POLYTOPES OF POLYNOMIALS WITH ZEROS IN A PRESCRIBED REGION*

Minyue Fu

Department of Electrical and Computer Engineering
Wayne State University, Detroit, MI 48202

and

B. Ross Barmish

Department of Electrical and Computer Engineering
University of Wisconsin, Madison, WI 53706

Abstract. In Bartlett, Hollot and Lin [2], a fundamental result is established on the zero locations of a family of polynomials. It is shown that the zeros of a polytope P of n-th order real polynomials is contained in a simply connected region D if and only if the zeros of all polynomials along the exposed edges of P are contained in D. This paper is motivated by the fact that the requirement of simple connectedness of D may be too restrictive in applications such as dominant pole assignment and filter design where the separation of zeros is required. In this paper, we extend the "edge criterion" in [2] to handle any region D whose complement D^c has the following property: Every point $d \in D^c$ lies on some continuous path which remains within D^c and is unbounded. This requirement is typically verified by inspection and allows for a large class of disconnected regions. We also allow for polynomials with complex coefficients.

1 Introduction

In this paper we address a special case of the following problem: Given a family of n-th order polynomials P (real or complex) and a region D in the complex plane, determine whether all polynomials $p(s)$ in P have all their zeros interior to D. When this is the case, P is said to be *D-stable*. A first seminal result on this problem is given in a paper by Kharitonov [1] for the special case when P corresponds to a family of real interval polynomials and D is the left half plane. More precisely, bounding intervals $[\alpha_i, \beta_i]$ are specified a priori and polynomials $p(s) \in P$ are of the form

$$p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

with $a_i \in [\alpha_i, \beta_i]$ for $i = 1, 2, \cdots, n$. Subsequently, Kharitonov's Theorem indicates that D-stability of only four extreme polynomials (generated using the α_i and β_i) are sufficient to guarantee the D-stability of P.

From a system theoretical point of view, there are two fundamental limitations of Kharitonov's Theorem: The first fundamental limitation stems from the assumption that D is the left half plane. Hence, the result does not apply to discrete-time systems or problems where specifications on pole locations must be satisfied. For example, for so-called dominant pole location problem, it is desirable to have two closed loop poles within some prescribed ϵ-neighborhoods of a given target $\alpha \pm j\beta$ ($\alpha < 0$) with the remaining poles having real part less than some specified $\sigma \ll \alpha$. A second example is the Butterworth filtering problem where the set of ideal poles should be uniformly distributed on the circle with radius ω where ω is the cutoff frequency of the filter. In view of the fact that variations in the filter parameters may lead to perturbations in the pole locations, the following robustness problem is of interest: Given a prescribed $\epsilon > 0$ and a range of variations for the filter parameters, determine if the poles of the perturbed filter stay within the ϵ-neighborhoods of their ideal locations.

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The second fundamental limitation of Kharitonov's result stems from the assumption that coefficients vary within prescribed intervals $[\alpha_i, \beta_i]$. This assumption is tantamount to "independence" between coefficient variations and is rarely met in practice. For example, in a mechanical system, perturbations in a coefficient of friction typically enter into more than one coefficient in the transfer function of the system.

An important result aimed at overcoming the limitations of Kharitonov's Theorem is given in Bartlett, Hollot and Lin [2]. These authors take D to be simply connected and allow for *linearly dependent* coefficient perturbations by taking P to be a polytope of real n-th order monic polynomials. That is, they consider a polytope of monic n-th order polynomials P generated by polynomials $p_1(s), p_2(s), \cdots, p_m(s)$. Hence, P is described by

$$P = \{p(s) = \sum_{i=1}^{m} r_i p_i(s) : \sum_{i=1}^{m} r_i = 1;\ r_i \geq 0\ \forall\ i\}. \qquad (1)$$

Subsequently, it is shown that P is D-stable if and only if all exposed edges of P are D-stable. Hence, to determine if P is D-stable, it is suffices to show that $rp_i(s) + (1-r)p_j(s)$ is D-stable for all $i, j \in \{1, 2, \cdots, m\}$ and all $r \in [0, 1]$. This result is further refined (see, for example, [3] and [4]) where it is shown that the r-sweep associated with the D-stability test above can be replaced by a "one-shot" test if D is the open left half plane.

The main motivation for this short paper is derived from the fact that the assumption of simple connectedness of D might be too restrictive in many applications. Recalling the motivating examples (dominant pole specification and Butterworth filter design) given above, notice that although D violates the simple connectedness requirement in [2], its complement D^c satisfies the following condition: Through every point D^c, there is an unbounded continuous path which remains within D^c. More precisely, we say that D^c is *pathwise connected on the Riemann sphere*. This will be the fundamental property of D which we exploit in the derivation of our main result. Indeed, we extend the "edge criterion" in [2] to accommodate this class of D-regions. For examples of practical interest, it is not hard to see that simple connectedness of D implies pathwise connectedness of D^c on the Riemann sphere; i.e., this theory not only handles disconnected regions but also those considered in [2]. Other (perhaps less important) differences between this paper and [2] are that we do not require the generating polynomials $p_i(s)$ for P to be monic and that we allow for polynomials with complex coefficients.

2 Preliminary Notation

A complex n-th polynomial $p(s)$ is described by

$$p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n;\ a_0 \neq 0 \qquad (2)$$

with $a_i = \alpha_i + j\beta_i$; $\alpha_i, \beta_i \in \mathbf{R}$ for all i. We denote the coefficient vector of $p(s)$ by

$$\mathbf{p} = [\ \alpha_0\ \ \beta_0\ \ \alpha_1\ \ \beta_1\ \ \cdots\ \ \alpha_n\ \ \beta_n\ ]^T. \qquad (3)$$

Given a polytope of n-th order polynomials P (not necessarily monic, with $n \geq 1$) generated by $p_1(s), p_2(s), \cdots, p_m(s)$, we denote the set of coefficients by

$$\mathbf{P} = \{\sum_{i=1}^{m} r_i \mathbf{p}_i : \sum_{i=1}^{m} r_i = 1;\ r_i \geq 0\ \forall\ i\} \qquad (4)$$

where $\mathbf{p}_i$ is the coefficient vector for $p_i(s)$. Note that if $\mathbf{P}$ is a polytope of real polynomials, then P is n-th order if and only if all the generating polynomials $p_i(s)$ are n-th order with the same sign of their highest order coefficients. In general, a polytope of polynomials P is n-th order if and only if the highest order coefficients of all the generating polynomials $p_i(s)$ stay within any half plane which does not include the origin. We denote the affine hull of $\mathbf{P}$ by aff($\mathbf{P}$). We call s is a zero of P if s is a zero of some polynomial $p(s) \in P$. Equivalently, there exists some $\mathbf{p} \in \mathbf{P}$ such that

$$K(s)\mathbf{p} = 0$$

where

$$K(s) \doteq \begin{bmatrix} \mathrm{Re}(s^n) & \mathrm{Im}(s^n) \\ -\mathrm{Im}(s^n) & \mathrm{Re}(s^n) \\ \mathrm{Re}(s^{n-1}) & \mathrm{Im}(s^{n-1}) \\ -\mathrm{Im}(s^{n-1}) & \mathrm{Re}(s^{n-1}) \\ \cdots & \cdots \\ \mathrm{Re}(s) & \mathrm{Im}(s) \\ -\mathrm{Im}(s) & \mathrm{Re}(s) \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T \in \mathbf{R}^{2\times 2(n+1)}. \qquad (5)$$

3 Main Result

Theorem 3.1 *Consider a polytope of n-th order (real or complex) polynomials P and a region*

D in the complex plane such that D^c is pathwise connected on the Riemann sphere. Then, P is D-stable if and only if the exposed edges of P are D-stable.

Proof: Throughout the proof we use $E(\Omega)$ to denote the exposed edges of a polytope Ω.

(Necessity) Suppose P is D-stable. Then it follows trivially that $E(P)$ is D-stable because $E(P)$ is a subset of P.

(Sufficiency) We assume that $E(P)$ is D-stable and must show that P is D-stable. First, we dispose with the trivial case when $\dim \operatorname{aff}(\mathbf{P}) = 1$ because $E(\mathbf{P}) = P$ in this situation. Hence, we assume $\dim \operatorname{aff}(\mathbf{P}) \geq 2$ and proceed by contradiction. Indeed, assume that P is not D-stable. Then, there exists some $\mathbf{p} \in \mathbf{P}$ and some $\alpha \in D^c$ such that

$$K(\alpha)\mathbf{p} = 0. \tag{6}$$

To obtain the desired contradiction, we need to show that there exists some $\mathbf{q} \in E(\mathbf{P})$ and some $\beta \in D^c$ such that

$$K(\beta)\mathbf{q} = 0. \tag{7}$$

To this end, we consider two cases. In case 1, we assume $\dim \operatorname{aff}(\mathbf{P}) = 2$. Subsequently, for Case 2 when $\dim \operatorname{aff}(\mathbf{P}) > 2$, we argue that the problem can be reduced to Case 1.

Case 1: $\dim \operatorname{aff}(\mathbf{P}) = 2$. First we express $\operatorname{aff}(\mathbf{P})$ as

$$\operatorname{aff}(\mathbf{P}) = \{\mathbf{p} + Ax : x \in \mathbf{R}^2\}$$

for some appropriate $2(n+1) \times 2$ dimensional matrix A. We now consider two subcases.

Subcase 1A: $\operatorname{rank}(K(\alpha)A) \leq 1$. Notice that the set of coefficients of polynomials associated with $\operatorname{aff}(\mathbf{P})$ having α as a zero is

$$\begin{aligned}\mathbf{P}_\alpha &= \{\mathbf{p} + Ax : K(\alpha)(\mathbf{p} + Ax) = 0; x \in \mathbf{R}^2\}\\ &= \{\mathbf{p} + Ax : K(\alpha)Ax = 0; x \in \mathbf{R}^2\}.\end{aligned}$$

Furthermore, $\mathbf{P}_\alpha$ is contained in $\operatorname{aff}(\mathbf{P})$ and since $\mathbf{P}_\alpha$ has dimension 1 or 2, it follows that $\mathbf{P}_\alpha$ intersects $E(\mathbf{P})$. Choosing $\mathbf{q} \in \mathbf{P}_\alpha \cap E(\mathbf{P})$, we obtain the desired contradiction with $\beta = \alpha$.

Subcase 1B: $\operatorname{rank}(K(\alpha)A) = 2$. Now, since D^c is pathwise connected on the Riemann sphere, there exists some unbounded continuous path Γ in D^c passing through α. Furthermore, by compactness of $\mathbf{P}$, there must exists some $\gamma \in \Gamma$ which is not a zero of any polynomial in P. Now let $f(\cdot) : [0,1] \to \Gamma$ be a continous function associated with the segment of Γ between α and γ, i.e., $f(0) = \alpha$ and $f(1) = \gamma$. Furthermore, we define

$$\lambda^* \doteq \sup\{\lambda \in [0,1] : \operatorname{rank}(K(f(\varsigma))A) = 2 \; \forall \varsigma \in [0,\lambda)\}.$$

By definition of λ^*, the equation

$$K(f(\lambda))(\mathbf{p} + Ax) = 0$$

has a unique solution

$$x_\lambda = -[K(f(\lambda))A]^{-1}K(f(\lambda))\mathbf{p}$$

for all $\lambda \in [0, \lambda^*)$. This solution generates a continuous path in $\operatorname{aff}(\mathbf{P})$ described by

$$\mathbf{p}_\lambda = \mathbf{p} + Ax_\lambda; \lambda \in [0, \lambda^*).$$

There are two possibilities: The first possibility is that $\mathbf{p}_\varsigma$ does not belong to $\mathbf{P}$ for some $\varsigma \in (0, \lambda^*)$. In this situation, there must exist some $\delta \in [0, \varsigma)$ such that $\mathbf{p}_\delta \in E(\mathbf{P})$. Hence, we obtain the desired contradiction with $\mathbf{q} = \mathbf{p}_\delta$ and $\beta = f(\delta)$.

The second possibility is that $\mathbf{p}_\lambda \in \mathbf{P}$ for all $\lambda \in [0, \lambda^*)$. By compactness of $\mathbf{P}$ and continuity of $\mathbf{p}_\lambda$, there must exist some sequence $\{\lambda_n\}$ in $[0, \lambda^*)$ converging to λ^* and some $\mathbf{p}^* \in \mathbf{P}$ such that

$$\mathbf{p}^* = \lim_{n \to \infty} \mathbf{p}_{\lambda_n}.$$

Furthermore, we have

$$K(f(\lambda^*))\mathbf{p}^* = 0 \tag{8}$$

because

$$K(f(\lambda^*))\mathbf{p}^* = \lim_{n \to \infty} K(f(\lambda_n))\mathbf{p}_{\lambda_n}$$

and

$$K(f(\lambda_n))\mathbf{p}_{\lambda_n} = 0$$

for each n. Since $\mathbf{p}^* \neq 0$ (note that the highest order coefficient is nonzero), from (8), it follows that

$$\operatorname{rank}(K(f(\lambda^*))A) \leq 1.$$

Now, by repeating the analysis used in Subcase 1A (with $\mathbf{p} = \mathbf{p}^*$ and $\alpha = f(\lambda^*)$), we obtain some $\mathbf{q} \in E(\mathbf{P})$ and $\beta \in D^c$ such that $K(\beta)\mathbf{q} = 0$.

Case 2: $\dim \operatorname{aff}(\mathbf{P}) = r > 2$. In view of Case 1, it suffices to prove the following: There exists an $(r-1)$–dimensional exposed face $\mathbf{F}$ of $\mathbf{P}$, some $\mathbf{f} \in \mathbf{F}$ and some $\gamma \in D^c$ such that

$$K(\gamma)\mathbf{f} = 0$$

Once $\mathbf{F}$ and $\mathbf{f}$ are found, it is apparent that this argument can be repeated(note $\mathbf{F}$ is a polytope) until we obtain a 2-dimensional exposed face of $\mathbf{P}$ containing the coefficient vector for a polynomial which is *not* D-stable. Then Case 1 applies.

Indeed, let $\underline{P}$ denote any 2-dimensional affine set passing through **P** and notice that

$$\mathbf{P}' \doteq \mathbf{P} \cap \underline{\mathrm{P}}$$

is a subpolytope of **P** of dimension 2 containing **p**. Hence, from Case 1, it follows that there exists some $\mathbf{f} \in E(\mathbf{P}')$ and some $\gamma \in D^c$ such that $K(\gamma)\mathbf{f} = 0$. The proof is completed by noting that $E(\mathbf{P}')$ is contained in some $(r-1)$–dimensional exposed face **F** of **P**. □

4 Conclusion

The next step in this line of research is to develop stability criteria for more general family of polynomials. The polytopic assumption on P clearly restricts the class of physical perturbations which can be handled. Another important point to note is that the edge criterion given here does not easily degenerate into Kharitonov's Theorem for the special case when the polytope corresponds to a family of real interval polynomials; i.e., in this special case, it is not obvious (from the theory given here) why it is suffices to test four polynomials in lieu of all the edges. This leaves open the possibility that *for polytopes of polynomials*, there is some alternative to the edge criterion which specializes to Kharitonov's Theorem in the "correct manner." Besides having aesthetic appeal, such an alternative would be desirable for two reasons. First, as the number of extreme points of P increases, one might be able to avoid the "combinatoric explosion" in computation associated with checking stability of all convex combinations of extreme points taken two at a time. Secondly, such an alternative for the polytopic case might suggest approaches to stability analysis for more general families of polynomials.

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