ROBUST STABILITY UNDER A CLASS OF NONLINEAR PARAMETRIC PERTURBATIONS

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Abstract
This paper considers the Hurwitz invariance of LTI systems under a class of nonlinear parametric perturbations. The setting is one of determining the closed loop stability of systems whose open loop transfer functions consist of powers, products and ratios of polytopes of polynomials. We give several results on the construction of value sets and Hurwitz invariance checking, for this general setting and important special cases.

1. INTRODUCTION

The following problem is of interest in the verification of robust stability of linear time-invariant systems. Determine as simply as possible if all members of a family of transfer functions, parameterized by a real vector γ, have roots contained entirely in the open left half plane (i.e., the family is Hurwitz invariant), when γ takes values from a given set. Such a family, $H(\Gamma)$, can be described by

$$H(\Gamma):=\{h(s,\gamma):\ \gamma\in\Gamma\} \tag{1.1}$$

where Γ is a connected set in $\mathbf{R}^N$, and for each $\gamma\in\Gamma$, $h(s,\gamma)$ is a transfer function. Generally, the transfer function coefficients depend nonlinearly on γ.

One approach to this problem is to treat it in its broadest generality, as is done in [1]. Alternatively, one can consider particular parametrizations reflecting specific forms of structural information supplied by the modelling process. This allows formulation of stability verification schemes which are computationally less demanding. Examples of this approach include [2], which considers a family of polynomials admitting independent variations in the coefficients; [3-5], which account for affinely dependent variations; and [6-8], which consider multilinear dependence (see [9,10] for surveys). Each of [2-8] exploits the underlying structural information and demonstrates consequent simplifications.

This paper adopts the second approach by focussing on a special class of nonlinear parametric dependence. To keep the presentation simple, only Hurwitz invariance is investigated, though the results do in fact generalize to more general stability regions.

In (1.1), assume Γ to be a hyperrectangle (axis parallel box) and partition the parameter vector γ as

$$\gamma=(\gamma_1^T\ \gamma_2^T\cdots\gamma_n^T)^T \tag{1.2}$$

where $\gamma_i\in\Gamma_i$, $i=1,2,...,n$; with each Γ_i an axis parallel box in $\mathbf{R}^{N_i}$. Then the function $h(s,\gamma)$, $\gamma\in\Gamma$, under consideration is given as

$$h(s,\gamma)=g_1(s)+g_2(s)\prod_{i=1}^{n}(p_{i0}(s)+\gamma_i^T\mathbf{P}_i(s))^{k_i}, \tag{1.3}$$

where $g_1(s)$ and $g_2(s)$ are real rational functions in s, $p_{i0}(s)$ are real scalar polynomials, $\mathbf{P}_i(s)$ are real vector polynomials with dimension N_i, and k_i are fixed nonzero integers *allowed to be positive and negative*. The quantities $g_i(s)$, $p_{i0}(s)$, $\mathbf{P}_i(s)$, n, N_i, k_i and Γ are assumed known. The j-th element of γ_i (resp. $\mathbf{P}_i(s)$) is denoted by γ_{ij} (resp. $\mathbf{P}_{ij}(s)$). Since Γ is an axis parallel box, each γ_{ij} varies independently of the others within an interval

$$\gamma_{ij}^-\le\gamma_{ij}\le\gamma_{ij}^+ . \tag{1.4}$$

Thus, each factor $(p_{i0}(s)+\gamma_i^T\mathbf{P}_i(s))$ forms a *polytope* of polynomials as γ_i varies in Γ_i. Notice, in the case where the exponents k_i are restricted to be +1 and/or -1, the stability verification of (1.3) is equivalent to that of a subclass of the multilinear problem.

One motivation for considering the function (1.3) is that in many cases a plant consists of a cascade of subsystems. Separate information may be available for each and their physical independence may lead to independent parameter variations. Further, one may wish to verify if each member of a family of controllers, parameterized in a manner similar to the component subsystems, stabilizes the entire set of plants. It is clear that such a stability verification reduces to the problem we consider.

To further illustrate the scope of (1.3), two more examples are considered. The first example is of a plant with independent real zero, pole and gain variations and is as follows:

$$h(s)=d\,h_0(s)\frac{(s+\lambda_1)^2(s+\lambda_2)}{(s+\lambda_3)(s+\lambda_4)(s+\lambda_5)^3} \tag{1.5}$$

Here $h_0(s)$ is a fixed rational function and d and λ_i vary independently within given bounds. The objective is to verify if a given compensator c(s) stabilizes the plant for all possible parameter variations. Then, the characteristic function of the closed loop system fits the function in (1.3). To allow for complex zero and pole variations, one may include factors of the form $(s^2+a_is+b_i)^{k_i}$, with a_i and b_i also varying independently in intervals. With some abuse of terminology, the version of (1.3) given in (1.6) below will be referred to as the complex zero-pole-gain variation problem:

$$h(s,\gamma)=1+d\,c(s)\prod_{i=1}^{\tau}(s+\lambda_i)^{k_i}\prod_{j=\tau+1}^{n-1}(s^2+a_js+b_j)^{k_j} \tag{1.6}$$

with $\gamma:=(\lambda_1\ \lambda_2\ ...\lambda_\tau\ a_{\tau+1}\ b_{\tau+1}\ ...\ a_{n-1}\ b_{n-1}\ d)^T$. In this paper, the following assumptions are made.

Assumption 1.1 The function $h(s,\gamma)$ has no unstable zero-pole cancellation for any $\gamma\in\Gamma$.

Assumption 1.2 Continuous variations of γ result in continuous changes in the roots of $h(s,\gamma)$.

Assumption 1.3: The function $h(s,\gamma)$ has no purely imaginary poles for any $\gamma\in\Gamma$. $\nabla\nabla\nabla$

The need for these assumptions is explained in section 1.1 below.

1.1 Approach and Main Results: As in [7,8,11,12], we follow a powerful approach to robust stability analysis -- the so-called *value set approach*. For the family of rational functions (1.1), the value set, $H(\omega,\Gamma)$, at a frequency ω is defined as

$$H(\omega,\Gamma):=\{h(j\omega,\gamma):\ \gamma\in\Gamma\}. \tag{1.8}$$

This method exploits the *Zero Exclusion Principle* [8], which under assumptions (1.1,1.2), states that the following conditions are necessary and sufficient for the Hurwitz

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invariance of $H(\Gamma)$: (i) at least one member of $H(\Gamma)$ is Hurwitz; and (ii) $0 \notin H(\omega, \Gamma)$, for all $\omega \in \mathbf{R}$. Assumption (1.3) ensures that at all finite frequencies $H(\omega, \Gamma)$ is a bounded set. In fact further refinements to (ii) are possible.As $h(s,\gamma)$ is a real function, a slight variation of an argument in [13] shows that (ii) above can be simplified to be(ii-a) $0 \notin \partial(H(\omega, \Gamma))$, $\forall\ \omega \in \mathbf{R}$, where ∂ denotes the boundary.

Besides aiding robust stability analysis, value sets can be regarded as a generalized Nyquist plot [14]. Consequently, they can be effectively employed in designing controllers which meet performance considerations that go beyond mere closed loop stabilization. Accordingly, in this paper we consider both the determination of value sets for (1.3) as well as its Hurwitz invariance. Throughout, to avoid trivialities, we will always assume that at least one member of $H(\Gamma)$ is Hurwitz. In particular, the following results are derived.

(1) We show that for the general family (1.3), at each fixed frequency ω, each member of $\partial H(\omega,\Gamma)$ has preimages in certain line segments in the parameter set Γ. These critical line segments are simply characterized. Some of them are the edges of Γ and the rest, frequency dependent intenal lines. The mathematical descriptions of these line segments are independent of the exponents k_i (see (1.3)) although certain combinations of k_i help to drastically reduce their number.

(2) The ability to link the value set boundary to (one dimensional) line segments considerably simplifies the verification of Hurwitz invariance. In particular, every line segment can be represented in terms of a single real variable. This fact is exploited to show that Hurwitz invariance (1.3) is equivalent to checking a finite number of continuous and piecewise differentiable scalar functions in ω for avoidance of the negative real axis. Generally, each such function can be linked to one particular value set boundary defining frequency dependent line segment mentioned in (1).

(3) For the special case of real zero-pole-gain varations, we show that the critical segments are *frequency independent*, and are in fact the edges of Γ plus certain simply constructable 45° line segments in the parameter space. Consequently, the Hurwitz invariance of $H(\Gamma)$ is guaranteed by that of these frequency independent lines segments (including the edges). We also show via an example that such 45° line segments are in fact necessary for determining Hurwitz invariance, in that despite the Hurwitzness of the edges some of these 45^0 segments may have non-Hurwitz members.

(4) For the complex zero-pole-gain variation case the critical lines determining the value set boundaries are either frequency independent or as frequency varies, vary on certain (2-dimensional) planes and certain (3-dimensional) boxes in Γ. To check for robust Hurwitzness it then suffices to check these frequency independent lines, planes and boxes.

With respect to (1)-(4), we argue that, often many of the critical lines, planes and boxes may not pass through the parameter set Γ. For example, in (3), this occurs when the real poles and zeros vary in non-overlapping intervals. In such cases only the edges of Γ need be considered. Section 2 presents preliminaries; (3) and (4) are considered in sections 3 and 4 respectively. Section 5 addresses (1) and (2) for the general family (1.3). Detailed proofs are in [15].

2. PRELIMINARIES

In this section, we present two preliminary Lemmas which help prove the main results of this paper. Each of these concerns the extraction of a critical subset of the parameter box whose image forms the boundary of the value set.

In Lemma 2.1 we consider the hyperrectangle

$$Q := \{\ \mathbf{q}=(q_1\ q_2 \ldots q_m)^T:\ q_i^- \le q_i \le q_i^+,\ i=1,2,\ldots m\} \quad (2.1)$$

in $\mathbf{R}^m$ and a complex function $f(\cdot)$: $Q \to \mathbf{C}$:

$$f(\mathbf{q}) = \Delta_1 + \Delta_2 \prod_{i=1}^{m} (q_i + \alpha_i)^{k_i} \quad (2.2)$$

Here, the Δ_i are complex constants and α_i are real constants. In the sequel the set $f(Q)$ will denote $f(Q) := \{\ f(\mathbf{q}) : \mathbf{q} \in Q\}$ The following assumption is required.

Assumption 2.1 If $\beta_i = 0$ and $k_i < 0$ for some i, then the associated interval $[q_i^- + \alpha_i\ ,\ q_i^+ + \alpha_i]$ does not contain zero; i.e. $f(Q)$ is a bounded set. ∇∇∇

It is our objective to extract a subset of Q which determines the boundary of $f(Q)$. It turns out that the following affine line in $\mathbf{R}^m$ plays a crucial role in this task:

$$L_f := \{(q_1..q_m)^T = \rho[\beta_1..\beta_m]^T - [\alpha_1..\alpha_m]^T:\ -\infty < \rho < \infty\} \quad (2.3)$$

In the Lemma the term *open projection,* defined below, is used. Consider a k-dimensional boundary Q_k of Q. Note that there are (m-k) elements of $\mathbf{q}$ fixed at extreme values on this boundary. Modify L_f by fixing these (m-k) parameters in $(q_1\ q_2 \ldots q_m)^T$ to the extreme values as they assume in Q_k. Then the intersection of the *interior* of Q_k and this modified affine line is called the *open projection* of L_f on Q_k. For example, if $m = 3$, $Q = [0, 2] \times [0, 4] \times [0,5]$ and $L_f := \{(q_1, q_2, q_3)^T = \rho[1, 1, 1]^T - [0, 1, 1]^T: -\infty < \rho < \infty\}$, then the projection of L_f on Q is $\{(q_1, q_2, q_3)^T = \rho[1, 1, 1]^T - [0, 1, 1]^T: 1 < \rho < 2\}$. Equally, the projection on the face defined by $q_1 = 2$ is $\{(q_1, q_2, q_3)^T = \rho[0, 1, 1]^T - [-2, 1, 1]^T: 1 < \rho < 5\}$.

In the sequel we will use certain short hand expressions whose meaning we now make clear. We say that "(f(Q)) *is mapped from a subset* Q* of Q" if *every point on* $\partial(f(Q))$ *has at least one preimage in* Q*. Such a set Q* will be called *critical*. We have the following Lemma.

Lemma 2.1: Consider the hyperrectangle Q in $\mathbf{R}^m$ given in (2.1) and the complex function $f(\cdot)$: $Q \to \mathbf{C}$ in (2.2). Then under Assumption 2.1, there exists a collection of line segments in Q from which $\partial(f(Q))$ is mapped. These line segments consist of (a) all the edges of Q and (b) the open projections of the affine line L_f in (2.3) on all k-dimensional boundaries Q_k of Q, $2 \le k \le m$, except those for which the following condition holds: let Ψ be the subset of $\{1,2,\ldots,m\}$ associated with Q_k such that q_σ is set at an extreme value for all $\mathbf{q}=(q_1, q_2,\ldots, q_m)^T \in Q_k$, $\sigma \in \Psi$. Then, either

(i) $\beta_i = 0$, for some $i \notin \Psi$; or

(ii) $\sum_{i \notin \Psi} k_i = 0$. (2.4)

Remark 2.1: in order to illustrate this Lemma, especially conditions (b-i) and (b-ii), we consider the following example:

$$f(q) = \frac{(q_1 + 1 + j)}{(q_2 + 2 + 3j)(q_3 + 4)}.$$

Then projections on the interior of Q and the faces $q_1 = q_1^-$ or q_1^+ and the faces $q_2 = q_2^-$ or q_2^+ are excluded by the restriction (i) above; likewise the projections on the faces at which q_3 takes extreme values are excluded by (ii). Thus for this example the edges comprise the critical set.

Remark 2.3: The exponents k_i do not affect the equations of the internal segments. Certain combinations of k_i, however, may reduce the number of segments which are critical ((b-ii)).

Remark 2.4: It can be shown that, for each $1 \le k \le m$, there are $C_m^k 2^{m-k}$ k-dimensional boundaries. Therefore, the total number of critical segements which determine the boundary of f(Q) is at most $\sum_{k=1}^{m} C_m^k 2^{m-k} = 3^m - 2^m$. From Remark 2.1, many of these are unnecessary.

To understand the significance of Lemma 2.1 to the problem at hand, note that our objective is to extract a set $\Gamma^*(\omega)$ from Γ, such that at a given ω, the boundary of the value set of (1.3) $H(\omega,\Gamma)$, is mapped from $\Gamma^*(\omega)$. In this context, the following Lemma is shown in [15].

Lemma 2.2: At any frequency, the $\Gamma^*(\omega)$ defined above, lies in the union of certain n-dimensional boundaries of Γ. On each of these boundaries all but one element of each γ_i (see (1.3)), are fixed at their extreme values. ∇∇∇

These boundaries will be denoted B_j, $j \in \{1,...,\nu\}$, where ν, the total number of these boundaries is

$$\nu = 2^{N-n} \prod_{i=1}^{n} N_i.$$

To find $\Gamma^*(\omega)$, one simply needs to find critical subsets of each B_j, such that the boundary of the image of B_j in the value set space is mapped from these subsets. Together, these subsets then define $\Gamma^*(\omega)$. Using Lemma 2.1 we will identify these critical subsets of B_j, with certain line segments which in turn are the projection of some affine lines on the boundaries of B_j.

3. REAL ZERO-POLE-GAIN-VARIATIONS

Consider,

$$h(s,\gamma) = 1 + d\, h_0(s)((s+\lambda_1)^{k_1}(s+\lambda_2)^{k_2}...(s+\lambda_{n-1})^{k_{n-1}}) \quad (3.1)$$

where

$$\gamma := (\lambda_1\ \lambda_2\ ..\lambda_{n-1}\ d)^T, \quad (3.2)$$

$h_0(s)$ is a fixed rational function, $k_i \neq 0$ are fixed integers which can be either positive or negative, and d and λ_i vary independently within given bounds, i.e.,

$$d^- \le d \le d^+;\ \lambda_i^- \le \lambda_i \le \lambda_i^+,\quad i=1,2,...n-1 \quad (3.3)$$

and $\Gamma = [\lambda_1^-, \lambda_1^+]x...x[\lambda_{n-1}^-, \lambda_{n-1}^+]x[d^-, d^+]$. Assumption 1.3 implies that if a given k_i is negative, the corresponding interval of λ cannot include zero. At each fixed frequency ω, the function $h(j\omega, \gamma)$ has the same form as the function f(q) in (2.2) with obvious identifications with the Δ_i, α_i and β_i. Thus, Lemma 2.1 applies. The critical line L_f given in (2.3) becomes

$$L_f = \{\lambda = \rho[\omega\ \omega\ ...\ \omega\ 0]^T: -\infty < \rho < \infty\}. \quad (3.4)$$

For $\omega \neq 0$, renaming ρ to be $\rho\omega$, we have

$$L_f = \{\lambda = \rho[1\ 1\ ...\ 1\ 0]^T: -\infty < \rho < \infty\}. \quad (3.5)$$

This line must be projected to certain boundaries of Γ. Observe $\beta_n = 0$. Thus from (b-i) of Lemma 2.1, these boundaries are such that on each of them the gain d must take extreme values. Consider any such boundary, in the interior of which λ_i, $i \in \Psi$, take extreme values. From Lemma 2.1, the projection of L_f on this boundary is critical only if (2.4) does not hold. Under this condition, the critical projection obeys

$$\lambda_i = \lambda_j,\ \ \forall\ i,j \notin \Psi. \quad (3.6)$$

with d and the other λ_i fixed at extreme values. At $\omega=0$, all $\beta_i=0$. Then, Lemma 2.1 ensures that $\partial H(0,\Gamma)$ is mapped from the edges. Thus we have the following Theorem.

Theorem 3.1: Consider the parameter box Γ in $\mathbf{R}^n$ given in (3.4) and the family (3.1). Then, at each ω the $\partial H(\omega,\Gamma)$ can be mapped from certain line segments in Γ, namely the edges of Γ and the projections given in (3.6). Furthermore, the family of functions $H(\Gamma)$ is Hurwitz invariant iff $h(s,\gamma)$ is Hurwitz invariant on these line segments. ∇∇∇

A case with special interest is when all the λ_i vary in intervals that do not overlap. Then the projections are empty. Thus, Corollary 3.1, below, follows.

Corollary 3.1 (An Edge Theorem): Consider the parameter box Γ in (3.4) and the family of function $H(\Gamma)$ in the form of (3.1). Suppose $(\lambda_i^-, \lambda_i^+) \cap (\lambda_j^-, \lambda_j^+)$ are empty for all $1 \le i < j \le n$. Then the boundary of the value set $H(\omega,\Gamma)$ at any frequency ω is mapped from the edges of Γ. Thus, $H(\Gamma)$ is Hurwitz invariant iff all the edges of $H(\Gamma)$ are Hurwitz invariant. ∇∇∇

For non-overlapping intervals of λ_i, however, the 45° line segments are indeed necessary. To show this, we provide the following simple example. For $\lambda_1, \lambda_2 \in [-30, 0]$

$$h(s,\lambda_1,\lambda_2) = 1 + \frac{0.1(0.8s^2+0.8s+4.4)(s+\lambda_1)(s+\lambda_2)}{s^4+10s^3+11.8s^2+11.8s+0.2}.$$

In this example, there are four edges with the associated transfer functions given by: $h(s,0,\lambda_2), \lambda_2 \in [-30, 0]$; $h(s,-30, \lambda_2), \lambda_2 \in [-30, 0]$; $h(s, \lambda_1, 0), \lambda_1 \in [-30,0]$; and $h(s,\lambda_1,-30), \lambda_1 \in [-30, 0]$. There is only one 45° line segment given by $h(s, \lambda, \lambda)$, $\lambda \in [-30, 0]$. It is straightforward to verify that the transfer functions on all the edges are Hurwitz but some on the 45° line segment are not. For example at $\lambda=-15$, $h(s,\lambda,\lambda)$ has the unstable roots $0.2424 \pm 1.8914j$.

4. COMPLEX ZERO-POLE-GAIN VARIATIONS

In this section (1.6) is treated. Assume first that $\omega \neq 0$. It becomes clear from Lemma 2.2 that at any ω, $\partial H(\omega,\Gamma)$ comes from boundaries of Γ on which one and only one parameter from each factor of (1.6) does not take an extreme value. Consider the interior of any such boundary B: on it certain $a_i, i \in \psi_a$ and b_i, $i \in \psi_b$ take extreme values. Then, with γ on this boundary, for some $\hat{c}(j\omega)$,

$$h(j\omega,\gamma)=1+\hat{c}(j\omega)\prod_{i=1}^{\tau}(\lambda_i+0+j\omega)^{k_i}$$

$$\{\prod_{i\in\psi_a}(b_i-\omega^2+j\omega a_i)^{k_i}\}\{\prod_{i\in\psi_b}(a_i+0+j\frac{\omega^2-b_i}{\omega})^{k_i})\}(d+j0)$$

As $a_i, i\in\psi_a$ and b_i, $i\in\psi_b$ take fixed extreme values on B, we have precisely the situation of Lemma 2.1, with the following definitions in vogue: Q=B,

$$(q_i, \alpha_i, \beta_i) = (\lambda_i, 0, \omega);\ \forall\ i\in \{1,....,\tau\} \qquad (4.1)$$

$$(q_i, \alpha_i, \beta_i) = (b_i, -\omega^2, \omega a_i);\ \forall\ i\in \psi_a \qquad (4.2)$$

$$(q_i, \alpha_i, \beta_i) = (a_i, 0, \frac{\omega^2-b_i}{\omega});\ \forall\ i\in \psi_b \qquad (4.3)$$

$$(q_n,\alpha_n,\beta_n)= (d,0,0). \qquad (4.4)$$

Notice in (4.1-4.4), all the α_i ,β_i are fixed real numbers for a fixed ω. Thus Lemma 2.1, states that the critical subset of B consists of its edges (which comprise some of the edges of Γ) and the projection of the affine line in (2.3) on all the boundaries of B, subject to (b-i and ii) in the Lemma.

Thus, a typical critical segment is the intersection of the following line with Γ

$$\hat{\gamma} = -\alpha(\omega) + \rho\beta(\omega);\ -\infty < \rho < \infty, \qquad (4.5)$$

where the elements of $\hat{\gamma}$ are certain λ_i, a_i and b_i; all parameters not in $\hat{\gamma}$ are fixed at extreme values; the respective elements of $\alpha(\omega)$ and $\beta(\omega)$ are corresponding α_i ,β_i in (4.1-4.3); d can never be an element of $\hat{\gamma}$, neither can a_i if b_i is an element or if $b_i = \omega^2$ ((i) of Lemma 2.1 holds) nor b_i if a_i is an element or if $a_i = 0$. Further if Ψ is the set indexing the q_i of (4.1-4.3) not in $\hat{\gamma}$, then (2.4) does not hold. Then the following Theorem follows.

Theorem 4.1: For, the (1.6), $\partial(H(\omega,\Gamma))$ at $\omega\neq 0$ is mapped from the edges of Γ and the internal segments characterized above. At $\omega=0$, $\partial(H(\omega,\Gamma))$ is mapped from the edges of Γ alone. ∇∇∇

The segments on which only λ_i vary and all a_i and b_i are fixed at extreme values are frequency invariant and are in fact 45^0 lines of section 3. *The set of all such segments will be called L_1*. Segments on which some of the a_i or b_i vary are, however, frequency dependent but happen to be contained entirely in certain 2-dimensional planes and 3-dimensional boxes which are *frequency independent*. To explain this we will consider three types of frequency dependent segments.

Type 1: No a_i is a variable. Consider a representative segment of this type obeying (4.5). We will show that for all frequencies (4.5) reduces to

$$\hat{\gamma} = \rho_1 C_1 + \rho_2 C_2;\ \forall\ [\rho_1,\rho_2] \in R^2 \qquad (4.6)$$

with the C_i *frequency independent*. Equation (4.6) is a plane and its intersection with Γ describes the critical set which contains the relevant segment at all frequencies. For, as on this segment all a_i are fixed at extreme values, and not elements of $\hat{\gamma}$, the variable λ_i, b_i obey

$$\lambda_i = 0 + \rho\omega \qquad (4.7)$$

$$b_i = -\omega^2 + \rho\omega a_i \qquad (4.8)$$

Then with C_1 and C_2 obviously defined (4.5) reduces to

$$\hat{\gamma} = \omega^2 C_1 + \rho\omega C_2 \qquad (4.9)$$

whence the description in (4.6) with $\rho_1= \omega^2$ and $\rho_2=\rho\omega$.

Type 2: No b_i is a variable. All b_i are fixed and the variable λ_i and a_i obey (4.7) and

$$a_i = 0 + \rho\omega(1-\frac{b_i}{\omega^2}). \qquad (4.10)$$

Then with the vectors C_i obviously defined, (4.5) becomes

$$\hat{\gamma} = \rho\omega C_1 + \frac{\rho}{\omega} C_2 \qquad (4.11)$$

whence (4.6) obtains with $\rho_1=\rho\omega$ and $\rho_2=\rho/\omega$.

Type 3: For some i, a_i is a variable for some others b_i is a variable. In this case for the elements in $\hat{\gamma}$ corresponding to the variables λ_i, b_i and a_i obey (4.8,4.9, 4.11) respectively. Note in the latter two, the a_i or b_i appearing in the right hand side are fixed for the segment in question. Then with the C_i obviously defined, (4.5) reduces to

$$\hat{\gamma} = -\omega^2 C_1 + \rho\omega C_2 + \frac{\rho}{\omega} C_3 \qquad (4.12)$$

Notice, now this segment is confined at all frequencies on the intersection of the following *three dimensional box* with Γ.

$$\hat{\gamma} = \rho_1 C_1 + \rho_2 C_2 + \rho_3 C_3,\ \forall\ [\rho_1,\rho_2,\rho_3] \in R^3. \qquad (4.13)$$

The sets of planes and boxes characterized above will be called L_2 and L_3 respectively. The zero exclusion principle then immediately yields the following Theorem.

Theorem 4.1: With L_1,L_2 and L_3 defined as above, the family (1.6) is Hurwitz invariant iff all the edges, the internal segments in L_1, the planes in L_2 and the boxes in L_3 are Hurwitz invariant. ∇∇∇

As in section 3 many of these segments, planes and boxes will be empty.

5. THE GENERAL CASE

In Sections 5.1we show that for the general family (1.3), the boundary of the value set can be mapped from certain critical line segments in Γ, though these vary with frequency and are not necessarily confined to 2 and 3-dimensional spaces. In section 5.2 we show that (1.3) is Hurwitz invariant iff certain scalar functions in *one variable* avoid the negative real axis.

5.1 Value Set Boundaries: From Lemma 2.2 and subsequent discussion, we must show that the critical subsets of B_j are certain line segments denoted $L_{ij}(\omega)$. The use of Lemma 2.1 becomes possible because on each B_j only one member of each γ_i varies. To elaborate, consider

$$h(j\omega,\gamma)=[(\omega^2-2)\gamma_{11}+(j\omega+1)\gamma_{12}][(-\omega^2+1+j\omega)\gamma_{21}+j\omega\gamma_{22}] (5.1)$$

Consider, the boundary B_1 given by $\gamma_{12}=\gamma\backslash S(-,12)=1$ and $\gamma_{22}=\bar{\gamma}_{22}=1$. At any fixed ω, $\omega^2\neq 2$, $\forall\gamma\in B_1$, $h(j\omega,\gamma)$ equals

$$(\omega^2-2)\{1-\omega^2+j\omega\}[\gamma_{11}+\frac{1}{(\omega^2-2)}+\frac{j\omega}{(\omega^2-2)}][\gamma_{21}+\frac{j\omega(1-\omega^2)}{(1-\omega^2)^2+\omega^2}+\frac{\omega^2}{(1-\omega^2)^2+\omega^2}]$$

and has the form demanded by Lemma 2.1 with γ_{11} and γ_{21} identified with q_1 and q_2; $h(j\omega,\gamma)$ with $f(q)$ and B_1 with Q. Thus for this choice of B_1 the internal segments for B_1 are the projections of the following line

$$[\gamma_{11},\gamma_{21}]^T = [-\frac{1}{(\omega^2-2)}, -\frac{\omega^2}{(1-\omega^2)^2+\omega^2}]^T + \rho\,[\frac{\omega}{(\omega^2-2)}, \frac{\omega(1-\omega^2)}{(1-\omega^2)^2+\omega^2}]^T.$$

on all the boundaries of B_1 of dimension 2 or more. Of course in this case there is only one such boundary namely B_1 itself. Further, at $\omega=0$ or 1, $\beta_2=0$ and the segment is not critical. At $\omega^2=2$, $h(j\omega,\gamma)$ is independent of γ_{11}. At this frequency one can thus assign an extreme value to γ_{11} without altering $h(j\omega,\gamma)$. The resulting γ is on an edge of B_1. Thus, at $\omega^2=2$, the number of critical segements is reduced. These facts are formalized in the following Theorem and the precise description of critical segments that follows the theorem statement.

Theorem 5.1: The subset $\Gamma^*(\omega)$ obeys

$$[\bigcup_{j=1}^{v} \bigcup_{i=1}^{v_j(\omega)} L_{ij}(\omega)] \supset \Gamma^*(\omega) \qquad (5.2)$$

where $v_j(\omega) \le 3^n - 2^n$ and $L_{ij}(\omega)$ are the edges of Γ and some open internal line segments.∇∇∇

The lines $L_{ij}(\omega)$: Consider a typical B on which for each i, only one $\gamma_{i\sigma_i}$ is not fixed at an extreme value and is thus a variable. Express

$$p_{i0}(j\omega) + \gamma_i^T P_i(j\omega) = \gamma_{i\sigma_i} f_{i1}(\omega) + f_{i2}(\omega) \qquad (5.3)$$

with obvious definitions of $f_{i1}(\omega)$ and $f_{i2}(\omega)$. Consider the following two cases.

Case I: $f_{i1}(\omega)\neq 0$, $\forall i\in \{1,\ldots,n\}$. We apply Lemma 2.1. To make the notation consistent, define $\mathbf{q} = [q_1,\ldots,q_n]^T = [\gamma_{1\sigma_1},\ldots,\gamma_{n\sigma_n}]^T$. Observe, with

$$\{\Delta_2(\omega),\alpha_i(\omega),\beta_i(\omega)\} = \{g_2(j\omega)\prod_{i=1}^{n}(f_{i1}(\omega)),\ \frac{Re[f_{i2}(\omega)f_{i1}^*(\omega)]}{|f_{i1}(\omega)|^2}, \frac{Im[f_{i2}(\omega)f_{i1}^*(\omega)]}{|f_{i1}(\omega)|^2}\},$$

$h(j\omega,\gamma)$ has precisely the form of $f(q)$ in Lemma 2.1 for all $\gamma \in B$. It follows that the critical segments comprise the edges of B and the open projections of the line L_f of Lemma 2.1 on certain boundaries of B. Consider a boundary of B and Ψ such that in the interior of this boundary, only $\gamma_{i\sigma_i}$ $\forall i\notin\Psi$, are not fixed at extreme values. Then the projection of L_f on this boumdary is critical only if (2.4) and (5.4) below hold.

$$Im[f_{i2}(\omega)f_{i1}^*(\omega)] \neq 0\ ,\ \forall i\notin \Psi \qquad (5.4)$$

Under these conditions, for each σ_i, $i\notin\Psi$ define

$$\{u_i(\omega),v_i(\omega)\} := \{Re[f_{i2}(\omega)f_{i1}^*(\omega)], Im[f_{i2}(\omega)f_{i1}^*(\omega)]\}; \qquad (5.5)$$

$U(\omega)$ as the vector of $u_i(\omega)$, $\forall i\notin\Psi$; $V(\omega)$ similarly; and the matrix $F(\omega) = diag\{|f_{i1}(\omega)|^2, \forall i\notin\Psi\}$. Then a typical $L_{ij}(\omega)$ is the open segment

$$F(\omega)\hat{\gamma} = -U(\omega) + \rho\, V(\omega), \quad \mu_1(\omega) < \rho < \mu_2(\omega) \qquad (5.6)$$

where the $\mu_i(\omega)$ are given as follows. Consider the quantities

$$\frac{|f_{i1}(\omega)|^2 \hat{\gamma}_i^- + Re[f_{i2}(\omega)f_{i1}^*(\omega)]}{Im[f_{i2}(\omega)f_{i1}^*(\omega)]} \qquad (5.7)$$

$$\frac{|f_{i1}(\omega)|^2 \hat{\gamma}_i^+ + Re[f_{i2}(\omega)f_{i1}^*(\omega)]}{Im[f_{i2}(\omega)f_{i1}^*(\omega)]} \qquad (5.8)$$

For each $i\notin\Psi$ call the smaller of these to be $x_i(\omega)$ and the larger $y_i(\omega)$. Then

$$\{\mu_1(\omega),\mu_2(\omega)\} = \{\ \sup_{i\notin\Psi}[x_i(\omega)]\ ,\ \inf_{i\notin\Psi}[y_i(\omega)]\ \}. \qquad (5.9)$$

If $\mu_1(\omega) \ge \mu_2(\omega)$ the open segment is empty.

Case II: $f_{i1}(\omega)=0$, $\forall i\in S$ where S is a subset of $\{1,\ldots,n\}$. In this case the value set is *independent of* $\gamma_{i\sigma_i}$, $\forall i\in S$. Thus these parameters can be assigned extreme values without affecting the value set. A slight variation of the preceding argument shows that a typical internal critical segment is as above with the added restriction that $\gamma_{i\sigma_i}$ *is not a variable if* $f_{i1}(\omega) = 0$.

5.2 Hurwitz invariance:

Consider the segment of (5.4-5.9). At certain frequencies it may either cease to exist or may no longer be critical. Such situations will be called degeneracies. There are four types of frequencies at which this happens. Three of these have already been covered; i.e. (a) when for some i, $f_{i1}(\omega)=0$; (b) when $Im[f_{i2}(\omega)f_{i1}^*(\omega)]=0$ or (c) when $\mu_1(\omega) \ge \mu_2(\omega)$. For the fourth situation consider

$$h(j\omega,\gamma) = g_1(j\omega) + G_2(\omega)\prod_{i\notin\Psi}(p_{i0}(s) + \gamma_i^T P_i(s))^{k_i}, \qquad (5.10)$$

(all the factors of (1.3),devoid of variables for the segment being considered are subsumed in the fixed function $G_2(\omega)$). Suppose, $G_2(\omega)=0$. At such a frequency $h(j\omega,\gamma) = g_1(j\omega)$, irrespective of the choice of γ_i $\forall i\notin\Psi$; in particular elements of these vectors could all take extreme values without altering the value of $h(j\omega,\gamma)$. Thus the value set of the entire segment is covered by the image of a corner and at these frequencies, this segment is not critical. So the fourth degeneracy represents frequencies at which $G_2(\omega)=0$. The set of frequencies representing degeneracies will be called Ω.

Then one can associate with each $L_{ij}(\omega)$ a scalar continuous, piecewise differentiable function $\xi_{ij}(\omega)$ having the properties: (i) for the Ω associated with a given $L_{ij}(\omega)$, $\xi_{ij}(\omega)=0$, $\forall\omega\in\Omega$; (ii) at all other ω the image of $L_{ij}(\omega)$ in the value set space is zero exclusive, iff $\xi_{ij}(\omega)\notin(-\infty,0)$. With the $\xi_{ij}(\omega)$ so defined the following obtains.

Theorem 5.2: The set $H(\Gamma)$ is Hurwitz invariant iff (i) all the edges of this set are Hurwitz invariant , and (ii) $\forall$ $j\in\{1,\ldots,v\}$, $i\in\{1,\ldots,v_j\}$, real ω and $\xi_{ij}(\omega)$ as above

$\xi_{ij}(\omega)\notin(-\infty,0)$. ∇∇∇

Though these functions are easy to construct their formal characterization is notationally involved and can be found in [15]. Here, we outline the essentials of their construction.

Consider a segment $L_{ij}(\omega)$ with just two variables, γ_{11} and γ_{21} (i.e. Ψ does not include 1 and 2). Using (5.10) and

(5.3), for any $\omega \notin \Omega$ and $\gamma \in L_{ij}(\omega)$

$$h(j\omega,\gamma)=g_1(j\omega)+ G_2(\omega)\prod_{i=1}^{2}\left(\frac{|f_{i1}(\omega)|^2\gamma_{i1}+f_{i2}(\omega)f_{i1}^*(\omega)}{f_{i1}^*(\omega)}\right)^{k_i}, \quad (5.11)$$

whence, from (5.5), (5.6), for $\rho \in (\mu_1(\omega),\mu_2(\omega))$

$$h(j\omega,\gamma)=g_1(j\omega)+G_2(\omega)\prod_{i=1}^{2}\left(\frac{\mathrm{Im}[f_{i2}(\omega)f_{i1}^*(\omega)]\,(\rho+j)}{f_{i1}^*(\omega)}\right)^{k_i}$$

Thus, $h(j\omega,\gamma)\neq 0$ for all $\omega \notin \Omega$ and γ on this segment $\Leftrightarrow$ the following function avoids ($\mu_1(\omega)$, $\mu_2(\omega)$)

$$\{-\frac{g_1(j\omega)}{G_2(\omega)}\prod_{i=1}^{2}\left(\frac{f_{i1}^*(\omega)}{\mathrm{Im}\,[f_{i2}(\omega)f_{i1}^*(\omega)]}\right)^{k_i}\}^{1/M} - j \quad (5.12)$$

where $M = k_1 + k_2$. Notice that criticality of this segment ensures that $M\neq 0$ (see (5.5)).

Observe the following easily established fact. For reals π,θ_1,θ_2

$$\pi \notin (\theta_1,\theta_2) \Leftrightarrow \frac{(\pi-\theta_2)(\theta_2-\theta_1)}{(\pi-\theta_1)} \notin (-\infty,0).$$

Calling the function in the left side of (5.12) $\eta(\omega)$, we thus have,that $h(j\omega,\gamma)\neq 0$ for all $\omega \notin \Omega$ iff

$$\frac{(\eta(\omega)-\mu_2(\omega))}{(\eta(\omega)-\mu_1(\omega))}\,(\mu_2(\omega)-\mu_1(\omega)) \in (-\infty,0) \quad (5.13)$$

This function is continuous,piecwise differentiable in $\omega \notin \Omega$. However for $\omega \in \Omega$ it is undefined.To obtain the final $\xi_{ij}(\omega)$ with the required properties, a further set of transformations,described in [15] are availed of. Note for each $\xi_{ij}(\omega)$ one should carry M functions, one for each Mth derivative.

6. CONCLUSIONS

In this paper, we have considered both the verification of Hurwitz invariance and the construction of value sets for (1.3). The value set boundary of the characteristic function at each fixed frequency is shown to be determined by the edges and some frequency dependent internal line segmentsof Γ. A continous frequency sweeping function is given such that Hurwitz invariance of (1.3) is equivalent to this function's avoidance of the negative real axis. For the special case of real zero-pole-gain varations, the critical line segments are all frequency independent, whence the determination of the robust stability is even simpler. For the case of complex zero-pole-gain variations, the critical internal lines are either frequency independent or vary in certain (2-dimensional) planes or (3-dimensional) boxes.

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REFERENCES

[1] Vicino, A., Tesi, A. and Milanese, M., "An algorithm for nonconserative stability bounds computation for systems with nonlinearly correlated parametric uncertainties," *Proc. 27th CDC,* Austin, TX, vol. 3, pp. 1761-1766, 1988.

[2] Kharitonov,V.L., "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential Equations,* vol. 14, pp. 1483-1485, 1979.

[3] Bartlett, A. C., Hollot, C. V. and Huang, L., "Root locations of an entire polytope of polynomials: it suffices to check the edges," *MCSS,* vol. 1, pp. 61-71, 1988.

[4] Fu, M. and Barmish, B. R., "Polytopes of polynomials with zeros in a prescribed set," *IEEE Trans.Auto. Contr* vol. 34, no. 5, pp. 544-546, 1989.

[5] Chapellat,H.and Bhattacharyya, S. ,"A generalization of Kharitonov's theorem for robust stability of interval plants," *IEEE Trans Auto. Cont.* vol. 34, pp 306-312, 1989.

[6] Kraus, F.J., Mansour, M. and Anderson, B.D.O., "Robust stability of polynomials with multilinear parametric dependence," submitted to *IJ C,* 1989.

[7] Barmish,B.R. and Z.Shi, "Robust stability of a class of polynomials with coefficients depending multilinearly on perturbations", *Tech. Rept,* Univ. Wisc., Madison, 1988.

[8] de Gaston,R.R.E. and Safonov,M.G.,"Exact calculation of the multiloop stability margin," *IEEE Trans.Auto. Cont.,* vol. 33, no. 2, pp.156-171, 1988.

[9] Barmish, B. R., "New tools for robustness analysis," *Proc. 27th CDC,* Austin, TX, vol. 1, pp. 1-6, 1988.

[10] Bose, N.K. and Zeheb, E., "Kharitonov's theorem and stability test for multi-dimensional digital filters", *IEE Proc. G,* vol. 133, pp 187-190, 1986.

[11] Barmish, B. R., "A generalization of kharitonov's four polynomial concept for robust stability problems with linear dependent coefficient perturbations," *IEEE Trans. Auto. Contr,* vol. 34, pp. 157-165, 1989.

[12] Anagnost, J. J., Desoer C. A. and Minnichelli R. J., "Kharitonov's theorem and a graphical stability test for linear time-invariant systems," in *Robustness in Identification and Control,* eds. Milanese,Tempo and Vicino, 1989, to appear.

[13] Dasgupta,S., Parker,P.J., Anderson, B.D.O., Kraus, F.J. and Mansour,M.,"Frequency domain conditions for the robust stability of linear and nonlinear dynamic systems," *Proc. JACC,* Atlanta, GA, pp. 1863-1868, 1988.

[14] Fu, M.,"Computing the frequency response of a transfer function with parametric perturbations," *Tech. Rept.,* Wayne State University, 1989.

[18] Fu, M., Dasgupta, S. and Blondel V, "Robust stability verification under a class of nonlinear parametreic perturbations", *Tech. Rept,* Univ. Catholique de Louvain - la- Neuve, 1989.