Guaranteed Cost Analysis and Control for a Class of Uncertain Nonlinear Discrete-Time Systems[1]

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Abstract

This paper deals with the problem of guaranteed cost analysis and control of a class of nonlinear discrete-time systems with uncertain parameters. We use polynomial Lyapunov functions to derive stability conditions with a guaranteed bound on the 2-norm of the performance output in terms of linear matrix inequalities (LMIs). We then extend this approach to control design by considering parameter-dependent Lyapunov functions and nonlinear (state- and parameter-dependent) multipliers.

1 Introduction

In the last decade or so, there has been significant interest in using the LMI (linear matrix inequality) framework in control of continuous-time nonlinear systems. Design approaches range from using quadratic Lyapunov functions ([1]) to those based on polynomial Lyapunov functions ([2]) in both guaranteed cost and $\mathcal{H}_\infty$ settings. In general, non-quadratic Lyapunov functions are less conservative in dealing with uncertain and nonlinear systems than the quadratic ones at the expense of extra computation [3]. However, most of the robust control results using non-quadratic Lyapunov functions for nonlinear systems involve solving matrix inequalities which are non-convex in the decision variables.

On the other hand, for nonlinear discrete-time systems we have another fundamental difficulty with non-quadratic Lyapunov functions which lies in the fact that the difference between the Lyapunov functions at time $k+1$ and k is highly nonlinear. To make this point clear, consider the system

$$x(k+1) = A(x(k), \delta(k))x(k),$$

and a Lyapunov function

$$V(x(k), \delta(k)) = x(k)P(x(k), \delta(k))x(k),$$

where $\delta(k)$ represents uncertain parameters, and the matrices $A(x(k), \delta(k))$ and $P(x(k), \delta(k))$ depend on $x(k)$ and $\delta(k)$. In the Lyapunov function difference (which we will refer to as a Lyapunov inequality) appears the term

$$A(x(k), \delta(k))'P(x(k+1), \delta(k+1))A(x(k), \delta(k)) \quad (1)$$

which is typically a highly nonlinear function of $x(k)$, $\delta(k)$ and $\delta(k+1)$. In contrast, if we consider a similar continuous-time system $\dot{x}(t) = A(x(t), \delta(t))x(t)$ and a similar Lyapunov function, we have much less product terms between $x(t)$ and $\delta(t)$.

An alternative approach to the robust control of discrete-time systems is the quasi-LPV (linear parameter-varying) representation of nonlinear systems, i.e. systems described by

$$x(k+1) = A(\theta(x(k)))x(k),$$

where the state-dependent parameter $\theta(x(k)) \in \Theta$ with Θ being a given polytope. For example, the works of *Tu and Shamma* in [4] and *Tuan et. al* in [5] use LPV techniques for guaranteed cost and $\mathcal{H}_\infty$ control of nonlinear discrete-time systems, respectively. In spite of the fact that the quasi-LPV approach overcomes the problem caused by the term in (1), there are some shortcomings in this methodology. More specifically, it is computationally feasible only for a small number of nonlinear terms and when considering parameter-dependent Lyapunov functions (e.g. $V(x(k)) = x(k)'P(\theta(x(k)))x(k)$) the forward-shift parameter $\theta(x(k+1))$ that appears in the Lyapunov inequality has to belong to the polytope Θ increasing the computational effort.

The approach used in this paper is motivated by the work of *Oliveira et. al.* [6] which proposed a new test of stability using LMIs for linear discrete-time systems with polytopic uncertainties. In this approach, the system matrix and the Lyapunov matrix are assumed to be affine in uncertain parameters, i.e. $A(\delta)$ and $P(\delta)$ are used and they are affine in δ. The Lyapunov inequality is modified by introducing an auxiliary matrix that separates the system matrix $A(\delta)$ from the Lyapunov matrix $P(\delta)$. This introduces some conservatism, but significantly reduce the nonlinearity. Further, the resulting inequality can be expressed as an LMI which is affine in δ. Although, one can think of many possible auxiliary matrices with the above feature, the particular one introduced in [6] appears to be excellent in

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terms of the conservatism it brings. This method has been also extended to performance analysis and control design [7].

The purpose of this paper is to devise a technique for guaranteed cost analysis and robust controller design for a class of nonlinear discrete-time systems with (time-varying) uncertain parameters. We employ polynomial Lyapunov functions and give regional stability conditions with guaranteed cost in terms of LMIs. However, suitable Lyapunov inequalities will be used to simplify the numerical computations which will be done by generalizing the work of [6]. We then extend this result to control design by using a parameter-dependent Lyapunov function and nonlinear multipliers.

2 Problem Statement

Consider the following class of discrete-time nonlinear systems:

$$\begin{cases} x_+ &= f(x(k),\delta(k)) = A(x(k),\delta(k))x(k), \\ z(k) &= c(x(k),\delta(k)) = C(x(k),\delta(k))x(k) \end{cases} \quad (2)$$

where $x_+ = x(k+1)$, $x(k) \in \mathbb{R}^n$ is the state vector, $z(k) \in \mathbb{R}^{n_z}$ represents the performance output and $\delta(k) \in \mathbb{R}^l$ denotes the vector of bounded uncertain parameters with bounded variation $v(k) = \delta(k+1) - \delta(k)$, and the system matrices $A(x,\delta), C(x,\delta)$ are allowed to depend on $x(k)$ and $\delta(k)$. It is assumed that $A(x,\delta)$ and $C(x,\delta)$ are continuous functions in $\mathbb{R}^n \times \mathbb{R}^l$ and $\mathbb{R}^{n_z} \times \mathbb{R}^l$, respectively. We further assume that the parameter vector $\delta(k)$ and its variation $v(k)$ lie in a given polytope Δ with known vertices.

Note that the values of the forward-shift parameter $\delta(k+1)$ depends on the values of $\delta(k)$ and $v(k)$, i.e. $\delta(k+1) = \delta(k) + v(k)$. Assuming that the values of $\delta(k)$ are independent of k and thus are the same for $\delta(k+1)$, the admissible values of $v(k)$ have to satisfy the constraint $\delta(k+1) = \delta(k) + v(k)$. A polytope satisfying this condition is called a *consistent domain* [8]. For simplicity, we assume throughout this paper that Δ represents a consistent domain.

The problem of concern in this paper is of two fold: (i) to determine a region in the state-space (that we will refer to as a domain of performance) in which a bound on the 2-norm of the output signal is guaranteed , and (ii) to design a robust controller such that the domain of performance is maximized for a given cost bound. To this end, we first introduce the notion of domain of performance (DOP).

Given a region $\mathcal{R} \subset \mathbb{R}^n$, we say that $\mathcal{R}$ is a DOP for system (2) if for every $x(0) \in \mathcal{R}$ and $\delta \in \Delta$, the 2-norm of the output signal satisfies $\|z(k)\|_2 < \lambda$ and the trajectory $x(k)$ remains in $\mathcal{R}$ for all $k \geq 0$ and approaches to the origin as $k \to \infty$.

From the Lyapunov theory, we have the following result:

Lemma 1 *Consider system (2). Let $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ be a given Lyapunov function candidate, where $\mathcal{P}(x,\delta)$ is a matrix function of (x,δ). Define a region in the state-space as follows:*

$$\mathcal{X} \triangleq \{x : x \in \mathbb{R}^n,\ x'\mathcal{P}(x,\delta)x \leq 1,\ \forall\, \delta \in \Delta\} \quad (3)$$

Suppose there exists positive scalars ϵ_1, ϵ_2 and λ such that:

$$\epsilon_1 x'x \leq x'\mathcal{P}(x,\delta)x \leq \epsilon_2 x'x,\ \forall\, x \in \mathcal{X},\ \delta \in \Delta \quad (4)$$

$$x'\left(A(x,\delta)'\mathcal{P}(x_+,\delta+v)A(x,\delta) - \mathcal{P}(x,\delta)\right)x < -\frac{z'z}{\lambda}, \quad \forall\, x \in \mathcal{X},\ (\delta,v) \in \Delta \quad (5)$$

where x_+ is as defined in (2). Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is an estimate of the DOP.

A possible approach to simplifying the product term $A(x,\delta)'\mathcal{P}(x_+,\delta+v)A(x,\delta)$ is to use the idea of Schur complement. However, we still have a very complicated condition to be checked because of two problems: (1) coupling of $A(x,\delta)$ and $\mathcal{P}(x_+,\delta+v)$ still gives nonconvex terms; (2) checking the conditions over $\mathcal{X} \times \Delta$ is highly nontrivial [9]. These are the problems we will address in the next section.

3 Preliminary Results

We give two results in this section. The first one, Lemma 2, is a nonlinear version for guaranteed cost analysis of the result in [6, Theorem 1] which will allow to remove the coupling between the system and the Lyapunov matrices. The second one, Lemma 3, give a way to remove the nonlinear dependence on x in the conditions of Lemma 1 by a relaxation technique.

Lemma 2 *Consider system (2) and $V(x,\delta)$ and $\mathcal{X}$ as defined in Lemma 1. Suppose (4) and the following inequality holds for some auxiliary matrix function $\mathcal{G}(x,\delta)$ and a positive scalar λ:*

$$\begin{bmatrix} x \\ y \end{bmatrix}' \begin{bmatrix} -\mathcal{P} & A'\mathcal{G}' \\ \mathcal{G}A & \mathcal{P}_+ - \mathcal{G} - \mathcal{G}' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} < -\frac{z'z}{\lambda}, \quad \forall\, x \in \mathcal{X},\ (\delta,v) \in \Delta,\ y \in \mathbb{R}^n,\ z \in \mathbb{R}^{n_z} \quad (6)$$

where $\mathcal{P} = \mathcal{P}(x,\delta)$, $A = A(x,\delta)$, $\mathcal{G} = \mathcal{G}(x,\delta)$, and $\mathcal{P}_+ = \mathcal{P}(x_+,\delta+v)$. Then, $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$ and $\mathcal{X}$ is an estimate of the DOP with $\|z(k)\|_2 < \lambda$ for all $(\delta,v) \in \Delta$.

Remark 1 The conservativeness of Lemma 2 lies in the choice of the auxiliary matrix $\mathcal{G}(x,\delta)$. Observe that we can recover Lemma 1 by considering $\mathcal{G}(x,\delta) = \mathcal{P}(x_+,\delta+v)$. But this choice of $\mathcal{G}(x,\delta)$ leads to complicated conditions. Consequently, we will have a compromise between the conservatism and complexity when choosing $\mathcal{G}(x,\delta)$.

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Note that when $A(x,\delta)$, $\mathcal{P}(x,\delta)$ and $\mathcal{G}(x,\delta)$ do not depend on x and $v = 0$ and $\lambda = 0$, Lemma 2 reduces to the original result in [6].

Next, we aim to remove the dependence on x in conditions (4) and (6). To this end, we use a version of the well-known Finsler's Lemma (see, e.g. [10]).

Lemma 3 *Consider the following nonlinear matrix inequality:*

$$\mathcal{T}(\xi) > 0,\ \mathcal{T}(\xi) = \mathcal{T}(\xi)',\ \forall\ \xi \in \mathcal{E} \tag{7}$$

where $\xi \in \mathbb{R}^{n_\xi}$ denotes a parameter vector, and the matrix $\mathcal{T}(\xi) \in \mathbb{R}^{n_t \times n_t}$ is a nonlinear function of ξ and $\mathcal{E} \subset \mathbb{R}^{n_\xi}$ is a polytopic region with known vertices. Suppose $\mathcal{T}(\xi)$ can be decomposed as follows:

$$\mathcal{T}(\xi) = \begin{bmatrix} I_{n_t} \\ \mathcal{M}(\xi) \end{bmatrix}' T \begin{bmatrix} I_{n_t} \\ \mathcal{M}(\xi) \end{bmatrix} \tag{8}$$

where $T \in \mathbb{R}^{m_t \times m_t}$ is a constant symmetric matrix, $\mathcal{M}(\xi) \in \mathbb{R}^{m_t \times n_t}$ is a nonlinear matrix function of ξ with the property that

$$\Xi_1(\xi) + \Xi_2(\xi)\mathcal{M}(\xi) = 0 \tag{9}$$

for some matrices $\Xi_1(\xi) \in \mathbb{R}^{m_\xi \times n_t}, \Xi_2(\xi) \in \mathbb{R}^{m_\xi \times m_t}$ which are affine functions of ξ with $\Xi_2(\xi)$ having column full rank for all ξ of interest. Then, (7) is satisfied if there exists a constant matrix L such that

$$T + L\Xi(\xi) + \Xi'(\xi)L' > 0,\ \xi \in \mathcal{V}(\mathcal{E}) \tag{10}$$

where $\Xi(\xi) = [\ \Xi_1(\xi) \quad \Xi_2(\xi)\ ]$ and $\mathcal{V}(\mathcal{E})$ is the set of all vertices of $\mathcal{E}$.

Note that (10) is affine in ξ, so it can be checked by setting ξ at the vertices of $\mathcal{E}$. We point out that the decomposition conditions (8) and (9) are very general and can be satisfied for many nonlinear matrix inequalities such as those with rational nonlinearities.

Conservativeness of NLMIs

The use of standard LMI techniques for testing state-dependent matrix inequalities can be quite conservative [11]. For example, consider the condition:

$$\xi'\mathcal{T}(\xi)\xi > 0,\ \forall\ \xi \in \mathcal{E}. \tag{11}$$

Here we have $n_t = n_\xi$. This condition may be checked by applying Lemma 3 to $\mathcal{T}(\xi) > 0$. If there exists a solution to (10) for all $\xi \in \mathcal{V}(\mathcal{E})$, then the following is satisfied:

$$t'\mathcal{T}(\xi)t > 0,\ \forall\ \xi \in \mathcal{E},\ \forall\ t \in \mathbb{R}^{n_\xi}. \tag{12}$$

Obviously, this is too conservative. To relax this, the notion of *linear annihilators* is introduced in [11] as below:

Definition 1 *A matrix $\mathcal{N}(\xi)$ is called a linear annihilator of ξ if it is a linear function of ξ and $\mathcal{N}(\xi)\xi = 0$.*

The basic idea in this approach is to associate a multiplier to the constraint $\mathcal{N}(\xi)\xi = 0$, hence reducing the conservativeness of (12). In this paper, we will consider the following linear annihilator:

$$\mathcal{N}(\xi) = \begin{bmatrix} \xi_2 & -\xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_3 & -\xi_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \xi_{n_\xi} & -\xi_{n_\xi - 1} \end{bmatrix} \tag{13}$$

Similarly to [3, Lemma 3.1], we modify the condition (10) to the following:

$$T + L_a\Xi + \Xi' L_a' + L_b\mathcal{N}(\xi)Q_m + Q_m'\mathcal{N}(\xi)'L_b' > 0, \tag{14}$$

for all $\xi \in \mathcal{V}(\mathcal{E})$, where $Q_m = [\ I_{n_\xi} \quad 0_{n_\xi \times m_t}\ ]$, and L_a and L_b are constant matrices to be determined.

4 Performance Analysis

In order to apply the results in the previous section, we need to re-parameterize the system model (2) and define the structure of the Lyapunov matrix $\mathcal{P}(x,\delta)$ and the auxiliary matrix $\mathcal{G}(x,\delta)$ accordingly. These are detailed below.

4.1 System Model Representation

We further assume that system (2) can be decomposed as follows:

$$\begin{cases} x_+ &= (A_0 + A_1\Pi_1(x,\delta))\,x, \\ z &= (C_0 + C_1\Pi_1(x,\delta))\,x, \\ 0 &= \Omega_0(x,\delta) + \Omega_1(x,\delta)\Pi_1(x,\delta) \end{cases} \tag{15}$$

where $A_0 \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times m}, C_0 \in \mathbb{R}^{n_z \times n}$ and $C_1 \in \mathbb{R}^{n_z \times m}$ are constant matrices; $\Pi_1(x,\delta) \in \mathbb{R}^{m \times n}$ is a nonlinear matrix function of (x,δ); and $\Omega_0(x,\delta) \in \mathbb{R}^{p \times n}$ and $\Omega_1(x,\delta) \in \mathbb{R}^{p \times m}$ are affine matrix functions of (x,δ). We assume that the matrix $\Omega_1(x,\delta)$ has column full rank for all x and δ of interest.

For simplicity of notation, we may hereafter represent the matrices $\Pi_1(x,\delta), \Omega_0(x,\delta)$ and $\Omega_1(x,\delta)$ without their respective dependence on x and δ, and the system (15) may be also described in the following compact form:

$$x_+ = \mathcal{A}\Pi x,\ z = \mathcal{C}\Pi x,\ \Omega\Pi = 0$$

where

$$\begin{aligned} \mathcal{A} &= [\ A_0 \quad A_1\ ], \\ \mathcal{C} &= [\ C_0 \quad C_1\ ], \end{aligned} \quad \Pi = \begin{bmatrix} I_n \\ \Pi_1(x,\delta) \end{bmatrix}, \tag{16}$$

and $\Omega = [\ \Omega_0(x,\delta) \quad \Omega_1(x,\delta)\ ]$.

Note that the choice of $\mathcal{A}, \mathcal{C}, \Pi$ and Ω is not unique and there is no a systematic way to compute them. However, this allows the representation of a large class of nonlinear systems. In this paper, we will use the aforementioned degree of freedom in order to parameterize the Lyapunov matrix in terms of the above nonlinear decomposition in order to test the conditions of Lemma 2 via an optimization problem over a set of LMIs.

Remark 2 In fact, a wrong choice of the matrices $\mathcal{A}, \mathcal{C}, \Pi$ and Ω may lead to a poor performance estimation (see Example 3.2 of [12]). The conservativeness of choosing these matrices is in part reduced by the inclusion in the stability conditions of the constraint $\mathcal{N}(x)x = 0$ adding free variables to the problem, similarly to [12, Lemma 3.1] and [3, Lemma 3.1].

4.2 Lyapunov Function Candidate

Consider the following Lyapunov matrix:

$$\mathcal{P}(x,\delta) = \begin{bmatrix} \Theta(x) \\ I_n \end{bmatrix}' P(\delta) \begin{bmatrix} \Theta(x) \\ I_n \end{bmatrix} \qquad (17)$$

where $P(\delta) = P(\delta)'$ is an affine matrix functions of δ, and $\Theta(x) \in \mathbb{R}^{q \times n}$ is a given matrix function of x.

Observe from Lemma 2 that we need to compute the following matrix:

$$\mathcal{P}(x_+, \delta + v) = \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix}' P(\delta + v) \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix}.$$

To this end, we require the following constraints on $\Theta(x)$:

$$\begin{bmatrix} \Theta(x) \\ I_n \end{bmatrix} = F\Pi = \begin{bmatrix} F_1 \\ Q \end{bmatrix}\Pi, \begin{bmatrix} \Theta(x_+) \\ I_n \end{bmatrix} = H\Pi = \begin{bmatrix} H_1 \\ Q \end{bmatrix}\Pi \qquad (18)$$

where $Q = [\ I_n \quad 0_{n\times m}\ ]$, $F_1, H_1 \in \mathbb{R}^{q\times(n+m)}$ are constant matrices, and Π is the same matrix defined in (16).

Also, in accordance with the system and Lyapunov function parameterization, we choose the auxiliary matrix function $\mathcal{G}(x,\delta)$ to be of the following form:

$$\mathcal{G}(x,\delta) = \Pi' G(\delta) \qquad (19)$$

where $G(\delta) \in \mathbb{R}^{(n+m)\times n}$ is an affine matrix function of δ to be determined.

With the above choice of $\mathcal{G}(x,\delta)$, we may have a certain degree of conservatism (see Remark 1). Nevertheless, it can lead to a convex characterization of Lemma 1 as we will see later in this section.

In addition, taking into account the above definitions, we can rewrite the inequality (6) as follows:

$$\begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix}' \begin{bmatrix} \left(-F'P(\delta)F + \frac{\mathcal{C}'\mathcal{C}}{\lambda}\right) & \star \\ G(\delta)\mathcal{A} & \Sigma \end{bmatrix} \begin{bmatrix} \sigma_a \\ \sigma_b \end{bmatrix} < 0 \qquad (20)$$

for all $x \in \mathcal{X}$, $y \in \mathbb{R}^n$ and $(\delta, v) \in \Delta$, where $\sigma_a = \Pi x$, $\sigma_b = \Pi y$ and

$$\Sigma = H'P(\delta + v)H - G(\delta)Q - Q'G(\delta)'. \qquad (21)$$

In order to apply Lemma 2 in a numerically tractable manner, we also need a polytopic bounding set $\hat{\mathcal{X}}$ for $\mathcal{X}$. In this way, we will require (20) to hold for all $\hat{\mathcal{X}}$ instead of $\mathcal{X}$. Hence, we want to choose $\hat{\mathcal{X}}$ to be reasonably close to $\mathcal{X}$ to reduce the conservativeness but having a small number of vertices so the resulting conditions are easy to check. A possible way to achieve a good compromise is to define the shape of the bounding set and use a parameter to control its size. This parameter can be then adjusted through iterations to obtain an optimal size. But for the discussion in the sequel, we assume that the bounding set $\hat{\mathcal{X}}$ is given.

Without loss of generality, we assume that the bounding set is represented in terms of the following constrains:

$$\hat{\mathcal{X}} = \{x : a_j' x \le 1,\ j = 1, \ldots, n_e\} \qquad (22)$$

where $a_j \in \mathbb{R}^{n_x}$ are given vectors associated with the n_e edges of $\hat{\mathcal{X}}$.

Using the $\mathcal{S}$-procedure (see, e.g., Section 4 of [2]), the condition $\mathcal{X} \subset \hat{\mathcal{X}}$ is satisfied if the following inequality is satisfied for all j:

$$\begin{bmatrix} 1 \\ \Theta(x)x \\ x \end{bmatrix}' \begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & P(\delta) \end{bmatrix} \begin{bmatrix} 1 \\ \Theta(x)x \\ x \end{bmatrix} \ge 0. \qquad (23)$$

In order to ensure that the Lyapunov matrix function $\mathcal{P}(x,\delta)$ in (17) is positive definite for all $x \in \hat{\mathcal{X}}$, we apply Lemma 3 and obtain the following condition:

$$P(\delta) + L\Psi_1(x) + \Psi_1(x)'L' > 0,\ \forall\ (x,\delta) \in \mathcal{V}(\hat{\mathcal{X}} \times \Delta) \qquad (24)$$

where L is a free matrix to be determined and

$$\Psi_1(x) = [\ I_m \quad -\Theta(x)\ ]. \qquad (25)$$

In order to maximize the volume of $\mathcal{X}$, we normally approximate it by minimizing the trace of the Lyapunov matrix. However, $\mathcal{P}(x,\delta)$ is a nonlinear function of (x,δ) that leads to a non-convex condition. To overcome this problem, we will approximate the volume maximization by

$$\min \max_{(x,\delta,v)\in\Gamma} \operatorname{trace}\left(P(\delta) + L\Psi_1(x) + \Psi_1(x)'L'\right) \quad \text{subject to (20), (23) and (24),} \qquad (26)$$

where $\Gamma = \mathcal{V}(\hat{\mathcal{X}} \times \Delta)$.

Now, with above analysis we can state the following theorem which gives a convex solution to the DOP problem for system (2) in terms of LMIs.

Theorem 1 *Consider the nonlinear discrete-time system (2) as decomposed in (15). Let $\Theta(x)$ be a given affine matrix function of x satisfying (18) and the Lyapunov matrix $\mathcal{P}(x,\delta)$ be in the form of (17). Let $\hat{\mathcal{X}}$ and Δ be given polytopes such that Δ is consistent. Let λ be an upper-bound on the output 2-norm. Define $\Psi_1(x)$ as in (25) and*

$$\Psi_2(x,\delta) = \operatorname{diag}\{\ \Omega,\ \Omega\ \}. \qquad (27)$$

Suppose there exist affine matrices $G(\delta)$, $P(\delta)$; constant matrices L, N_1, N_2 and M_j $(j = 1, \ldots, n_e)$; and *a positive scalar η solving the following optimization problem where the LMIs are constructed at $\mathcal{V}(\hat{\mathcal{X}} \times \Delta)$.*

$$\min \eta \quad \textit{subject to:}$$

$$\eta - \mathsf{trace}\left(P(\delta) + L\Psi_1(x) + \Psi_1(x)'L'\right) \geq 0 \tag{28}$$

$$P(\delta) + L\Psi_1(x) + \Psi_1(x)'L' > 0 \tag{29}$$

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & a_j' \end{bmatrix} \\ \begin{bmatrix} 0 \\ a_j \end{bmatrix} & (P(\delta) + M_j\Psi_1(x) + \Psi_1(x)'M_j') \end{bmatrix} \geq 0 \tag{30}$$

$$\begin{bmatrix} \begin{pmatrix} -F'P(\delta)F + \lambda^{-1}\mathcal{C}'\mathcal{C} \\ N_1\mathcal{N}(x)Q + Q'\mathcal{N}(x)'N_1' \end{pmatrix} & \star \\ G(\delta)\mathcal{A} + N_1\mathcal{N}(x)Q & \Sigma \end{bmatrix} + \\ +N_2\Psi_2(x,\delta) + \Psi_2(x,\delta)'N_2' < 0 \tag{31}$$

where $\mathcal{V}(\cdot)$ denotes the set of all vertices of $(\cdot)$ and Σ is as defined in (21). Then, $V(x,\delta) = x'\mathcal{P}(x,\delta)x$ is a Lyapunov function in $\mathcal{X}$. Moreover, $\mathcal{X}$ given by (3) is an estimate of the DOP, i.e. for all $x(0) \in \mathcal{X}$ and $(\delta, v) \in \Delta$ the trajectory $x(k) \in \mathcal{X}$ for all k and approaches the origin as $k \to \infty$ with $\|z(k)\|_2 < \lambda$.

5 Control Design

The LMI methods for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control synthesis of discrete-time LPV systems use the dual version of Lemma 1 (i.e. stability conditions in terms of the Lyapunov matrix inverse) and then parameterize the control gain leading to convex conditions (see e.g. [7]). However, an extension of this technique to deal with nonlinear systems yields non-convex conditions since it appears in the stabilization matrix inequalities the vector x_+ which is a function of the control matrix loosing the convexity. To make this point clear, consider in the following the class of nonlinear system to be considered in this section and a dual version of Lemma 2 for control design.

Consider the following nonlinear system:

$$\begin{aligned} x_+ &= A(x(k),\delta(k))x(k) + B(x(k),\delta(k))u(k), \\ z(k) &= C(x(k),\delta(k))x(k) + D(x(k),\delta(k))u(k), \\ u(k) &= K(x(k),\delta(k))x(k) \end{aligned} \tag{32}$$

where $u(k) \in \mathbb{R}^r$ is the control input, $B(x,\delta) \in \mathbb{R}^{n\times r}, D(x,\delta) \in \mathbb{R}^{n_z\times r}$ and $K(x,\delta) \in \mathbb{R}^{r\times n}$ are nonlinear matrix functions of $x(k)$ and $\delta(k)$. Also, for simplicity, assume that the parameters $\delta(k)$ are known on-line to the controller as in the gain-scheduling control [13].

Now, consider the following basic result for control design (a nonlinear version of [14, Theorem 1] for guaranteed cost control).

Lemma 4 *Consider system (32), $V(x,\delta) = x'(\mathcal{Y}(x,\delta))^{-1}x$ and $\mathcal{X} = \{x : x \in \mathbb{R}^n, V(x,\delta) \leq 1, \ \forall\ \delta \in \Delta\}$. Suppose the following inequalities are satisfied for $\mathcal{Y}(x,\delta)$, some auxiliary matrix functions $\mathcal{G}(x,\delta)$ and $K(x,\delta)$ of appropriate dimensions, and a positive scalar λ:*

$$s'\mathcal{Y}(x,\delta)s > 0, \ \forall\ x \in \mathcal{X}, \ s \in \mathbb{R}^n, \ \delta \in \Delta \tag{33}$$

$$\begin{bmatrix} s \\ w \\ \rho \end{bmatrix}' \begin{bmatrix} \Sigma_a & \star & \star \\ \Sigma_b & \Sigma_c & \star \\ \Sigma_d & 0 & -\lambda I_{n_z} \end{bmatrix} \begin{bmatrix} s \\ w \\ \rho \end{bmatrix} < 0, \tag{34}$$

for all $x \in \mathcal{X}$, $s \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, $\rho \in \mathbb{R}^{n_z}$, $(\delta, v) \in \Delta$. Where $\Sigma_a = \mathcal{Y}(x,\delta) - \mathcal{G}(x,\delta) - \mathcal{G}(x,\delta)'$, $\Sigma_b = (A(x,\delta) + B(x,\delta)K(x,\delta))\mathcal{G}(x,\delta)$, $\Sigma_c = -\mathcal{Y}(x_+,\delta+v)$ and $\Sigma_d = (C(x,\delta) + D(x,\delta)K(x,\delta))\,\mathcal{G}(x,\delta)$. Then, the following holds: (i) the closed-loop system with $u = K(x,\delta)x$ is regionally asymptotically stable; (ii) $V(x,\delta)$ is a Lyapunov function in $\mathcal{X}$; (iii) $\mathcal{X}$ is an estimate of the closed-loop DOP; and (iv) $\|z(k)\|_2 < \lambda$ for all $x(0) \in \mathcal{X}$ and $(\delta, v) \in \Delta$.

Observe from above that the matrix $\mathcal{Y}(x_+, \delta + v)$ is a nonlinear function of x_+, i.e. it is a function of the control matrix $K(x,\delta)$, not allowing a parameterization as proposed in Section 4 for the Lyapunov function candidate. A possible solution could be decoupling the vectors x and x_+ by considering that x_+ as a parameter that belongs to a known polytope turning the vector x_+ independent of x as in the quasi-LPV representation (see e.g. [4, 5]). On the one hand we are considering polynomial Lyapunov functions for control purposes, on the other hand we are adding a certain degree of conservativeness since we are not taking into the account the system dynamics.

To overcome this problem, we will use in this paper a parameter-dependent Lyapunov matrix of the form $\mathcal{Y}(x,\delta) = Y(\delta)$ and a nonlinear multiplier $\mathcal{G}(x,\delta)$ leading to a parameterization $\mathcal{Z}(x,\delta) = K(x,\delta)\mathcal{G}(x,\delta)$. From these definitions and similarly to the analysis problem, we can transform the stabilization conditions in Lemma 4 into convex ones. To this end, consider that system (32) can be rewritten as follows:

$$\begin{aligned} x_+ &= \left(\tilde{A}_0 + \tilde{\Pi}_1(x,\delta)'\tilde{A}_1\right)x + \left(B_0 + \tilde{\Pi}_1(x,\delta)'B_1\right)u \\ z &= \left(\tilde{C}_0 + \Phi_1(x,\delta)'\tilde{C}_1\right)x + \left(D_0 + \Phi_1(x,\delta)'D_1\right)u \\ 0 &= \tilde{\Omega}_0(x,\delta) + \tilde{\Omega}_1(x,\delta)\tilde{\Pi}_1(x,\delta) \\ 0 &= \Lambda_0(x,\delta) + \Lambda_1(x,\delta)\Phi_1(x,\delta) \end{aligned} \tag{35}$$

where $\tilde{A}_0 \in \mathbb{R}^{n\times n}, \tilde{A}_1 \in \mathbb{R}^{\tilde{m}\times n}, B_0 \in \mathbb{R}^{n\times r}, B_1 \in \mathbb{R}^{\tilde{m}\times r}, \tilde{C}_0 \in \mathbb{R}^{n_z\times n}, \tilde{C}_1 \in \mathbb{R}^{m_z\times n}, D_0 \in \mathbb{R}^{n_z\times r}$ and $D_1 \in \mathbb{R}^{m_z\times r}$ are constant matrices; $\tilde{\Pi}_1(x,\delta) \in \mathbb{R}^{\tilde{m}\times n}$ and $\Phi_1(x,\delta) \in \mathbb{R}^{m_z\times n_z}$ are nonlinear matrix functions of (x,δ); and $\tilde{\Omega}_0(x,\delta) \in \mathbb{R}^{\tilde{p}\times n}, \tilde{\Omega}_1(x,\delta) \in \mathbb{R}^{\tilde{p}\times\tilde{m}}, \Lambda_0(x,\delta) \in \mathbb{R}^{q_z\times n_z}$ and $\Lambda_1(x,\delta) \in \mathbb{R}^{q_z\times m_z}$ are affine matrix functions of (x,δ). Further, assume that $\tilde{\Omega}_1(x,\delta)$ and $\Lambda_1(x,\delta)$ have column full rank for all x and δ of interest.

In order to simplify the notation, we may represent the above matrix functions without their respective dependence on x and δ and also we may represent system (35) in the following compact form:

$$x_+ = \tilde{\Pi}'(\tilde{\mathcal{A}}x + \mathcal{B}u), \ z = \Phi'(\tilde{\mathcal{C}}x + \mathcal{D}u), \ \tilde{\Omega}\tilde{\Pi} = 0, \ \Lambda\Phi = 0,$$

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where $\tilde{\Omega} = [\ \tilde{\Omega}_0(x,\delta) \quad \tilde{\Omega}_1(x,\delta)\ ]$, $\Lambda = [\ \Lambda_0(x,\delta) \quad \Lambda_1(x,\delta)\ ]$ and

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A}_0 \\ \tilde{A}_1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \tilde{\mathcal{C}} = \begin{bmatrix} \tilde{C}_0 \\ \tilde{C}_1 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} D_0 \\ D_1 \end{bmatrix},$$
$$\tilde{\Pi} = \begin{bmatrix} I_n \\ \tilde{\Pi}_1(x,\delta) \end{bmatrix}, \Phi = \begin{bmatrix} I_{n_z} \\ \Phi_1(x,\delta) \end{bmatrix}.$$

Also, consider the following structure for the matrices $\mathcal{G}(x,\delta)$ and $\mathcal{Z}(x,\delta)$:

$$\mathcal{G}(x,\delta) = G(\delta)\tilde{\Pi} \quad \text{and} \quad \mathcal{Z}(x,\delta) = Z(\delta)\tilde{\Pi} \qquad (36)$$

where $G(\delta) \in \mathbb{R}^{n\times(n+\tilde{m})}$ and $Z(\delta) \in \mathbb{R}^{r\times(n+\tilde{m})}$ are affine matrix of δ to be determined. In addition, define the following matrices:

$$\tilde{Q} = [\ I_n \quad 0_{n\times\tilde{m}}\ ], Q_z = [\ I_{n_z} \quad 0_{n_z\times m_z}\ ]. \qquad (37)$$

Thus, we can state the following convex characterization of Lemma 4.

Theorem 2 *Consider the nonlinear discrete-time system (32) as decomposed in (35), and the matrices $\tilde{Q}, Q_z$ as defined in (37). Further, define:*

$$\Psi = \text{diag}\{\tilde{\Omega}, \tilde{\Omega}, \Lambda\}. \qquad (38)$$

Let $\hat{\mathcal{X}}$ and Δ be given polytopes such that Δ is consistent. Suppose there exist affine matrices $Y(\delta), G(\delta), Z(\delta)$, and a constant matrix L solving the following optimization problem where the LMIs are constructed at $\mathcal{V}(\hat{\mathcal{X}} \times \Delta)$.

$$\max \text{trace}\{Y(\delta)\} \quad \textit{subject to:}$$
$$1 - a_j' Y(\delta) a_j > 0,\ j = 1,\ldots,n_e \qquad (39)$$
$$\begin{bmatrix} \Sigma_{11} & \star & \star \\ \Sigma_{21} & \Sigma_{22} & \star \\ \Sigma_{31} & 0 & -\lambda Q_z' Q_z \end{bmatrix} + L\Psi + \Psi' L' < 0 \quad (40)$$

where $\Sigma_{11} = \tilde{Q}'Y(\delta)\tilde{Q} - \tilde{Q}'G(\delta) - G(\delta)'\tilde{Q}$, $\Sigma_{21} = \tilde{\mathcal{A}}G(\delta) + \mathcal{B}Z(\delta)$, $\Sigma_{22} = -\tilde{Q}'Y(\delta+v)\tilde{Q}$, $\Sigma_{31} = \tilde{\mathcal{C}}G(\delta) + \mathcal{D}Z(\delta)$. Then, the following holds: (i) the closed-loop system with $K(x,\delta) = Z(\delta)\tilde{\Pi}(G(\delta)\tilde{\Pi})^{-1}$ is regionally asymptotically stable; (ii) $V(x,\delta) = x'(Y(\delta))^{-1}x$ is a Lyapunov function in $\mathcal{X}$; (iii) $\mathcal{X} = \{x : V(x,\delta) \le 1,\ \forall\ \delta \in \Delta\}$ is an estimate of the closed-loop DOP; and (iv) $\|z(k)\|_2 < \lambda$ for all $x(0) \in \mathcal{X}$ and $(\delta, v) \in \Delta$.

6 Concluding remarks

This paper has generalized the results of [6] to the guaranteed cost control for a class of uncertain nonlinear discrete-time systems. We have used polynomial Lyapunov functions to reduce the conservatism in the performance analysis and applied a decomposition technique to both the nonlinear system and the Lyapunov function candidate in order to make the computations feasible. We have also extended these results to designing stabilizing controllers with a guaranteed upperbound on the 2-norm of the output signal.

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