A CLASS OF KHARITONOV REGIONS FOR ROBUST STABILITY OF LINEAR UNCERTAIN SYSTEMS

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Abstract. In this paper the Kharitonov's Theorems [1,2] are generalized to the problem of so-called Kharitonov regions for robust stability of linear uncertain systems. Given a polytope of (characteristic) polynomials P and a stability region D in the complex plane, P is called D-stable if the zeros of every polynomial in P are interior to D. It is of interest to know whether D is a Kharitonov region, that is, whether the D-stability of the vertices of P implies the D-stability of P. A simple approach is developed which unifies and generalizes many known results on this problem.

Notation

$\mathbf{C}$	=	the complex plane
$\mathbf{C}_-$	=	the open left plane
D	=	an open set in $\mathbf{C}$
∂D	=	boundary of D
D^C	=	$\{d \in \mathbf{C} : d \notin D\}$, the complement of D
$p(s)$	=	a polynomial with real or complex coefficients
P	=	a family of polynomials
V_P	=	the set of vertex polynomials of P
$p_0(s)$	=	nominal polynomial
$p_i(s)$	=	perturbation polynomials, $i = 1, 2, \cdots, m$
$\mathbf{p}$	=	polynomial vector $(p_0(s), p_1(s), \cdots, p_m(s))$
q_i	=	real or complex "perturbation parameters", $i = 1, 2, \cdots, m$
Q_i	=	rectangle in $\mathbf{C}$ or interval in $\mathbf{R}$ which q_i belongs to, $i = 1, 2, \cdots, m$
q_{ik}	=	vertex of Q_i, $k = 1, 2, 3, 4$
$\deg p(s)$	=	degree of the polynomial $p(s)$
$\arg c$	=	angle of a complex number c
$\mathrm{Re}\, c$	=	real part of a complex number c
$\mathrm{Im}\, c$	=	imaginary part of a complex number c
$[x]$	=	integer part of a non-negative real number x

1 Introduction

Consider a family of characteristic polynomials P associated with a linear dynamic system containing parameter perturbations:

$$P \doteq \{p(s,q) = \sum_{i=0}^{n} a_i(q) s^i : q \in Q\} \tag{1}$$

where

$$q \doteq [q_1, q_2, \cdots, q_m]^T \tag{2}$$

is the vector of *perturbation parameters* with each q_i varying in the *bounding rectangle*

$$Q_i \doteq \{t_i + jw_i : \underline{t}_i \le t_i \le \bar{t}_i, \underline{w}_i \le w_i \le \bar{w}_i\} \subset \mathbf{C}, \tag{3}$$

$$Q \doteq Q_1 \times Q_2 \times \cdots \times Q_m \tag{4}$$

is the bounding set of q, and $a_i(q)$ is the i-th coefficient of $p(s,q)$. It is assumed that $a_i(q)$ are affine functions of q and that each Q_i contains zero. Under these assumptions, we can rewrite $p(s,q)$ in (1) as

$$p(s,q) = p_0(s) + \sum_{i=1}^{m} q_i p_i(s) \tag{5}$$

where $p_0(s)$ is the *nominal polynomial* which is obtained from $p(s,q)$ by setting $q = 0$, and $p_i(s)$ are the *perturbation polynomials*, obtained from $p(s,q) - p_0(s)$ by setting $q_i = 1$ and $q_k = 0, k \ne i$. Accordingly, the family of polynomials P in (1) can be rewritten as

$$P = \{p_0(s) + \sum_{i=1}^{m} q_i p_i(s) : q_i \in Q_i, i = 1, 2, \cdots, m\}. \tag{6}$$

For engineering motivation of this type of polynomials, the reader is referred to, among numerous papers and books, [3,4,5] and the references thereof.

For convenience, we denote

$$\mathbf{p} \doteq (p_0(s), p_1(s), \cdots, p_m(s)). \tag{7}$$

The set of vertex polynomials of P is given by

$$V_P \doteq \{p(s,q) : q_i \in \{q_{i1}, q_{i2}, q_{i3}, q_{i4}\}, i = 1, 2, \cdots, m\} \tag{8}$$

where q_{i1}, q_{i2}, q_{i3}, and q_{i4} are the vertices of Q_i. Note that if the perturbation parameter q_i is purely real, then Q_i becomes an interval and the number of its vertices is dropped to two.

Given the family of (characteristic) polynomials as in (1) and a stability region D in $\mathbf{C}$, it is of interest to determine whether the zeros of every polynomial in P are interior to D. The stability regions are usually subsets of $\mathbf{C}_-$ for continuous-time systems, and subsets of the open unit disk for discrete-time systems.

We now give the definitions of D-stability, anti-D-stability and Kharitonov regions.

Definition 1.1 [3,6] *Given an open set $D \subset \mathbf{C}$, a polynomial $p(s)$ is called D-stable (resp. anti-D-stable) if every zero of $p(s)$ is interior to D (resp. D^C, including ∂D). A family of polynomials P is called D-stable (resp. anti-D-stable) if every polynomial in P is D-stable (resp. anti-D-stable).*

where c and r are the center and the radius of D, respectively. Let z_1 and z_2 be any zeros of $f(s)$ and $p_i(s)$, respectively; see Fig. 5. We claim that $\arg (s-z_2)/(s-z_1)$ is monotonously descreasing. To see this, we divide ∂D into L_1 and L_2 according to the tangent points A and B in Fig. 5. When s traverses on L_1, $\arg (s-z_2)/(s-z_1)$ is obviously decreasing because $\arg (s-z_1)$ is increasing and $\arg (s-z_2)$ is decreasing. Now suppose s traverses on L_2 and θ is increased by $d\theta$. Note that both $\arg (s-z_1)$ and $\arg (s-z_2)$ is increased. Therefore, we need to prove that the increment $d\phi_1$ of $\arg (s-z_1)$ is greater than the increment $d\phi_2$ of $\arg (s-z_2)$. This is not difficult to see from Fig. 5 because $d\phi_1 > d\phi_3$, $d\phi_2 \leq d\phi_4$, and $d\phi_3 = d\phi_4 = d\theta/2$. Consequently, $\arg (s-z_2)/(s-z_1)$ is monotonously decreasing on L_2. Hence, our claim holds. Finally, we conclude that $\arg p_i(s)/f(s)$ is monotonously descreasing on ∂D because number of zeros of $p_i(s)$ is less than or equal to that of $f(s)$. □

Corollary 3.3 [9] *Any open circular region of the form shown in Fig. 6 with $\sigma \geq r$ is a Kharitonov region with respect to $\mathbf{p} = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of n-th order.*

Theorem 3.3 *Any open region of the form shown in Fig. 7 and any hyperbolic region in Fig. 8 are Kharitonov regions with respect to $p(s) = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of n-th order.*

Proof: The proof is very similar to that of Theorem 3.1 and therefore omitted. □

Theorem 3.4 [11] *Any region D of the form shown in Fig. 9 is a Kharitonov region with respect to $p(s) = (p_0(s), 1, s, \cdots, s^n)$ for any $p_0(s)$ of n-th order provided that the parameters $q_i, i = 1, 2, \cdots, n$ and the coefficients of $p_0(s)$ are real.*

Proof: Let L_1, L_2, L_3 and L_4 be the four line segments of ∂D (see Fig. 10) and $f(s)$ be any n-th order D-stable polynomial with zeros given by $z_1, z_1^*, z_2, z_2^*, \cdots$, where z_k^* denotes the complex conjugate of z_k. From Theorem 2.1, it is sufficient to show that $\arg s^i/f(s)$ is monotonously descreasing when s traverses on ∂D for any $0 \leq i \leq n$. We first observe that $\arg s^i$ is fixed on each L_t. Therefore, we only need to show that $\arg f(s)$ is monotonously increasing on each L_t. Note that this holds trivially on L_1 and L_4. On L_2, this holds because $\arg (z-z_k)$ increases faster than $\arg (z-z_k^*)$ decreases following from the fact that $d\theta_k \geq d\phi_k$ in Fig. 10. Similar argument applies to L_3 too. Therefore, $\arg s^i/f(s)$ is monotonously descreasing on ∂D. □

Theorem 3.5 *Every open convex region $D \subset \mathbf{C}$ is a Kharitonov region with respect to $(p_0(s), 1)$ for any $p_0(s)$.*

Proof: Let $f(s)$ be any D-stable polynomial. It is obvious that $\arg 1/f(s)$ is monotonously descreasing on ∂D. Therefore, it follows from Theorem 2.1 that D is a Kharitonov region with respect to $(p_0(s), 1)$. □

Corollary 3.4 *Let $D \subset \mathbf{C}$ be an open convex region holding the following property: For any $d \in D$, d is nonzero and $d^{-1} \in D$. Then, D is a Kharitonov region with respect to $(p_0(s), s^n)$ for any n-th order $p_0(s)$. An example of such a D region is given in Fig. 11.*

Proof: Using the property of D, it follows that, for any $q \in \mathbf{C}$, $p_0(s)+qs^n$ is D-stable if and only if $s^n p_0(1/s)+q$ is D-stable. Therefore, D is a Kharitonov region with respect to $(p_0(s), s^n)$ if and only if D is a Kharitonov region with respect to $(s^n p_0(1/s), 1)$. The latter holds from Theorem 3.5 because D is convex. □

Theorem 3.6 [7] *Let $\mathbf{p}$ be given in (7) satisfying the following condition: For each $i = 1, 2, \cdots, m$, either $\mathrm{Re}\, p_i(j\omega) \equiv 0$ or $\mathrm{Im}\, p_i(j\omega) \equiv 0$. Then $\mathbf{C}_-$ is a Kharitonov region with respect to $\mathbf{p}$.*

Proof: Let $f(s)$ be an arbitrary n-th order $\mathbf{C}_-$-stable polynomial. From Theorem 2.1, it is sufficient to show that $\arg p_i(j\omega)/f(j\omega)$ is monotonously descreasing except at $p_i(j\omega) = 0$ when ω increases, $i = 1, 2, \cdots, m$. This is obvious because $\arg f(j\omega)$ is monotonously increasing and $p_i(j\omega)$ is either purely real or purely imaginary. □

4 Kharitonov Regions for Discrete-time Systems

Corollary 4.1 *Any open circular region D of the form shown in Fig. 12 with $\sigma \geq r$ and $\sigma + r \leq 1$ is a Kharitonov region with respect to $p(z) = (p_0(z), 1, z, \cdots, z^n)$ for any $p_0(z)$ of n-th order. In particular, σ and r can be set to $1/2$.*

Proof: This is a direct consequence of Theorem 3.2. □

Corollary 4.2 *The open unit disk or any open circular region D inside of it is a Kharitonov region with respect to $(p_0(z), p_1(z), \cdots, p_m(z))$ if $p_i(z), i = 1, 2, \cdots, m$ are anti-D-stable.*

Proof: Also a direct consequence of Theorem 3.2. □

Theorem 4.1 [13] *The open unit disk is a Kharitonov region with respect to*

$$\mathbf{p} = (p_0(z), 1+z^n, 1-z^n, z+z^{n-1}, z-z^{n-1}, \\ \cdots, z^{[n/2]}+z^{n-[n/2]}, z^{[n/2]}-z^{n-[n/2]})$$

for any $p_0(z)$ of n-th order.

Proof: Let D be the open unit disk and $f(z)$ be any n-th order D-stable polynomial. Note that

$$\partial D = \{\exp(j\theta) : 0 \leq \theta \leq 2\pi\}.$$

By applying Theorem 2.1, it is sufficient to show that $\arg (f(z), z^i \pm z^{|n-i|})$ is monotonously descreasing for any $i \leq [n/2]$ when z traverses on ∂D. Let $z_k, k =$

Definition 1.2 *Let* $\mathbf{p}$ *be given in* (7). *A set* $D \subset \mathbf{C}$ *is called a Kharitonov region with respect to* $\mathbf{p}$ *if the following condition holds: For an arbitrary bounding set* Q *of the form* (4) *and* (3), P *in* (6) *is* D*-stable if and only if* V_P *in* (8) *is* D*-stable.*

The objective of this paper is as follows: Given the family of polynomials P in (6) and an open set $D \subset \mathbf{C}$, determine whether D is a Kharitonov region, i.e., whether the D-stability of V_P implies the D-stability of P. The first important results related to this problem were given by Kharitonov [1,2]. In his seminal work, Kharitonov considered a special case where $D = \mathbf{C}_-$ and P is a so-called *interval polynomial*, i.e.,

$$p(s,q) = p_0(s) + \sum_{i=0}^{n} q_i s^i, \tag{9}$$

or equivalently, $p_i(s) = s^{(i-1)}, i = 1,2,\cdots,n$. He showed that for this special case P is $\mathbf{C}_-$-stable if and only if V_P is $\mathbf{C}_-$-stable, and furthermore, if and only if eight special vertex polynomials in V_P are D-stable or four special ones when the coefficients of the interval polynomials are purely real. Following his work, a number of interesting results have emerged. In Bialas and Garloff [7], it was shown that the convex combinations of two complex polynomials $f_1(s)$ and $f_2(s)$ are $\mathbf{C}_-$-stable if either Re $f_1(j\omega) \equiv$ Re $f_2(j\omega)$ or Im $f_1(j\omega) \equiv$ Im $f_2(j\omega)$. In Bartlett, Hollot and Lin [3], it was discovered that for any simply connected region D, a polytope of real polynomials is D-stable if and only if every edge of P is D-stable. This result is referred to as the "Edge Theorem", and is extended by Fu and Barmish [4] to include disconnected stability regions and complex polynomials and by Fu, Olbrot and Polis [8] to delay systems.

The most pertinent results to this paper are those by Petersen [9], Soh and Berger [10], and Soh [11]. In these papers, the problem of Kharitonov regions for interval polynomials was studied. In [9], the regions in $\mathbf{C}_-$ which can be mapped onto $\mathbf{C}_-$ by the so-called *strongly admissible* rational functions [12,9] were considered. A number of interesting regions (e.g., Figs. 2,3,7) are found to be Kharitonov regions. In [10,11], it was shown by using a different approach that any sector of the form shown in Fig. 3 with $a = b = c = 0$ or in Fig. 10 is a Kharitonov region provided that the polynomial coefficients are real.

In this paper, a new approach to the problem of Kharitonov regions is developed using the concept of decreasing phase property defined below:

Definition 1.3 *Given a stability region* $D \subset \mathbf{C}$ *and the polynomial vector* $\mathbf{p}$ *in* (7), $\mathbf{p}$ *is called to hold the* decreasing phase property *if, for an arbitrary* n*-th order* D*-stable polynomial* $f(s)$ *and* $1 \le i \le m$, $\arg p_i(s)/f(s)$ *is monotonously decreasing (except at* $p_i(s) = 0$*) when* s *traverses on* ∂D *in the counter-clockwise direction (or, for short, monotonously decreasing on* ∂D*).*

Our key theorem (Theorem 2.1) shows that a given open set D is a Kharitonov region with respect to $\mathbf{p}$ in (7) if $\mathbf{p}$ holds the descreasing phase property above. Using this key theorem, many known results on the problem of Kharitonov regions are unified and generalized.

2 Key Theorem

Theorem 2.1 (see Appendix for proof) *Let an open set* $D \subset \mathbf{C}$ *and* $\mathbf{p} = (p_0(s), p_1(s), \cdots, p_m(s))$ *be given. Then* D *is a Kharitonov region with respect to* $\mathbf{p}$ *if* $\mathbf{p}$ *holds the descreasing phase property.*

3 Kharitonov Regions for Continuous-time Systems

In this section, Theorem 2.1 is used to derive a number of useful Kharitonov regions. These Kharitonov regions are mainly for continuous-time systems, but some of them will be used in the next section to develop Kharitonov regions for discrete-time systems.

Theorem 3.1 *Any open half plane of the form shown in Fig. 1 is a Kharitonov region with respect to* $\mathbf{p}$ *in* (7) *for any* $p_0(s)$ *if* $p_i(s), i = 1,2,\cdots,m$ *are anti-*D*-stable.*

Proof: Suppose $p_i(s), i = 1,2,\cdots,m$ are anti-D-stable. Let $f(s)$ be an arbitrary n-th order D-stable polynomial. It is straightforward to see that $\arg p_i(s)/f(s)$ is monotonously decreasing on ∂D because $\arg p_i(s)$ (resp. $\arg f(s)$) are monotonously non-increasing (resp. increasing). Therefore, it follows from Theorem 2.1 that D is a Kharitonov region with respect to $\mathbf{p}$. □

Corollary 3.1 [9] *Any open half plane of the form shown in Fig. 2 with* $a \le 0$ *is a Kharitonov region with respect to* $\mathbf{p} = (p_0(s), 1, s, \cdots, s^n)$ *for any* $p_0(s)$ *of* n*-th order. In particular,* $\mathbf{C}_-$ *is a Kharitonov region with respect to the* $\mathbf{p}$ *above* [1].

Corollary 3.2 [9,10] *Any open region of the form shown in Fig. 3 with* $a, b, c \le 0$ *is a Kharitonov region with respect to* $\mathbf{p} = (p_0(s), 1, s, \cdots, s^n)$ *for any* $p_0(s)$ *of* n*-th order.*

Proof: The proof is very similar to that of Theorem 3.1 and therefore omitted. □

Theorem 3.2 *Any open circular region* D *of the form shown in Fig. 4 is a Kharitonov region with respect to* $\mathbf{p}$ *in* (7) *for any* $p_0(s)$ *if* $p_i(s)$, $i = 1,2,\cdots,m$ *are anti-*D*-stable.*

Proof: Let $f(s)$ be any n-th order D-stable polynomial. From Theorem 2.1, it is sufficient to show that $\arg p^i(s)/f(s)$, $1 \le i \le m$, is monotonously descreasing when D traverses on

$$\partial D = \{c + r\exp(j\theta) : 0 \le \theta \le 2\pi\}$$

$1, 2, \cdots, n$ be the zeros of $f(z)$, $z = exp(j\theta)$ and suppose θ is increased by $d\theta$, as shown in Fig. 12. Note that

$$\begin{aligned} & \exp(j\theta) \pm \exp(j(n - i)\theta) \\ = \; & \exp(j\frac{n}{2}\theta)\left(\exp(j(i - \frac{n}{2})\theta) \pm \exp(j(\frac{n}{2} - i)\theta)\right) \\ = \; & \begin{cases} 2\cos((i - \frac{n}{2})\theta)\exp(j\frac{n}{2}\theta) & \text{or} \\ 2j\sin((i - \frac{n}{2})\theta)\exp(j\frac{n}{2}\theta) \end{cases} \end{aligned}$$

and its phase is either $n\theta/2$ or $(-\pi + n\theta)/2$. It follows that $\arg z^i \pm z^{|n-i|}$ is increased by $nd\theta/2$. On the other hand, $\arg f(z)$ is increased by at least $nd\theta/2$ because

$$d\theta_k > d\phi_k = d\theta/2 ;$$

see Fig. 13. Consequently, $\arg\,(z^i \pm z^{|n-i|})/f(z)$ is monotonously descreasing on ∂D. □

Theorem 4.2 [14] *The unit disk is a Kharitonov region with respect to* $\mathbf{p} = (p_0(z), 1, z, \cdots, z^{\lfloor n/2 \rfloor})$ *for any* $p(z)$ *of n-th order.*

Proof: The proof of this Theorem is exactly the same as that of Theorem 4.1 except $z^i \pm z^{|n-i|}$ is replaced by z^i and we need to use the fact that $\arg z^i$ is increased by only $id\theta$ when θ is increased by $d\theta$. For brevity, the detail is omitted. □

5 Summary

In this paper, a number of Kharitonov regions for robust stability of linear uncertain systems are given based on a new approach which unifies and generalizes many results in [1,2,9,10,11,13,7,14]. To summarize, the Kharitonov regions given in Sections 3-4 are tabulated in Tables 1 and 2. It should be noted, however, that these tables are not complete because more Kharitonov regions can be constructed by 1) applying Theorem 2.1 on other special uncertain polynomials (e.g., low order polynomials), 2) using the fact that the intersection of Kharitonov regions is a Kharitonov region [9], and 3) relaxing the requirement of the descreasing phase property in Theorem 2.1.

Appendix: Proof of Theorem 2.1

The following lemma is essential in the proof of Theorem 2.1.

Lemma 1 *Given a stability region* $D \subset \mathbf{C}$ *and n-th order D-stable polynomials* $f_0(s)$ *and* $f_0(s) + f_1(s)$ *with positive leading coefficients. Suppose* $\arg f_1(s)/f_0(s)$ *is monotonously descreasing on* ∂D. *Then, the polynomial below*

$$f(s, \alpha) = f_0(s) + \alpha f_1(s)$$

is D-stable for all $0 < \alpha < 1$.

Proof: Let $\Gamma \subset \mathbf{C}$ denote the trajectory of $f_1(s)/f_0(s)$ as s traverses on ∂D; i.e.,

$$\Gamma = \{f_1(s)/f_0(s) : s \in \partial D\} .$$

Since $f_0(s)$ is D-stable and $\deg f_1(s) \leq \deg f_0(s)$, Γ is a bounded and closed curve. Therefore, $\arg f_1(s)/f_0(s)$ being monotonously descreasing implies that Γ encloses the origin. On the other hand, the point -1 is not encircled by Γ because $f_0(s) + f_1(s)$ is D-stable (Principle of Argument). Consequently, using the facts that $\arg f_1(s)/f_0(s)$ is monotonously descreasing again and that Γ encloses the origin, the interval $(-\infty, -1]$ is not enclosed by Γ. In particular, the point $-1/\alpha$ is not enclosed by Γ. Therefore, $f_0(s) + \alpha f_1(s)$ is D-stable (Principle of Argument). □

Proof of Theorem 2.1: Suppose V_P is D-stable. Define, for $i = 1, 2, \cdots, m$,

$$\begin{aligned} g_{2i-1}(s) &= (\bar{t}_i - \underline{t}_i)p_i(s) \\ g_{2i}(s) &= j(\bar{\omega}_i - \underline{\omega}_i)p_i(s) \\ \alpha_{2i-1} &= \begin{cases} \frac{t_i - \underline{t}_i}{\bar{t}_i - \underline{t}_i} & \bar{t}_i \neq \underline{t}_i \\ 0 & \bar{t}_i = \underline{t}_i \end{cases} \\ \alpha_{2i} &= \begin{cases} \frac{\omega_i - \underline{\omega}_i}{\bar{\omega}_i - \underline{\omega}_i} & \bar{\omega}_i \neq \underline{\omega}_i \\ 0 & \bar{\omega}_i = \underline{\omega}_i \end{cases} \end{aligned}$$

$$f_0(s) = p_0(s) + \sum_{i=1}^{m}(\underline{t}_i + j\underline{\omega}_i)p_i(s)$$

and, for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2m})^T$ and $1 \leq \ell \leq 2m$,

$$f_\ell(s, \alpha) = f_0(s) + \sum_{k=1}^{\ell} \alpha_k g_k(s) .$$

Note that

$$f_\ell(s, \alpha) = f_{\ell-1}(s, \alpha) + \alpha_\ell g_\ell(s)$$

and that any polynomial in P can be expressed by $f_{2m}(s, \alpha)$ for some α with all $0 \leq \alpha_k \leq 1, k = 1, 2, \cdots, 2m$. From the decreasing phase property of $\mathbf{p}$, we know that, for any n-th order D-stable polynomial $f(s)$, $\arg g_k(s)/f(s)$ is monotonously descreasing on $\partial D, k = 1, 2, \cdots, m$.

Given an arbitrary polynomial $f_{2m}(s, \alpha) \in P$, we need to prove that $f_{2m}(s, \alpha)$ is D-stable by *reductio ad absurdum*. That is, we assume $f_{2m}(s, \alpha)$ is anti-D-stable and show that there exists some vertex polynomial of P which is also anti-D-stable. Indeed, according to Lemma 1, $f_{2m}(s, \alpha)$ being anti-D-stable implies that either $f_{2m-1}(s, \alpha)$ or $f_{2m-1}(s, \alpha) + g_{2m}(s)$ is anti-D-stable. Without loss of generality, we may assume that $f_{2m-1}(s, \alpha)$ is anti-D-stable. Using Lemma 1 again, we further obtain that either $f_{2m-2}(s, \alpha)$ or $f_{2m-2}(s, \alpha) + \alpha_{2m-1} g_{2m-1}(s)$ is anti-D-stable. Continuing with this process repeatedly, we will eventually have either $f_0(s)$ or another vertex polynomial of P to be anti-D-stable. This conclusion contradicts the assumption that V_P is D-stable. Therefore, $f_{2m}(s, \alpha)$ must be D-stable. Since $f_{2m}(s, \alpha)$ is an arbitrary polynomial in P, D must be a Kharitonov region with respect to $\mathbf{p}$. □

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p	Kharitonov region D	condition
$(p_0(s), p_1(s), \cdots, p_m(s))$	Figs. 1, 4	$p_i(s)$ is anti-D-stable
	$\mathbf{C}_-$	Re $p_i(j\omega) \equiv 0$ or Im $p_i(j\omega) \equiv 0, 1 \le i \le m$
$(p_0(s), 1, s, \cdots, s^n)$	Figs. 2, 3, 6, 7, 8	no
	Fig. 9	real parameters and coefficients
$(p_0(s), 1)$	any open convex set	no
$(p_0(s), s^n)$	any open convex set such that if $d \in D$ then $d \neq 0$ and $d^{-1} \in D$ (e.g., Fig. 11)	no

Table 1: Kharitonov regions for continuous-time systems

p	Kharitonov region D	condition
$(p_0(z), p_1(z), \cdots, p_m(z))$	the open unit disk or any open circular region inside of it	$p_i(z)$ is anti-D-stable
$(p_0(z), 1, z, \cdots, z^n)$	Fig. 12	no
$(p_0(z), 1, z, \cdots, z^{\lfloor n/2 \rfloor})$	the open unit disk	no
$(p_0(z), 1 \pm z^n, \cdots, z^{\lfloor n/2 \rfloor} \pm z^{n - \lfloor n/2 \rfloor})$	the open unit disk	no

Table 2: Kharitonov regions for discrete-time systems

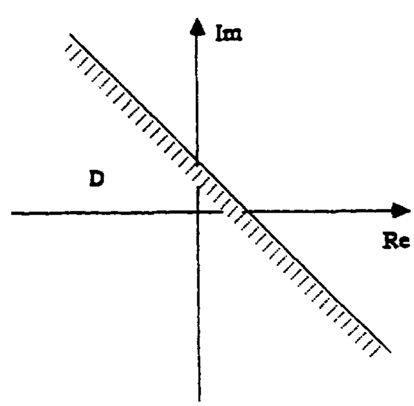


Figure 1

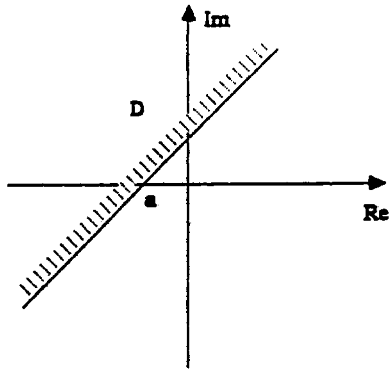


Figure 2

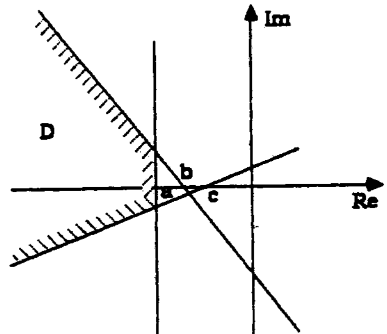
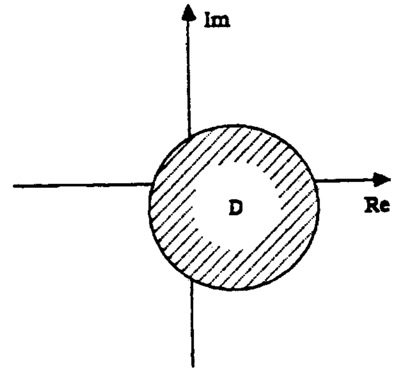


Figure 3



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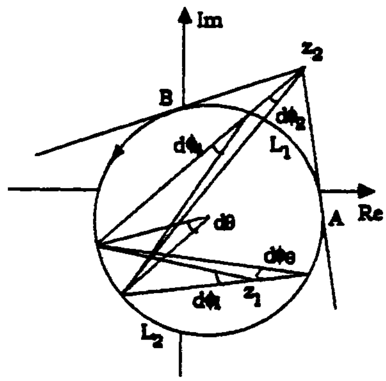

Figure 5

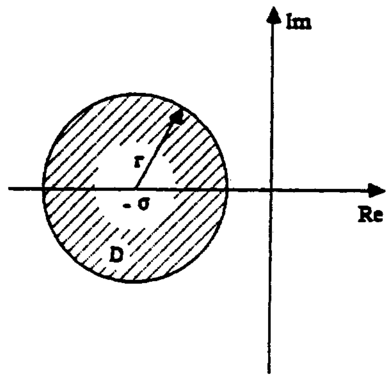

Figure 6

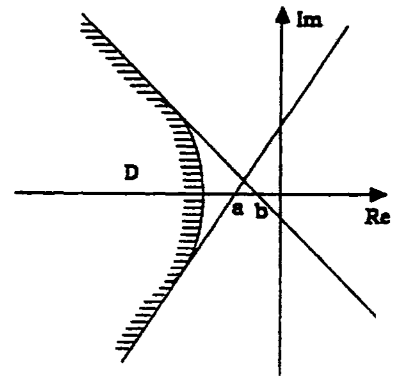

Figure 7

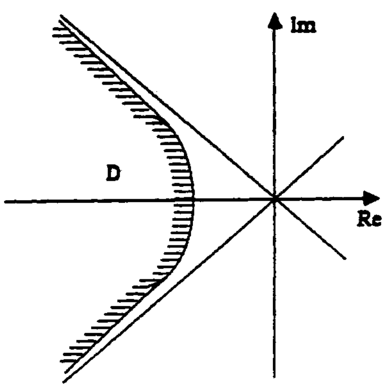

Figure 8

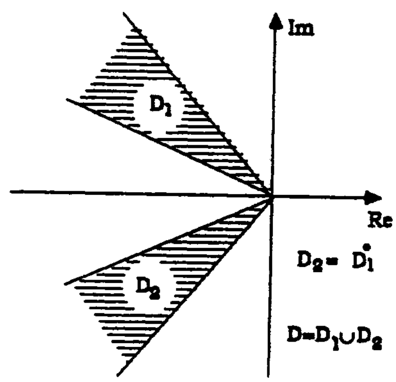

Figure 9

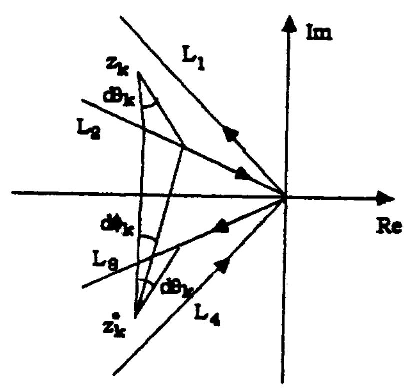

Figure 10

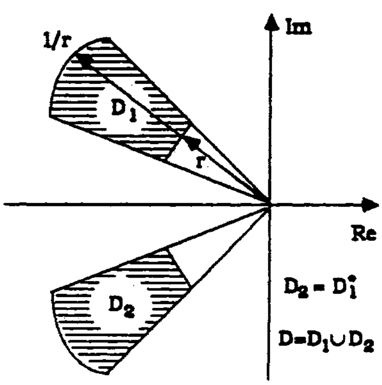

Figure 11

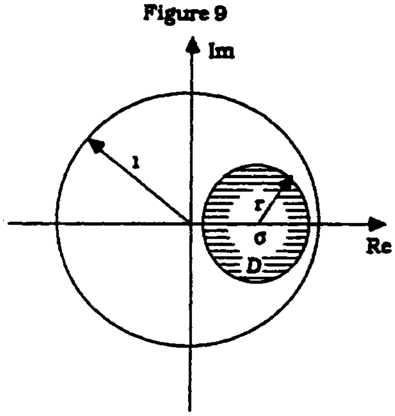

Figure 12

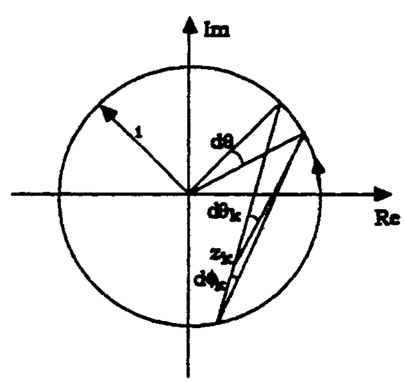

Figure 13