly, we give the state transition diagram for time-delay (see Fig. 1):

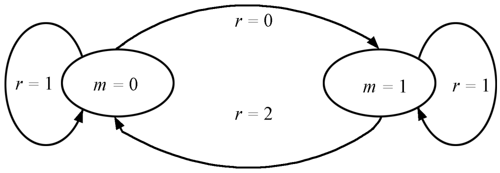

Fig. 1 The state transition for random time-delay

In Fig. 1, m is the number of delayed measurements at time k, and r is the number of the received packets at time k. An arrow indicates the change of m from time k to $k+1$. Thus there are the following cases:

1) At time k, $m = 0$. This means that measurements $\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1}$ all have been received. Then from Fig. 1, there are the following two cases according to the number of received measurements r:

Case 1. When $r = 0$, there is no packet arriving. Then time delay will happen, and $m = 1$ at time $k+1$, as indicated by an arrow from $m = 0$ to $m = 1$.

Case 2. When $r = 1$, packet $\boldsymbol{y}_k$ is received on time, and $m = 0$ at time $k+1$, as indicated by the arrow from $m = 0$ to $m = 0$.

2) $m = 1$. This means that $\boldsymbol{y}_{k-1}$ is missing at $k-1$ time, i.e., time-delay has happened at time k. Again, there are two possible cases according to r:

Case 3. When $r = 1$, due to the assumption on no packet loss, $\boldsymbol{y}_{k-1}$ must be received at time k, and we have $m = 1$ at time $k+1$, because $\boldsymbol{y}_k$ is not received at time k, and this is indicated by the arrow from $m = 1$ to $m = 1$;

Case 4. When $r = 2$, the received measurements must be $\boldsymbol{y}_k$ and $\boldsymbol{y}_{k-1}$ (but without known order). Subsequently, $m = 0$ at time $k+1$, as indicated by the arrow from $m = 1$ to $m = 0$.

Form these four cases, we know that when $r = 0$ or 1, the received measurement can be precisely deduced. The estimator can be presented easily in the first three cases (Cases $1 \sim 3$). But for Case 4 it is difficult because we do not know the correct order of the arrival sequence. In the next section, we mainly give the state estimator for Case 4.

2 Estimator design with out-of-sequence measurements

The problem in Case 4 is as follows: At time $k-1$, measurement $\boldsymbol{y}_{k-1}$ does not arrive, but there are two measured outputs $\{\boldsymbol{y}_{k-1}, \boldsymbol{y}_k\}$ arriving at time k. Because of the lack of time stamps, we do not know the order of $\{\boldsymbol{y}_{k-1}, \ \boldsymbol{y}_k\}$. Then the arrival sequences have two cases at time k:

1) The packets are received in the correct order

$$\tilde{\boldsymbol{y}}_k = \begin{bmatrix} \boldsymbol{y}_{k-1} \\ \boldsymbol{y}_k \end{bmatrix}.$$

2) The packets are received in a reversed order

$$\tilde{\boldsymbol{y}}_k = \begin{bmatrix} \boldsymbol{y}_k \\ \boldsymbol{y}_{k-1} \end{bmatrix}.$$

Thus, the observation processes of the measurements received by the estimator are modeled as:

$$\tilde{\boldsymbol{y}}_k = \begin{bmatrix} \boldsymbol{y}_k^{(1)} \\ \boldsymbol{y}_k^{(2)} \end{bmatrix}, \tag{3}$$

with

$$\boldsymbol{y}_k^{(1)} = (1-\gamma_k)\boldsymbol{y}_{k-1} + \gamma_k \boldsymbol{y}_k\ , \ \ \boldsymbol{y}_k^{(2)} = \gamma_k \boldsymbol{y}_{k-1} + (1-\gamma_k)\boldsymbol{y}_k, \tag{4}$$

where γ_k is a scalar quantity taking on values 0 and 1 with

$$p := Pr\{\gamma_k = 1\}, \quad 1-p := Pr\{\gamma_k = 0\}, \quad \mathrm{E}\gamma_k = \mathrm{E}\gamma_k^2 = p, \tag{5}$$

and we assume that $0 < p < 1$.

We want to obtain a linear state estimator as follows:

$$\hat{\boldsymbol{x}}_{k+1} = F_k \hat{\boldsymbol{x}}_{k-1} + [H_{k1} \ \ H_{k2}]\tilde{\boldsymbol{y}}_k. \tag{6}$$

It is useful to define the estimator error and error covariance:

$$\boldsymbol{e}_{k+1} \quad := \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1}, \tag{7}$$
$$\bar{P}_{k+1} \quad := \mathrm{E}_x \mathrm{E}_{\gamma_k}[\boldsymbol{e}_{k+1}\boldsymbol{e}_{k+1}^{\mathrm{T}}], \tag{8}$$

where E_x is the expectation with respect to $\boldsymbol{v}, \boldsymbol{\omega}$ and $\boldsymbol{x}_0$; and E_{γ_k} is the expectation with respect to γ_k.

The estimate $\hat{\boldsymbol{x}}_{k+1}$ needs to be optimal in the sense that it minimizes the error covariance, i.e., it is desired to find the estimator to minimize (8). We demand that the estimator is unbiased, i.e., $\mathrm{E}_x \mathrm{E}_{\gamma_k} \boldsymbol{e}_{k+1} = 0$, and we also want the estimation error covariance to be uniformly bounded, as defined below.

Definition 1. The estimation error covariance is called uniformly bounded if there exists a constant $M > 0$ independent of P_0, such that

$$\bar{P}_k \leq M, \tag{9}$$

for all $k = 0, 1, 2, \cdots$.

The estimator error $\boldsymbol{e}_{k+1}$ is defined in (7). Substituting (1), (3) and (6) into it, we get

$$\begin{aligned}
&\boldsymbol{e}_{k+1} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1} = \\
&A^2\boldsymbol{x}_{k-1} + A\boldsymbol{v}_{k-1} + \boldsymbol{v}_k - F_k\hat{\boldsymbol{x}}_{k-1} - \\
&[H_{k1} \quad H_{k2}]\begin{bmatrix} (1-\gamma_k)C + \gamma_k CA \\ \gamma_k C + (1-\gamma_k)CA \end{bmatrix}\boldsymbol{x}_{k-1} - \\
&[H_{k1} \quad H_{k2}]\begin{bmatrix} (1-\gamma_k)\boldsymbol{\omega}_{k-1} + \gamma_k C\boldsymbol{v}_{k-1} + \gamma_k\boldsymbol{\omega}_k \\ \gamma_k\boldsymbol{\omega}_{k-1} + (1-\gamma_k)C\boldsymbol{v}_{k-1} + (1-\gamma_k)\boldsymbol{\omega}_k \end{bmatrix}.
\end{aligned} \tag{10}$$

From the unbiased property $\mathrm{E}_x\mathrm{E}_{\gamma_k}\boldsymbol{e}_{k+1} = 0$, with the property of γ_k in (5) and the mean of noises being 0, we get:

$$F_k = A^2 - [H_{k1} \quad H_{k2}]\begin{bmatrix} (1-p)C + pCA \\ pC + (1-p)CA \end{bmatrix}. \tag{11}$$

Substituting F_k into (10), the error is rewritten as

$$\begin{aligned}
&\boldsymbol{e}_{k+1} = \left[A^2 - [H_{k1}\ H_{k2}]\begin{bmatrix} (1-p)C + pCA \\ pC + (1-p)CA \end{bmatrix}\right]\boldsymbol{e}_{k-1} - \\
&[H_{k1}\ H_{k2}]\begin{bmatrix} (p-\gamma_k)C + (\gamma_k - p)CA \\ (\gamma_k - p)C + (p-\gamma_k)CA \end{bmatrix}\boldsymbol{x}_{k-1} - \\
&[H_{k1} \quad H_{k2}]\begin{bmatrix} (1-\gamma_k)\boldsymbol{\omega}_{k-1} + \gamma_k C\boldsymbol{v}_{k-1} + \gamma_k\boldsymbol{\omega}_k \\ \gamma_k\boldsymbol{\omega}_{k-1} + (1-\gamma_k)C\boldsymbol{v}_{k-1} + (1-\gamma_k)\boldsymbol{\omega}_k \end{bmatrix} + \\
&A\boldsymbol{v}_{k-1} + \boldsymbol{v}_k.
\end{aligned} \tag{12}$$

Lemma 1. Considering the estimation error dynamic equation (12), if A is unstable, then a necessary condition for the estimation error to be unbiased and error covariance to be uniformly bounded is that $H_{k1} = H_{k2}$ for all k. Consequently, the optimal estimator has the form

$$\hat{\boldsymbol{x}}_{k+1} = F_k\hat{\boldsymbol{x}}_{k-1} + H_k\frac{1}{2}(\boldsymbol{y}_{k-1} + \boldsymbol{y}_k), \tag{13}$$

i.e., the average of $\boldsymbol{y}_{k-1}$ and $\boldsymbol{y}_k$ needs to be used.

Proof. To ensure that the estimator is unbiased, we get (11) and (12), as explained before. Since the system matrix A is assumed to be unstable, $\mathrm{E}[\boldsymbol{x}_k\boldsymbol{x}_k^{\mathrm{T}}] \to \infty$ as $k \to \infty$. With the assumption of uncorrelation, and from (5), using Definition 1, the expected estimation error covariance will be uniformly bounded only if

$$\mathrm{E}_{\gamma_k}[\Delta(\gamma_k)\Delta^{\mathrm{T}}(\gamma_k)] = 0, \tag{14}$$

where

$$\Delta(\gamma_k) = [H_{k1}\ H_{k2}]\begin{bmatrix} (p-\gamma_k)C + (\gamma_k - p)CA \\ (\gamma_k - p)C + (p-\gamma_k)CA \end{bmatrix}. \tag{15}$$

Rewriting the above, we have:

$$\Delta(\gamma_k) = (p - \gamma_k)(H_{k1} - H_{k2})(C - CA). \tag{16}$$

Hence,

$$\begin{aligned}
&\mathrm{E}_{\gamma_k}[\Delta(\gamma_k)\Delta^{\mathrm{T}}(\gamma_k)] = \mathrm{E}_{\gamma_k}[(p-\gamma_k)^2(H_{k1} - H_{k2})\times \\
&(C - CA) \times (C - CA)^{\mathrm{T}}(H_{k1} - H_{k2})^{\mathrm{T}}].
\end{aligned} \tag{17}$$

Since $\mathrm{E}_{\gamma_k}[(p-\gamma_k)^2] \neq 0$, we must have:

$$H_{k1}(C - CA) = H_{k2}(C - CA). \tag{18}$$

Any choice of $(H_{k1},\ H_{k2})$ satisfying (16) yields the same effect on (12) as the choice of $H_{k1} = H_{k2}$. Hence, $H_{k1} = H_{k2}$ is necessary for the estimation error to be unbiased and uniformly bounded.

We let $H_k = 2H_{k1} = 2H_{k2}$ and substitute it into (6), then the estimator (6) is finally equivalent to (13). □

Substituting $H_k = 2H_{k1} = 2H_{k2}$ into (1), we have:

$$\begin{aligned}
&F_k = A^2 - [H_k \quad H_k]\frac{1}{2}\begin{bmatrix} (1-p)C + pCA \\ pC + (1-p)CA \end{bmatrix} = \\
&A^2 - H_k\frac{1}{2}((1-p)C + pCA) + H_k\frac{1}{2}(pC + (1-p)CA) = \\
&A^2 - \frac{1}{2}H_k(C + CA).
\end{aligned} \tag{19}$$

The optimal estimation gain H_k is given in the following theorem.

Theorem 1. For system (1) and (2), and the estimator form (13), if the estimation error covariance $\bar{P}_{k-1}$ is given, then the estimation gain H_k for

$$\min_{H_k} \bar{P}_{k+1} \tag{20}$$

is given by

$$H_k = \frac{1}{2}(A^2\bar{P}_{k-1}(C + CA)^{\mathrm{T}} + AQ_{k-1}C^{\mathrm{T}})M_k^{-1}, \tag{21}$$

where

$$\begin{aligned}
&M_k = \\
&\frac{1}{4}[(C + CA)\bar{P}_{k-1}(C + CA)^{\mathrm{T}} + CQ_{k-1}C^{\mathrm{T}} + R_{k-1} + R_k].
\end{aligned} \tag{22}$$

The corresponding solution for $\bar{P}_{k+1}$ is given by

$$\bar{P}_{k+1} = A^2\bar{P}_{k-1}A^{2\mathrm{T}} - H_kM_kH_k^{\mathrm{T}} + AQ_{k-1}A^{\mathrm{T}} + Q_k, \tag{23}$$

$$P_0 = \mathrm{E}\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}. \tag{24}$$

Proof. From Lemma 1, the estimator error is

$$\begin{aligned}
\boldsymbol{e}_{k+1} = {} & (A^2 - \frac{1}{2}H_k(C + CA))\boldsymbol{e}_{k-1} + A\boldsymbol{v}_{k-1} + \boldsymbol{v}_k - \\
& \frac{1}{2}H_k(\boldsymbol{\omega}_{k-1} + C\boldsymbol{v}_{k-1} + \boldsymbol{\omega}_k).
\end{aligned} \tag{25}$$

Note that the noise of system $A\boldsymbol{v}_{k-1} + \boldsymbol{v}_k$ and the measurement noise $\boldsymbol{\omega}_{k-1} + C\boldsymbol{v}_{k-1} + \boldsymbol{\omega}_k$ are correlated. The estimation error covariance is given by

$$\begin{aligned}
&\bar{P}_{k+1} = \mathrm{E}_x\mathrm{E}_\gamma[\boldsymbol{e}_{k+1}\boldsymbol{e}_{k+1}^{\mathrm{T}}] = \\
&(A^2 - \frac{1}{2}H_k(C + CA))\bar{P}_{k-1}(A^2 - \frac{1}{2}H_k(C + CA))^{\mathrm{T}} + \\
&AQ_{k-1}A^{\mathrm{T}} - \frac{1}{2}AQ_{k-1}C^{\mathrm{T}}H_k^{\mathrm{T}} - \frac{1}{2}H_kCQ_{k-1}A^{\mathrm{T}} + Q_k + \\
&\frac{1}{4}H_kCQ_{k-1}C^{\mathrm{T}}H_k^{\mathrm{T}} + \frac{1}{4}H_kR_{k-1}H_k^{\mathrm{T}} + \frac{1}{4}H_kR_kH_k^{\mathrm{T}} = \\
&(H_k + H_k^*)M_k(H_k + H_k^*)^{\mathrm{T}} - H_kM_kH_k^{*\mathrm{T}} - H_k^*M_kH_k^{\mathrm{T}} - \\
&H_k^*M_kH_k^{*\mathrm{T}} + A^2\bar{P}_{k-1}A^{2\mathrm{T}} + AQ_{k-1}A^{\mathrm{T}} + Q_k - \\
&\frac{1}{2}H_kCQ_{k-1}A^{\mathrm{T}} - \frac{1}{2}AQ_{k-1}C^{\mathrm{T}}H_k^{\mathrm{T}} - \\
&\frac{1}{2}H_k(C + CA)\bar{P}_{k-1}A^{2\mathrm{T}} - \frac{1}{2}A^2\bar{P}_{k-1}(C + CA)^{\mathrm{T}}H_k^{\mathrm{T}},
\end{aligned} \tag{26}$$

where $M_k = \frac{1}{4}[(C+CA)\bar{P}_{k-1}(C+CA)^{\mathrm{T}} + CQ_{k-1}C^{\mathrm{T}} + R_{k-1} + R_k]$.

To minimize $\bar{P}_{k+1}$, H_k^* should be chosen as

$$H_k^* = -\frac{1}{2}(A^2\bar{P}_{k-1}(C+CA)^{\mathrm{T}} + AQ_{k-1}C^{\mathrm{T}})M_k^{-1},$$

and $H_k = -H_k^*$. Then the estimator gain (17) is obtained.

Substituting H_k and H_k^* back to (21), we get (18), with the initial condition $P_0 = \mathrm{E}\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}$. □

Remark 1. When the system noise and the measurement noise are uncorrelated, the error covariance equation (21) is monotonic in $\bar{P}_{k-1}$, and the estimation is optimal. This follows from the classical Kalman filter theory[23]. This monotonicity property is vital for recursion because it means that if we minimize $\bar{P}_{k-1}$ at time $k-1$, and use the minimized $\bar{P}_{k-1}$ to minimize $\bar{P}_{k+1}$, the resulting $\bar{P}_{k+1}$ is optimal over all $\bar{P}_{k-1}$.

In the following we will give the monotonicity property when the noises are correlated.

From (21), we have:

$$\begin{aligned}&\bar{P}_{k+1} = \\ &(A^2 - \frac{1}{2}H_k(C+CA))\bar{P}_{k-1}(A^2 - \frac{1}{2}H_k(C+CA))^{\mathrm{T}} + \\ &AQ_{k-1}A^{\mathrm{T}} - \frac{1}{2}AQ_{k-1}C^{\mathrm{T}}H_k^{\mathrm{T}} - \frac{1}{2}H_kCQ_{k-1}A^{\mathrm{T}} + \\ &\frac{1}{4}H_kCQ_{k-1}C^{\mathrm{T}}H_k^{\mathrm{T}} + Q_k + \frac{1}{4}H_kR_{k-1}H_k^{\mathrm{T}} + \\ &\frac{1}{4}H_kR_kH_k^{\mathrm{T}}.\end{aligned} \tag{27}$$

Denote the mapping (22) with (17) from $\bar{P}_{k-1}$ to $\bar{P}_{k+1}$ by $\mathcal{F}(\cdot) : S_+^n \rightarrow S_+^n$, i.e.,

$$\bar{P}_{k+1} = \mathcal{F}(\bar{P}_{k-1}). \tag{28}$$

Lemma 2. $\mathcal{F}(\cdot)$ is a monotonic function, i.e., if $\bar{P}_{k-1}^{(1)} \geq \bar{P}_{k-1}^{(2)} > 0$, then

$$\mathcal{F}(\bar{P}_{k-1}^{(1)}) \geq \mathcal{F}(\bar{P}_{k-1}^{(2)}). \tag{29}$$

Proof. Denote the mapping (22) from $\bar{P}_{k-1}$ and H_k to $\bar{P}_{k+1}$ by $G(\cdot,\cdot) : S_+^n \times \mathbf{R}^n \rightarrow S_+^n$: Then since the solution H_k in (17) is obtained by minimizing (22), that it is

$$H_k = \arg\min_{\tilde{H}_k} G(\bar{P}_{k-1}, \tilde{H}_k), \tag{30}$$

with the suppose $\bar{P}_{k-1}^{(1)} \geq \bar{P}_{k-1}^{(2)}$, let $H_k^{(1)}$ and $H_k^{(2)}$ be the corresponding H_k as obtained in (17) by (26), then

$$\begin{aligned}\bar{P}_{k+1}^{(2)} = G(\bar{P}_{k-1}^{(2)}, H_k^{(2)}) &\leq \\ G(\bar{P}_{k-1}^{(2)}, H_k^{(1)}) &\leq \\ G(\bar{P}_{k-1}^{(1)}, H_k^{(1)}) &= \bar{P}_{k+1}^{(1)}.\end{aligned} \tag{31}$$

Hence, the lemma holds.

In the above, the two equalities follow from (26). The first inequality follows from (26) as well. The second inequality follows from (22), i.e., $G(\bar{P}_{k-1}, \tilde{H}_k)$ is linear in $\bar{P}_{k-1}$ when H_k is fixed. □

Remark 2. From Lemma 2, we know that the estimator in Theorem 1 is optimal.

3 Optimal estimator for random delays bounded by $N = 1$

From Theorem 1, we know that when the number of the received measurements $r_k = 2$, the measurement adopted by the estimator in (13) is the average of $\boldsymbol{y}_{k-1}$ and $\boldsymbol{y}_k$. Then for Cases 1 and 2, the following model for the measurement received by the estimator is adopted:

$$\tilde{\boldsymbol{y}}_k = \begin{cases} \frac{1}{r_k}\sum\limits_{i=1}^{r_k}\boldsymbol{y}_{k-m_k+i-1}, & r_k = 1 \text{ or } 2, \\ \emptyset, & r_k = 0, \text{ there is no packet.}\end{cases} \tag{32}$$

Theorem 2. Consider system (1) ~ (2). Denote by m_k the random time delay and by r_k the number of the received packets at time k. Then the optimal estimator is given as follows:

1) When there is no packet received, i.e., $r_k = 0$, the most recent state estimate remains at $\hat{\boldsymbol{x}}_{k-m_k}$ and the error covariance at $\bar{P}_{k-m_k}$. The estimator is

$$\hat{\boldsymbol{x}}_{k+1} = A^{m_{k+1}}\hat{\boldsymbol{x}}_{k-m_k}, \tag{33}$$

with $m_{k+1} = m_k + 1$, and the error covariance recursive equation is

$$\bar{P}_{k+1} = A^{m_{k+1}}\bar{P}_{k-m_k}A^{m_{k+1}\mathrm{T}} + \sum_{i=0}^{m_{k+1}-1}A^iQ_{k-m_k+i}A^{i\mathrm{T}}. \tag{34}$$

2) When $r_k > 0$, the estimator is

$$\hat{\boldsymbol{x}}_{k+1-m_{k+1}} = \bar{F}_k\hat{\boldsymbol{x}}_{k-m_k} + \bar{H}_k\tilde{\boldsymbol{y}}_k, \tag{35}$$

and

$$\hat{\boldsymbol{x}}_{k+1} = A^{m_{k+1}}\hat{\boldsymbol{x}}_{k+1-m_{k+1}}, \tag{36}$$

with

$$m_{k+1} = m_k - r_k + 1, \tag{37}$$

$$\bar{F}_k = A^{r_k} - \frac{1}{r_k}\bar{H}_k\sum_{i=0}^{r_k-1}CA^i, \tag{38}$$

and

$$\begin{aligned}\bar{H}_k = &\frac{1}{r_k}(A^{r_k}\bar{P}_{k-m_k}(\sum_{i=0}^{r_k-1}CA^i)^{\mathrm{T}} + \\ &\sum_{i=1}^{r_k-1}\sum_{j=0}^{i-1}A^iQ_{k-m_k+r_k-i-1}A^{j\mathrm{T}}C^{\mathrm{T}})M_k^{-1},\end{aligned} \tag{39}$$

where

$$\begin{aligned}M_k = &\frac{1}{r_k^2}((\sum_{i=0}^{r_k-1}CA^i)\bar{P}_{k-m_k}(\sum_{i=0}^{r_k-1}CA^i)^{\mathrm{T}} + \sum_{i=0}^{r_k-1}R_{k-m_k+i} + \\ &\sum_{i=1}^{r_k-1}\sum_{j=0}^{i-1}CA^jQ_{k-m_k+i-j-1}A^{j\mathrm{T}}C^{\mathrm{T}}),\end{aligned} \tag{40}$$

and the error covariance update is given by

$$\begin{aligned}\bar{P}_{k+1-m_{k+1}} = &A^{r_k}\bar{P}_{k-m_k}A^{r_k\mathrm{T}} - \bar{H}_kM_k\bar{H}_k^{\mathrm{T}} + \\ &\sum_{i=0}^{r_k-1}A^iQ_{k-m_k+i}A^{i\mathrm{T}}.\end{aligned} \tag{41}$$

The initial error covariance is $\bar{P}_0 = \mathrm{E}\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}$.

Proof. From (1), (2) and (28), the state estimation error is

$$\begin{aligned}\boldsymbol{e}_{k+1-m_{k+1}} = \boldsymbol{x}_{k+1-m_{k+1}} - \hat{\boldsymbol{x}}_{k+1-m_{k+1}} = \\ A^{r_k}\boldsymbol{x}_{k-m_k} + \sum_{i=0}^{r_k-1} A^i\boldsymbol{v}_{k-m_k+r_k-i-1} - \bar{F}_k\hat{\boldsymbol{x}}_{k-m_k} - \\ \frac{1}{r_k}\bar{H}_k\sum_{i=0}^{r_k-1} CA^i\boldsymbol{x}_{k-m_k} - \frac{1}{r_k}\bar{H}_k\sum_{i=1}^{r_k-1} \times \\ \sum_{j=0}^{i-1} CA^j\boldsymbol{v}_{k-m_k+i-j-1} - \frac{1}{r_k}\bar{H}_k\sum_{i=0}^{r_k-1}\boldsymbol{\omega}_{k-m_k+i}.\end{aligned} \quad (42)$$

Using the estimator′s unbiased property, we get (30). Substituting (34) into (8), we have the estimator error covariance as follows:

$$\begin{aligned}\bar{P}_{k+1-m_{k+1}} = \mathrm{E}_x[\boldsymbol{e}_{k+1-m_{k+1}}\boldsymbol{e}_{k+1-m_{k+1}}^{\mathrm{T}}] = \\ (A^{r_k} - \frac{1}{r_k}\bar{H}_k\sum_{i=0}^{r_k-1} CA^i)\bar{P}_{k-m_k}(A^{r_k} - \frac{1}{r_k}\bar{H}_k\sum_{i=0}^{r_k-1} CA^i)^{\mathrm{T}} + \\ \sum_{i=0}^{r_k-1} A^iQ_{k-m_k+r_k-i-1}A^{i\mathrm{T}} + \frac{1}{r_k^2}\bar{H}_k\sum_{i=0}^{r_k-1} R_{k-m_k+i}\bar{H}_k^{\mathrm{T}} + \\ \frac{1}{r_k^2}\bar{H}_k\sum_{i=1}^{r_k-1}\sum_{j=0}^{i-1} CA^jQ_{k-m_k+i-j-1}A^{j\mathrm{T}}C^{\mathrm{T}}\bar{H}_k^{\mathrm{T}} - \\ \frac{1}{r_k}\sum_{i=1}^{r_k-1}\sum_{j=0}^{i-1} A^iQ_{k-m_k+r_k-i-1}A^{j\mathrm{T}}C^{\mathrm{T}}\bar{H}_k^{\mathrm{T}} - \\ \frac{1}{r_k}\bar{H}_k\sum_{i=1}^{r_k-1}\sum_{j=0}^{i-1} CA^jQ_{k-m_k+r_k-i-1}A^{i\mathrm{T}}.\end{aligned} \quad (43)$$

Similar to Theorem 1, by minimizing (35), (31) $\sim$ (33) are obtained. When there are no measurements, the estimator just updates as (17), and the covariance equation (29) can be obtained. Similar to Lemma 2, we know that $\bar{P}_{k+1-m_{k+1}}$ is monotonic in $\bar{P}_{k-m_k}$. Hence, the estimator (17), (29) and (30) are optimal. □

4 Simulation example

In this section, we present a numerical example to illustrate the previous theoretical results.

Consider a system described in (1) and (2) with the following specifications:

$$A = \begin{bmatrix} 1.1 & -0.1 \\ 0.5 & 0.9 \end{bmatrix},\ C = [1\ \ 2],$$

and $R = 0.1$, $Q = 0.25I_2$, $P_0 = 0.25I_2$, where I_2 is the identity matrix.

We know that r_k is obtained according to the transition diagram in Fig. 1, and we suppose the transition probabilities are as follows:

$$\begin{aligned}p_{00} = P(m(k+1) = 0|m(k) = 0) = 0.85,\\ p_{01} = P(m(k+1) = 1|m(k) = 0) = 0.15,\\ p_{10} = P(m(k+1) = 0|m(k) = 1) = 0.75,\\ p_{11} = P(m(k+1) = 1|m(k) = 1) = 0.25.\end{aligned}$$

Fig. 2 shows the comparison of the traces of the error covariance for three scenarios:

Method 1. The proposed method in this paper.

Method 2. The standard Kalman filtering, assuming that there is no time delay;

Method 3. When receiving two measurements, the estimator just uses the newest measurement.

It can be seen from the simulation results that the proposed estimator in the paper has a better performance than Method 3. We also show the curves of the true state values and estimated values using the proposed method. The simulation results are obtained as shown in Figs. 3 and 4. It can be seen from the simulation results that the proposed linear estimator tracks the real state value very well.

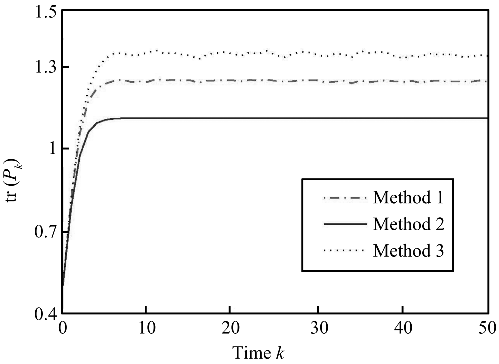

Fig. 2 Comparison of the traces of error covariance

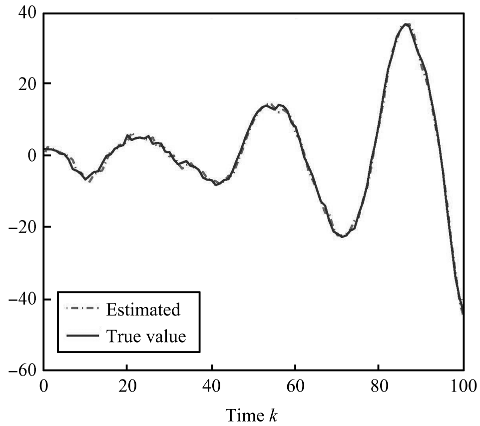

Fig. 3 The first state component of x_k and its estimate

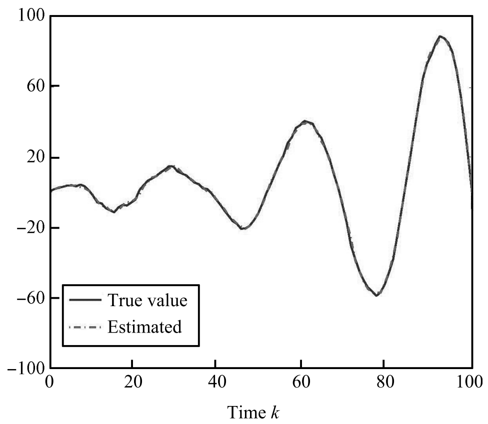

Fig. 4 The second state component of x_k and its estimate

5 Conclusion

In this paper, for the networked control systems with bounded random measurement delay of at most one step, the optimal estimator is proposed without using time stamps. The key to our development in the estimation of the networked control systems is to use the average of all the received measurements at each instant time. We have shown that the state estimator is optimal in the class of linear estimators with the properties of zero bias and uniformly bounded estimation error covariance. Furthermore, the proposed optimal filter is reduced to the standard Kalman filter when there are no random measurement delays.

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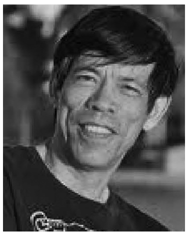

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