ON CONTROL OF LINEAR SYSTEMS USING QUANTIZED FEEDBACK

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Abstract

This paper studies a number of control problems for linear systems using quantized feedback. First, we revisit the work by Elia and Mitter on quadratic stabilization of linear systems using quantized state feedback and show that their result on coarsest quantization density can be simply obtained from known quadratic stabilization theory by treating the quantization error as sector-bounded uncertainty. This reinterpretation allows us to generalize their work to quantized output feedback and multi-input-multi-output systems.

1 Introduction

Control using quantized feedback has been an important research area for a long time. Even as early as in 1956, Kalman [1] studied the effect of quantization in a sampled data system and pointed out that if a stabilizing controller is quantized using a finite-alphabet quantizer, the feedback system would exhibit limit cycles and chaotic behavior. Most of the work on quantized feedback control concentrates on understanding and mitigation of quantization effects; see, e.g., [2, 3, 4].

A simple classical approach to analysis and mitigation of quantization effects is to treat the quantization error as uncertainty and bound it using a sector bound. By doing so, robustness analysis tools, such as absolute stability theory (see [5, 6]), can be applied to study the quantization effect. Further, control parameters can be optimized to minimize the quantization effect. We will call this the *sector bound method.*

There is a new line of research on quantized feedback control where an quantizer is regarded as an information coder. The fundamental question of interest is how much information needs to be communicated by the quantizer in order to achieve a certain control objective. Noticeable works include [7, 8, 9, 10, 11]. In [11], the problem of quadratic stabilization of discrete-time single-input-single-output (SISO) linear time-invariant (LTI) systems using quantized feedback is studied. The quantizer is assumed to be static and time-invariant (i.e. memoryless and with fixed quantization levels). It is proved in [11] that for a quadratically stabilizable system, the quantizer is the so-called *logarithmic* (i.e., the quantization levels are linear in logarithmic scale). Further, the coarsest quantization density is given explicitly in terms of the system's unstable poles. The work of [11] is also generalized to some extent to guaranteed performance control [12], stabilization of two-input systems [13], and multi-input systems [14].

Note that the required quantizer in the works above has an infinite number of quantization levels because of its time-invariance nature. When the quantizer is allowed to be dynamic and time-varying, it is known that only a finite number of quantization levels [9, 10].

The most pertinent work to this paper is [11]. In fact, this paper stems from the following motivations. First, the results in [11] (also those in [10]) are for SISO systems and for stabilization only. We want to generalize their results to multi-input-multi-output (MIMO) systems and to including performances. Secondly, the technique used in [11], although being novel, does not seem to have a simple interpretation. This is perhaps what makes the their results difficult to generalize.

In this paper, we first review the key result in [11] which is on quadratic stabilization of SISO linear systems using quantized state feedback. We show that coarsest quantization density for logarithmic quantizers can be simply obtained using the sector bound method. This not only gives a simpler interpretation of the result, but also provides the basis for generalization of the result. Secondly, we study the output feedback stabilization of SISO systems. Two cases are considered: observer-based quantized state feedback and dynamic feedback using quantized output. We show that the coarsest quantization density in the former case is the same as in quantized state feedback, whereas the latter case is related to a different H_∞ optimization problem and in general requires a finer quantization density. Thirdly, we generalize the quadratic stabilization problem to MIMO systems and show that quadratic stabilization with a set of logarithmic quantizers is the same as quadratic stabilization with a set of sector-bounded uncertainties. Because the latter problem has been well studied, the technical difficulty for the first problem is clearly revealed. A sufficient condition is

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then given, in terms of an H_∞ optimization problem, for the quantizers to render a quadratic stabilizer. Both state feedback and output feedback are considered.

2 Stabilization using Quantized State Feedback

In this section, we revisit the work of Elia and Mitter [11] on stabilization using quantized state feedback and show how to reinterpret their result. The simplest and most fundamental case considered in [11] is the problem of quadratic stabilization for the following system:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, x is the state and u is a quantized state feedback in the following form:

$$u(k) = f(v(k)) \tag{2}$$

$$v(k) = Kx(k) \tag{3}$$

In the above, $K \in \mathbf{R}^{1\times n}$ is the feedback gain, and $f(\cdot)$ is a quantizer which is assumed to be symmetric, i.e., $f(-v) = -f(v)$. Note that the quantizer is static and time-invariant.

The set of quantized levels is denoted by

$$\mathcal{U} = \{\pm u_i, i = 0, \pm 1, \pm 2, \cdots\} \cup \{0\} \tag{4}$$

Denote by $\#g[\epsilon]$ the number of quantization levels in $[\epsilon,\ 1/\epsilon]$. The quantizer density is defined as follows:

$$\eta_f = \limsup_{\epsilon\to 0} \frac{\#g[\epsilon]}{-\ln \epsilon} \tag{5}$$

Throughout this paper, we consider the so-called logarithmic quantizer below:

$$\begin{aligned} \mathcal{U} \;=\; & \{\pm u^{(i)} : u^{(i)} = \rho^i u^{(0)}, i = \pm 1, \pm 2, \cdots\} \\ & \cup \{\pm u^{(0)}\} \cup \{0\}, \quad 0 < \rho < 1, u^{(0)} > 0 \end{aligned} \tag{6}$$

For the quadratic stabilization problem, a quadratic Lyapunov function $V(x) = x^T Px, P = P^T > 0$, is used to assess the stability of the feedback system, i.e., the quantizer must satisfy

$$\nabla V(x) = V(Ax + Bf(Kx)) - V(x) < 0,\ \forall x \neq 0 \tag{7}$$

The coarsest quantizer is the one which minimizes η_f subject to (7).

The density of the quantizer depends on $V(x)$ (or P) and K. This raises the key question: What is the coarsest density among all possible P and K? In [11], the answer is given for

$$K = K_{GD} = -\frac{B^T PA}{B^T PB} \tag{8}$$

However, it turns out that the result remains the same even when K is allowed to be a free variable. The result is summarized below, but re-phrased with K being free.

Theorem 1 *Consider the linear system in (1). The coarsest quantization density is given by*

$$\rho = \frac{1-\delta}{1+\delta} \tag{9}$$

with

$$\delta^{-1} = \prod_i |\lambda_i^u| \tag{10}$$

where λ_i^u are the unstable eigenvalues of A. □

We prove the result above using an alternative method, i.e., the sector bound method.

Proof. Define the quantization error by

$$e = u - v = f(v) - v \tag{11}$$

Let the quantization levels be given by (6) for any $0 \leq \rho < 1$. It is straightforward to check that e is bounded by the following sector:

$$e = \Delta(v)v, \quad \|\Delta(v)\| \leq \delta \tag{12}$$

where δ is obtained from (9). Therefore, we can model the quantized feedback system as follows:

$$x(k+1) = Ax(k) + B(1 + \Delta(Kx))Kx(k) \tag{13}$$

The quadratic stabilization condition becomes

$$\nabla V(x) = V((A+B(1+\Delta(Kx))K)x) - V(x) < 0, \forall x \neq 0 \tag{14}$$

Let P and K be fixed for the moment. It is trivial to see that the above holds if the following holds:

$$\begin{aligned} \nabla P(\Delta) &= (A + B(1+\Delta)K)^T P(A + B(1+\Delta)K) - P \\ &< 0, \quad \forall |\Delta| \leq \delta \end{aligned} \tag{15}$$

where Δ is independent of the state. Next, we show below that the converse is also true, i.e., (14) implies (15). Indeed, suppose (14) holds but (15) is violated for some $|\Delta_0| \leq \delta$. Let x_0 be the eigenvector of $\nabla P(\Delta_0)$ corresponding to a non-negative eigenvalue, i.e., $x_0^T \nabla P(\Delta_0) x_0 \geq 0$. Note that $Kx_0 \neq 0$ because of (14). Now choose any $x_1 = \alpha x_0$ for some scalar $\alpha \neq 0$ such that $\Delta(Kx_1) = \Delta_0$, which is possible because $\Delta(\cdot)$ swings between $-\delta$ and δ. We have

$$\nabla V(x_1) = x_1^T \nabla P(\Delta(Kx_1)) x_1 \geq 0 \tag{16}$$

This contradicts the assumption that (14) holds. Hence, the converse is proved.

The result above means that the problem of coarsest quantization is equivalent to finding the maximum δ for the following system

$$x(k+1) = Ax(k) + B(1+\Delta)u(k), \quad |\Delta| \leq \delta \tag{17}$$

to be quadratically stabilizable. It is well-known [15] that this is equivalent to minimizing the H_∞-norm of the transfer function

$$G_c(z) = K(zI - A - BK)^{-1}B \tag{18}$$

More specifically,

$$\sup_{P,K} \delta = \frac{1}{\inf_K \|G_c(z)\|_\infty} \tag{19}$$

Hence, it remains to show that the solution to (19) leads to (10). To this end, we take (A, B) to be a controllable canonical form, which yields

$$G_c(z) = \frac{k(z)}{a(z) - k(z)} \tag{20}$$

where $a(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n = |zI - A|$ and $k(z) = k_0 + k_1 z + \cdots + k_{n-1}z^{n-1}$ is the control polynomial. The optimal control gain K must be such that it yields a stable $G_c(z)$ which is either all-pass or arbitrarily close to it. If $a(z)$ is strictly anti-stable, then solution to $k(z)$ is given by

$$K = \frac{a_0^2}{a_0^2 - 1}\left[a_0 - \frac{1}{a_0}, \cdots, a_{n-1} - \frac{a_1}{a_0}\right] \tag{21}$$

which gives $\|G_c(z)\|_\infty = |a_0| = \prod_i |\lambda_i^u|$. Indeed, (21) comes from solving the all-pass requirement for $G_c(z)$:

$$a(z) - k(z) = \alpha z^n k(z^{-1}) \tag{22}$$

for some α. Replacing z by z^{-1}, (22) becomes

$$a(z^{-1}) - k(z^{-1}) = \alpha z^{-n} k(z) \tag{23}$$

Combining (22)-(23) yields

$$k(z) = \frac{a(z) - \alpha z^n a(z^{-1})}{1 - \alpha^2} \tag{24}$$

Setting the nth order coefficient of $k(z)$ to zero results in $\alpha = 1/a_0$. It is straightforward to verify that (24) is the same as (21). Further, we claim $k(z)$ is strictly anti-stable. This is because (24) can be rewritten as

$$k(z) = \frac{a(z)}{1 - \alpha^2}(1 - \alpha\frac{z^n a(z^{-1})}{a(z)})$$

Because $a(z)$ is antistable, $|\alpha| < 1$ and $|z^n a(z^{-1})/a(z)| \le 1$ for any $|z| \le 1$, $k(z) \neq 0$ for any $|z| \le 1$. Hence, $k(z)$ is strictly anti-stable, which implies $G_c(z)$ is stable.

If $a(z)$ has a stable factor, we can write $a(z) = a_s(z)a_u(z)$, where $a_s(z)$ and $a_u(z)$ are the stable and unstable factors. Then, we should have $k(z) = a_s(z)k_1(z)$, which yields

$$G_c(z) = \frac{k_1(z)}{a_u(z) - k_1(z)}$$

and we can proceed as before. If $a_u(z)$ is strictly anti-stable, we still have $\|G_c(z)\|_\infty = \prod_i |\lambda_i^u|$. If $a_u(z)$ has marginally stable roots, then $k_1(z)$ can be chosen so that $\|G_c(z)\|_\infty$ is arbitrarily close to $\prod_i \|\lambda_i^u|$. Hence, we have verified (10). □

Remark 1 It is shown in [11] that the coarsest quantization density is related to the solution to the so-called "expensive" control linear quadratic problem:

$$\begin{array}{rl} \min_K & \sum_{k=0}^{\infty} |u(k)|^2 \\ \text{subject to} & \text{closed-loop stability with} \\ & u(k) = Kx(k) \end{array} \tag{25}$$

More specifically, the optimal ρ can be solved using the solution to the Riccati equation for the "expensive" control problem. However, the optimal control gain K for the quantization problem is different from the optimal control gain for the "expensive" control problem (This is also pointed out in [11]). From the proof above, we see that it is better to interpret the coarsest quantization problem as an H_∞ problem (19).

3 Stabilization using Quantized Output Feedback

We now generalize the technique above to quantized output feedback. Consider the following system:

$$\begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k) \\ y(k) & = & Cx(k) \end{array} \tag{26}$$

where A and B are the same as before and $C \in \mathbf{R}^{1\times n}$.

It turns out that there are two possible configurations for quantized output feedback, each leading to a different coarsest quantization density. These configurations correspond to:

Configuration I: The control signal is quantized but the measurement is not;

Configuration II: The measurement is quantized but the control signal is not.

Configuration I. This case has an interesting result:

Theorem 2 *Consider the system (26) with quantized control input. Suppose (A, C) is an observable pair. Then, the coarsest quantization density for quadratic stabilization by state feedback can also be achieved by output feedback. In particular, the corresponding output feedback controller can be chosen as an observer-based*

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controller below:

$$\begin{aligned} x_c(k+1) &= Ax_c(k) + Bu(k) + L(y(k) - Cx_c(k)) \\ v(k) &= Kx_c(k) \\ u(k) &= f(v(k)) \end{aligned} \tag{27}$$

where $f(\cdot)$ is the quantizer as before, K is the state feedback gain designed for any achievable quantization density via quantized state feedback, and L is a stabilizing observer gain. □

Proof. Let K be any state feedback gain that achieves any given quantization density. Choose L such that the observer is deadbeat, i.e., $e(k) = x(k) - x_c(k)$ only for a finite number of steps N. This can be always done because (A, C) is observable. Then, after N steps, the output feedback controller is the same as state feedback controller. Hence, the system is quadratically stabilized after N steps. Finally, it is a simple fact (although we do not give the details) that if a (nonlinear) system is quadratically stable after N steps and that the state is bounded in the first N steps (which clearly holds for the system (27)), it is quadratically stable. □

Configuration II. The controller now has the form

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c f(y(k)) \\ u(k) &= C_c x_c(k) + D_c f(y(k)) \end{aligned} \tag{28}$$

where $f(\cdot)$ is the quantizer as before. It is easy to verify that the closed-loop system is given by

$$\bar{x}(k+1) = \mathcal{A}(\Delta(y(k))\bar{x}(k) \tag{29}$$

where $\Delta(\cdot)$ is the same as in (12) and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$

$$\hat{I} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \tag{30}$$

and

$$\mathcal{A}(\Delta) = \bar{A} + \bar{B}\bar{K}(\bar{C} + \hat{I}\Delta\hat{C}) \tag{31}$$

The problem of concern is to find the coarsest quantizer for quadratic stabilization of the closed-loop system.

In fact, the coarsest quantization problem for (26)-(28) can be solved by generalizing the idea for the state feedback case. The result is given below.

Theorem 3 *Consider the system (26) and the controller structure (28). The coarsest quantizer for quadratic stabilization is given by (6)-(9) with*

$$\delta = \frac{1}{\inf_{\bar{K}} \|\bar{G}_c(z)\|_\infty} \tag{32}$$

where $\bar{K}$ is defined in (30) and

$$\bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z) \tag{33}$$

where $G(z) = C(zI - A)^{-1}B$ and $H(z) = D_c + C_c(zI - A_c)^{-1}B_c$. Further, if $G(z)$ has relative degree equal to 1 and no unstable zeros, then the coarsest quantization density for quantized state feedback can be reached via quantized output feedback. □

Proof. The proof is very similar to the proof of Theorem 1. The sector bound for the quantization error is done as in (11)-(12). The quadratic stability of the closed-loop system (26)-(28) requires the existence of some $\bar{P} = \bar{P}^T > 0$ such that

$$\bar{x}^T[\mathcal{A}(\Delta(y))^T \bar{P}\mathcal{A}(\Delta(y)) - \bar{P}]\bar{x} < 0 \tag{34}$$

for all $\bar{x} \neq 0$ and $y = Cx = \hat{C}\bar{x}$.

It is straightforward to verify that

$$\bar{G}_c(z) = \hat{I}\bar{K}\bar{C}(zI - \bar{A} - \bar{B}\bar{K}\bar{C})^{-1}\hat{B} \tag{35}$$

It is well-known [15] that the H_∞ optimization problem in (32) is equivalent to (34) if Δ is allowed to be arbitrary (but subject to $|\Delta| \leq \delta$). Hence, the solution to the H_∞ optimization problem implies the solution to (34).

To see the converse, we assume (34) holds but

$$\nabla\bar{P}(\Delta^0) = \mathcal{A}(\Delta^0)^T\bar{P}\mathcal{A}(\Delta^0) - \bar{P} \not< 0 \tag{36}$$

for some $|\Delta^0| \leq \delta$. Let $\bar{x}_0$ be the eigenvector corresponding to the non-negative eigenvalue of $\nabla\bar{P}(\Delta^0)$. Note that $y = \hat{C}\bar{x}_0 \neq 0$ because of (34). Take $\bar{x}_1 = \alpha\bar{x}_0$ for some $\alpha \neq 0$ such that $\Delta(\hat{C}\bar{x}_1) = \Delta^0$, which is always possible. Then, (34) is violated at $\bar{x} = \bar{x}_1$. This contradicts the assumption. Hence, the converse is proved.

Suppose $G(z)$ has relative degree 1 and no unstable zeros. Write $G(z) = b(z)/a(z)$. From the proof of Theorem 1, we know that the state feedback case corresponds to H_∞ optimization of $G_c(z)$ in (20). If we choose $H(z) = k(z)/b(z)$. Then, $\bar{G}_c(z)$ in (33) becomes $G_c(z)$. Hence, the quantization density for the quantized state feedback can be achieved by quantized output feedback. □

The example below shows that using quantized output requires a denser quantizer than using quantized state feedback.

Example 1 The system is given by (26) with $G(z) = C(zI - A)^{-1}B = (z-3)/z(z-2)$. Using quantized state feedback, $\delta = 2$ and $\rho = (2-1)/(2+1) = 0.3333$. For quantized output feedback, computing (32) yields $\delta = 10$ and $\rho = (10-1)/(10+1) = 0.8182$.

Remark 2 In [11], output feedback control design is done in two steps. In Step 1, coarsest quantization is solved for state estimation, which is a dual problem to the state feedback stabilization problem. In Step 2, the separation principle is applied, i.e., optimal state feedback is combined with optimal state estimation. The main result is that logarithmic quantization is sufficient for output feedback stabilization.

The drawback of the approach in [11] is that the physical meaning of the state estimation quantizer is not clear. Indeed, the problem of quantized state estimation is formulated to be:

$$e(k+1) = Ae(k) + Lf_e(Ce(k)) \tag{37}$$

where $e(k) = x(k) - x_c(k)$ is the state estimation error and $f_e(\cdot)$ is its quantizer. This can be unsatisfactory because the quantizer needs to know both $y(k)$ and its estimate $Cx_c(k)$. If the control signal is generated at the measurement end, there is obviously no need to use quantized $y(k)$. If the control signal is generated elsewhere using a quantized $y(k)$, it is difficult to imagine why its estimate needs to be sent back to the measurement end to form $Ce(k)$ for quantization. Hence, the validity of this formulation seems to be questionable.

4 Stabilization of MIMO Systems using Quantized Feedback

Now we generalize the quantization results in Section 2 to MIMO systems.

Configuration I. The system is still as in (26) (or (1) for state feedback) except that $u \in \mathbf{R}^m$, $y \in \mathbf{R}^r$. Suppose quantized state feedback (2)-(3) is used, where $K \in \mathbf{R}^{m\times n}$ and

$$f(v) = \operatorname{diag}\{f_1(v_1), f_2(v_2), \cdots, f_m(v_m)\} \tag{38}$$

where v_j is the jth component of v and $f_j(\cdot)$ is a quantizer of the form (6) but with $0 < \rho_j < 1$.

Because we have more than one quantizer, the notion of coarsest quantization is not well-defined. Instead, we ask the following question: Given a vector of quantization levels $\rho = [\rho_1\ \rho_2\ \cdots\ \rho_m]$, does there exist an quantized feedback controller that quadratically stabilizes the system (26)? The main result is given below:

Theorem 4 *Given the system in (26) and a quantization level vector ρ, consider the auxiliary system:*

$$x(k+1) = Ax(k) + B(I + \Delta(k))v(k) \tag{39}$$

where $|\Delta_j(k)| \leq \delta_j$ for all j and k, and δ_j are converted from ρ_j using (9), and $v(k)$ is a control input. Suppose the auxiliary system is quadratically stablizable via state feedback (3), then (26) is quadratically stabilizable via quantized state feedback. Conversely, suppose the system (26) is quadratically stabilizable via quantized state feedback and, in addition, suppose $\ln\rho_i/\ln\rho_j$ are irrational numbers for all $i \neq j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (39) with $|\Delta_j(k)| \leq \delta_j - \epsilon$ is quadratically stabilizable via state feedback (3). Further, the auxiliary system is quadratically stabilizable via state feedback (3) if the following state feedback H_∞ control has a solution K for some diagonal scaling matrix $\Gamma > 0$:

$$\|\Lambda\Gamma K(zI - A - BK)^{-1}B\Gamma^{-1}\|_\infty < 1 \tag{40}$$

where

$$\Lambda = \operatorname{diag}\{\delta_1, \cdots, \delta_m\} \tag{41}$$

In particular, any K that renders (40) is a solution to either quadratic stabilization problem. Finally, if (A, C) is an observable pair and (26) is quadratically stabilizable via quantized state feedback for the given ρ, then it is also quadratically stabilizable via observer-based quantized state feedback (27) for the same ρ. □

Remark 3 It is easy to see that if a given set of $\rho_j, j = 1, 2, \cdots, m$ do not satisfy the condition that $\ln\rho_i/\ln\rho_j$ are irrational for $i \neq j$, we can make it so by perturbing the ρ_j arbitrarily slightly. That is, the condition above holds generically.

Proof of Theorem 4. The basic idea is similar to the proof of Theorem 1. Although we are dealing with a MIMO system, the notation in that proof is still valid. In particular, the condition for quadratic stability of the closed-loop system is still given by (14). As in that proof, it is known [16] that the H_∞ optimization condition (40) implies quadratic stabilizability of the auxiliary system (39) via state feedback which in turn implies the quadratic stabilizability of (26) via quantized state feedback, and that the same controller applies to all three problems. The proof of the converse is are quite involved and omitted due to space limits.

The result on observer-based feedback is proved in a way similar to that of Theorem 2. □

Configuration II. When quantized measurements are available, we have the following result:

Theorem 5 *Given the system in (26) and a quantization level vector ρ, consider the auxiliary system:*

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \\ v(k) &= (I + \Delta(k))y(k) \end{aligned} \tag{42}$$

where $|\Delta_j(k)| \leq \delta_j$ for all $j = 1, 2, \cdots, m$ and k, and δ_j are converted from ρ_j using (9), and $v(k)$ is the output available for feedback. Suppose the auxiliary system is quadratically stablizable, then (26) is quadratically stabilizable via (28). Conversely, suppose the system (26) is quadratically stabilizable via (28) and, in addition, suppose $\ln \rho_i / \ln \rho_j$ are irrational numbers for all $i \neq j$ when $m > 1$. Then, for any (arbitrarily small) $\epsilon > 0$, the auxiliary system (42) with $|\Delta_j(k)| \leq \delta_j - \epsilon$ is quadratically stabilizable.

Further, the auxiliary system is quadratically stabilizable if the following state feedback H_∞ control has a solution $H(z)$ for some diagonal scaling matrix $\Gamma > 0$:

$$\|\Lambda\Gamma(I - G(z)H(z))^{-1}G(z)H(z)\Gamma^{-1}\|_\infty < 1 \qquad (43)$$

where Λ is given in (41). In particular, any $H(z)$ that renders (40) is a solution to either quadratic stabilization problem. □

Proof. The "equivalence" between the quadratic stabilization problems is similar to that of Theorem 4, but the details are not given due to space limits. The proof for the relation to H_∞ optimization is similar to the proof of Theorem 3. □

5 Conclusions

We have reinterpreted a key result in [11] on quadratic stabilization. This is done using the sector bound method, i.e., by treating the quantization error as a sector-bounded uncertainty. This simple interpretation allows us to generalize their result to quantized output feedback control and quantized control for MIMO systems. Finally, we point out that the sector bound method can also be applied to quantized control for H_∞ performance and quadratic performance.

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