Dynamic Decoupling of MIMO Systems: Linear Case

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Abstract. The decoupling problem has received much attention in past years. For systems which do not have a diagonal interactor, a bicausal precompensation or static state feedback is insufficient for decoupling. In this paper we characterise conditions under which a system can be made decouplable by a diagonal dynamic precompensation. More specifically, we determine necessary and sufficient conditions under which diagonal dynamic precompensation exists which achieves a diagonal interactor.

1 Introduction

There has been much effort applied in solving the problem of decoupling of multivariable systems. Decoupling is usually achieved by applying a precompensator to the plant, or by applying a state feedback when the full state of the system is available, see for example [1]–[6].

It is known that the decoupling of a multivariable linear system is closely related to the so-called interactor matrix [7] (or interactor for short) of the system, which serves as the generalisation of relative degree. For systems with a diagonal interactor, decoupling can be achieved by using a *bicausal* precompensator, or by a *static* state feedback [1]. For systems with a nondiagonal interactor, a nonbicausal precompensator or a *dynamic* state feedback is required. Some previous works formulate dynamic compensation using algebraic formulations, polynomial matrix factorizations, and noncausal differential schemes. These have the disadvantage of being involved and difficult to implement and may not achieve decoupling with stability; see [6] for a survey.

An alternative approach to decoupling is to study the following decouplability problem: given a general multivariable linear system in input-output form, search for a "minimal" diagonal precompensator $D(1/s)$ of the form $diag\{s^{-d_j}\}$, $d_j \geq 0$, such that the resulting system will have a diagonal interactor. Once this precompensation is found, the resulting system can be decoupled by using a bicausal precompensator or by a static state feedback, as already mentioned. The advantage of using diagonal precompensation is clear: a diagonal precompensator consists only of a specified number of integrators attached to each input to the system, and the compensation is independent of the system parameter variations.

This paper is concerned with the decouplability problem above, and provides a necessary and sufficient condition for the existence of diagonal precompensation which achieves a diagonal interactor. More specifically, it is shown that the existence of the diagonal compensation is characterised by the type of singularity of certain constant matrices related to the system transfer matrix. This result is similar to that of [8]. However, here we provide a clear derivation and a simple algorithm for finding the diagonal precompensation.

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2 Preliminaries

We begin by giving some preliminary definitions for a general square rational transfer matrix $T(s) : R^m \rightarrow R^m$; $s \in C$.

Definition 2.1 *The* relative degree *of a row of a transfer matrix is the maximum of the difference between the degrees of the denominator and numerator polynomials of each entry of that row.*

Definition 2.2 *A transfer matrix $T(s)$ is called* bicausal *if $T(s)$ is nonsingular (i.e., its determinant is nonzero for almost all finite complex numbers s) and both $T(s)$ and $T^{-1}(s)$ are proper (i.e., all entries of the matrices are proper).*

The following definitions describe the concepts of generic and nongeneric singularities of transfer matrices. These concepts are also mentioned, but not precisely defined in [8].

Definition 2.3 *A set of linearly dependent row/column vectors are called generically linearly dependent if the linear dependency is independent of the specific values of the nonzero elements of the vectors. Otherwise, the vectors are called nongenerically linearly dependent.*

Remark 2.1: It is easy to see from the definition above that the linear dependence of a set of nongenerically linear dependent vectors can be invalidated by slightly perturbing the values of the nonzero elements in the vectors. The number of vectors must exceed one in order to have nongeneric linear dependence. Furthermore, a set of row (resp. column) vectors are generically linearly dependent *if and only if* either of the following cases happens:

(i) there is a zero row (resp. column);

(ii) there exists a subset of vectors such that by grouping them as a matrix, the number of nonzero columns (resp. rows) in the matrix form a "tall" (resp. "wide") submatrix.

Definition 2.4 *A singular constant matrix is called nongenerically (resp. generically) singular if the singularity depends on (resp. is independent of) the particular values of the nonzero elements, i.e., its rows/columns are nongenerically (resp. generically) linearly dependent.*

As an illustration of these ideas, consider the following two singular matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{1}$$

The first matrix is generically singular because the first two rows are generically linearly dependent, whilst the second matrix is nongenerically singular because the second and third rows are nongenerically linearly dependent.

Definition 2.5 *[7] Let $T(s)$ be an $m \times m$ proper, nonsingular, rational transfer matrix. Suppose there exists a diagonal matrix $\xi_T(s) = \mathrm{diag}\{s^{d_i}\}, d_i \geq 0, 1 \leq i \leq m$ such that*

$$B(s) \triangleq \xi_T(s)T(s) \tag{2}$$

is a bicausal matrix, then $\xi_T(s)$ is called the diagonal interactor (which must be unique) of $T(s)$.

It is shown in [1] that a system which has a non-diagonal interactor cannot be decoupled by a bicausal precompensation, and consequently a dynamic compensation is first required to achieve a diagonal interactor.

3 Main Results

Given an $m \times m$ nonsingular transfer matrix $T(s)$, we wish to define conditions under which there exists a diagonal dynamic precompensator

$$D(1/s) = diag\{s^{-d_i}\}, \quad d_i \geq 0, \quad 1 \leq i \leq m \tag{3}$$

which ensures that $T(s)D(1/s)$ has diagonal interactor of the form

$$\overline{D}(s) = diag\{s^{\overline{d}_i}\}, \quad \overline{d}_i \geq 0, \quad 1 \leq i \leq m \tag{4}$$

i.e.

$$K(s) := \overline{D}(s)T(s)D(1/s) \tag{5}$$

is a bicausal matrix.

Theorem 3.1 *Given a nonsingular transfer matrix $T(s)$, one of the following two cases must occur and they are mutually exclusive:*

(i) There exists a pair $D(1/s)$ and $\overline{D}(s)$ of the forms (3) and (4) respectively such that

$$K_0 = \lim_{s \to \infty} \{\overline{D}(s)T(s)D(1/s)\} \tag{6}$$

is nongenerically singular. In this case, there does not exist any other diagonal precompensator of the form (3) which will achieve a diagonal interactor.

(ii) There exists a pair $D(1/s)$ and $\overline{D}(s)$ of the forms (3) and (4) respectively such that K_0 in (6) is nonsingular. In this case, the compensated system $T(s)D(1/s)$ has diagonal interactor $\overline{D}(s)$.

To aid this theorem, the following algorithm is required, which determines $\overline{D}(s) = diag\{s^{\overline{d}_i}\}$ and $D(1/s) = diag\{s^{-d_i}\}$ such that K_0 is nonsingular.

Algorithm 3.2 Initialize $D(1/s) = I$, i.e., $d_i = 0$, $1 \leq i \leq m$.

Step 1. Find $\overline{D}(s)$ such that every row of $K(s) = \overline{D}(s)T(s)D(1/s)$ has zero relative degree, and take $K_0 = \lim_{s \to \infty} K(s)$. There are three possibilities

1. K_0 is nongenerically singular: No diagonal compensator exists which will give a diagonal interactor.
2. K_0 is nonsingular: $D(1/s)$ is a diagonal precompensator for $T(s)$, and $\overline{D}(s)$ is the associated diagonal interactor.
3. K_0 is generically singular: Proceed to Step 2. (Note that $\overline{D}(s)$ guarantees that K_0 has no zero rows).

Step 2. Extract the maximum set i of rows for which the nonzero columns form a tall matrix. Denote the set of these nonzero columns by j, and the set of remaining columns by $j^\perp$, for which all elements in the rows in set i are zero. Then determine γ, the minimum relative degree of any element contained in the set i of rows and the set $j^\perp$ of columns of $K(s)$. Then for all $l \in j$, increment d_l by γ. Return to step 1.

The algorithm is complete when either case 1 or 2 is achieved.

Proof. Not provided in the Proceedings.

Example To illustrate the algorithm, we consider

$$T(s) = \begin{bmatrix} \frac{1}{s} & \frac{2}{s^3} & \frac{1}{s^4} \\ \frac{1}{s^2} & \frac{1}{s^2} & \frac{1}{s^2} \\ \frac{1}{s^3} & \frac{1}{s^4} & \frac{1}{s^6} \end{bmatrix} \tag{7}$$

Initially, let $D(1/s) = I$, i.e., $d_1 = d_2 = d_3 = 0$.

Iteration 1: To ensure that every row of $K(s)$ has zero relative degree, Step 1 gives $\overline{d}_1 = 1$, $\overline{d}_2 = 2$, $\overline{d}_3 = 2$ and

$$K(s) = \begin{bmatrix} 1 & \frac{2}{s^2} & \frac{1}{s^3} \\ 1 & 1 & 1 \\ 1 & \frac{1}{s^3} & \frac{1}{s^4} \end{bmatrix}, \quad \text{with} \quad K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{8}$$

Note that K_0 is generically singular. Applying Step 2 to $K(s)$, we find the set $i = \{1, 3\}$, that is rows 1 and 3 of K_0 are linearly dependent. We also find $j = \{1\}$ and accordingly $j^\perp = \{2, 3\}$. The minimum relative degree of any element of $K(s)$ which belongs to both sets i and $j^\perp$ is $\gamma = 2$. For all $l \in j$ we choose $d_l \triangleq d_l + \gamma$, then $d_1 = 2$, $d_2 = 0$ and $d_3 = 0$.

Iteration 2: We now return to Step 1 and formulate a new $K(s)$ with our new $D(1/s)$ by choosing $\overline{d}_1 = 3, \overline{d}_2 = 2$ and $\overline{d}_3 = 4$:

$$K(s) = \begin{bmatrix} 1 & 2 & \frac{1}{s} \\ \frac{1}{s^2} & 1 & 1 \\ 1 & 1 & \frac{1}{s^2} \end{bmatrix}, \quad \text{with} \quad K_0 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{9}$$

Since K_0 is nonsingular, the new $\overline{D}(s)$ is the diagonal interactor for the precompensated system $T(s)D(1/s)$.

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