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KALMAN FILTERING OVER LOSSY NETWORKS UNDER SWITCHING SENSORS

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ABSTRACT

In this paper, we study the mean square stability of Kalman filtering of a discrete-time stochastic system under two periodically switching sensors. The sensor measurements are sent to a remote estimator over a lossy channel whose packet loss process is independent and identically distributed. We prove that the problem can be converted into the stability analysis of Kalman filtering using two sensors at each time, and the measurements of each sensor are transmitted to the estimator via an independent lossy channel of the same packet loss rate. Some necessary and/or sufficient conditions for stability of the estimation error covariance matrices are derived. Moreover, the effect of the sensor switching on the filter stability is revealed. Their implications and relationships with related results in the literature are discussed.

Key Words: Kalman filtering, switching sensors, packet loss, error covariance matrix, stability.

I. INTRODUCTION

This work is a contribution to the stability analysis of Kalman filtering of a discrete-time stochastic system under two periodically switching sensors with random packet losses. In contrast to the current literature [1–3], the striking difference lies in the use of periodically switching sensors in the networked systems. Sensors of different nature, bandwidth, accuracies and noise levels usually have different performances in specific operating and/or environmental conditions. Thus, the use of different sensors may provide richer information to increase the estimation/control performance. This is particularly important in the situation where a single sensor may not be able to provide sufficient information to estimate the state of a dynamical system.

Specifically, we consider an estimation framework of a stochastic system over a lossy network under two periodically switching sensors. See the networked system in Fig. 1 for an illustration. A motivating example is given by sensors and the estimator communicating over a wireless channel where the quality of the communication network varies over time due to the random channel fading and/or congestion. This happens in resource limited wireless sensor networks where communications between devices are power constrained and therefore limited in range and reliability. As in [2,4], the packet loss process in this work is modeled as an independent and identically distributed (i.i.d.) Bernoulli process. Here the periodically switching sensors are used to observe the system, and result in a switching system. It is well known that the stability analysis of a switching system is usually more involved than that of a time-invariant system [5]. From this perspective, the problem of filter stability involving switching sensors for data transmission over a lossy network is more complicated than that of a single sensor.

Kalman filtering is of great importance in the networked systems due to its various applications ranging from tracking, detection to control. The stability analysis of Kalman filtering with intermittent measurements under a single sensor can be dated back to the influential work [2], which studies the optimal state estimation problem for a discrete-time linear stochastic system with the raw measurements being randomly dropped. By modeling the packet loss process as an i.i.d. Bernoulli process, it is proved that there exists a critical packet loss rate above which the mean state estimation error covariance matrices will diverge. This naturally raises a fundamental problem in quantifying the critical packet loss rate as it is a basic requirement on the lossy network to achieve filter stability. However, their approach for filter stability depends on the use of upper and lower bounds of the estimation error covariance matrices. Since both bounds typically do not coincide, they are unable to exactly quantify the critical loss rate for general vector systems, and only provide its lower and upper bounds, which are attainable

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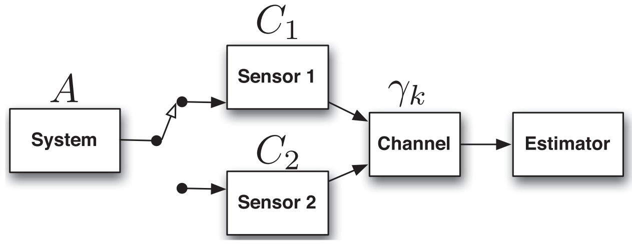

Fig. 1. Networked systems over lossy channels under two periodically switching sensors.

under some special cases, *e.g.*, the lower bound is tight if the observation matrix is invertible. To date, a large amount of effort has been made toward finding the critical packet loss rate [1,3,6,7]. Recently, we investigated the critical rate problem in [1] by developing a random down sampling approach, and obtain an exact quantification for a wider class of systems, including second-order systems and the so-called non-degenerate higher-order systems. A remarkable discovery in [8] is that there are examples of second-order systems for which the lower bound given by [2] is not tight. This approach extensively exploits the system structure, especially the presence form of the unstable open-loop pole [1,8].

In this paper, we extend this approach to the case involving two periodically switching sensors. Note that a periodically time-varying system can be transformed to a time-invariant system. Our idea is based on converting the original stability problem into the stability of the Kalman filter of another dynamical system observed by two sensors at each time, and each sensor's measurements are transmitted through an independent lossy channel with the same statistical property on the packet loss. Some sufficient conditions are given to guarantee the stability of the mean of the state estimation error covariance matrices. While for the second-order dynamical system, we derive the necessary and sufficient condition for the stability of the estimation error covariance matrices. The result exactly characterizes how the sensor switching affects the filter stability, and reveals an interesting tradeoff between the sensor information and network condition for the filter stability. In particular, the more information sent to the estimator per transmission, the less stringent is the condition required for the filter stability.

The rest of the paper is organized as follows. The problem is formulated in Section II. We derive the stability condition for Kalman filtering using periodically switching sensors without packet losses in Section III. Then, the effect of the lossy network on stability of the Kalman filter using two periodically switching sensors is studied in Section IV. In Section V, we derive the necessary and sufficient condition for the stability of the Kalman filter for the second-order system. Some conclusion remarks are drawn in Section VI. A preliminary version of this paper has been reported in [9].

II. PROBLEM FORMULATION

2.1 System description

Consider a linear discrete time-invariant stochastic system as follows:

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ denotes the system state at time k and w_k is a white Guassian noise with zero mean and positive definite covariance matrix Q. The initial state x_0 is a Gaussian random vector with mean $\bar{x}_0$ and covariance matrix P_0.

There are two switching sensors to cooperatively monitor the system, and at each time, one of them takes a noisy measurement from the system by

$$y_k = C_{\sigma_k} x_k + v_{\sigma_k}, \tag{2}$$

where $\sigma_k \in \{1, 2\}$ represents the index of which sensor is activated to take a measurement at time k, and v_{σ_k} is white Gaussian noise with zero mean and positive definite covariance matrix R_{σ_k}. Both C_1 and C_2 are of full row rank. The measurement y_k is directly transmitted to a remote estimator via an unreliable communication channel, see Fig. 1. Due to random fading and/or congestion of the communication channel, packets may be lost while in transit inside the network, which happens frequently in the wireless sensor networks. To examine this phenomenon, we use a binary random process γ_k to denote the packet loss process. Precisely, let $\gamma_k = 1$ indicate that the packet containing the information of y_k has been successfully delivered to the estimator while $\gamma_k = 0$ corresponds to the loss of the packet. In this paper, an erasure channel is employed for date communication, which implies that the random process γ_k is an i.i.d. process [10].

Different from [11,12], the present work ignores other effects such as quantization, transmission errors and data delays. In comparison with [1], the measurement matrix C_{σ_k} is now time-varying, which is used to alleviate the working load of one sensor for the purpose of prolonging the life time of the network or provide richer information for the estimator.

As an initial attempt, we consider a periodically switching rule in this work. To be precise,

$$\sigma_k = \begin{cases} 1, & \text{if } k \text{ is odd;} \\ 2, & \text{if } k \text{ is even.} \end{cases} \tag{3}$$

2.2 Intermittent Kalman filter

Since the switching rule is deterministic, the estimator is able to decide which sensor is in use at each time and whether the packet containing the measurement information y_k is received or not. Then, the information available to the estimator at time k is given by

$$\mathcal{F}_k = \left\{\sigma_i, \gamma_i, y_i\gamma_i, i \leq k\right\}.$$

Denote the minimum mean square error (MMSE) predictor and estimator by $\hat{x}_{k|k-1} = \mathbb{E}\left[x_k|\mathcal{F}_{k-1}\right]$ and $\hat{x}_{k|k} = \mathbb{E}\left[x_k|\mathcal{F}_k\right]$, respectively. Their corresponding estimation error covariance matrices are given by $P_{k|k-1} = \mathbb{E}\left[\left(x_k - \hat{x}_{k|k-1}\right)\left(x_k - \hat{x}_{k|k-1}\right)^T|\mathcal{F}_{k-1}\right]$ and $P_{k|k-1} = \mathbb{E}\left[(x_k - \hat{x}_{k|k})\left(x_k - \hat{x}_{k|k}\right)^T|\mathcal{F}_k\right]$. In view of [2], the above quantities can be computed via the following intermittent Kalman filter.

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \gamma_k K_k \left(y_k - C_{\sigma_k}\hat{x}_{k|k-1}\right); \tag{4}$$

$$P_{k|k} = P_{k|k-1} - \gamma_k K_k C_{\sigma_k} P_{k|k-1}, \tag{5}$$

where the Kalman gain $K_k = P_{k|k-1}C_{\sigma_k}^T\left(C_{\sigma_k}P_{k|k-1}C_{\sigma_k}^T + R_{\sigma_k}\right)^{-1}$, $\hat{x}_{0|-1} = \bar{x}_0$ and $P_{0|-1} = P_0$. In addition, $P_{k+1|k} = Ap_{k|k}A^T + Q$ and $\hat{x}_{k+1|k} = A\hat{x}_{k|k}$.

Let $P_k := P_{k|k-1}$, it follows that

$$\begin{aligned} P_{k+1} &= AP_kA^T + Q - \gamma_k AP_kC_{\sigma_k}^T \\ &\quad \times \left(C_{\sigma_k}P_kC_{\sigma_k}^T + R_{\sigma_k}\right)^{-1} C_{\sigma_k}P_kA^T \\ &:= g_k\left(P_k, R_{\sigma_k}\right). \end{aligned} \tag{6}$$

The goal of this paper is to derive the necessary and/or sufficient condition on the packet loss process for the mean square stability of the filter, *i.e.*, $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$, where the mathematical expectation is taken with respect to the packet loss process $\{\gamma_k\}_{k\in\mathbb{N}}$. In particular, there exists a positive definite matrix $\bar{P}$ such that $P_k \leq \bar{P}$ for all $k \in \mathbb{N}$. The matrix inequality $A \leq B$ means that $B - A$ is semi-positive definite. Similar notations will be made for $\prec, \succ$ and $\geq$ in the remainder of the paper.

Remark 1. In the sequel, the notion of filter stability is always in the sense that $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$.

III. STABILITY OF THE KALMAN FILTER

In this section, we consider the stability of the Kalman filter using two periodically switching sensors without packet losses. This allows us to reveal the effect of lossy channels on the stability of the Kalman filter in the next section. We recall the following result [13].

Lemma 1. $g_k(\cdot,\cdot)$ is monotonically increasing in the sense that

$$g_k(P_1, R) \leq g_k(P_2, R),\ \forall P_1 \leq P_2; \tag{7}$$

$$g_k(P, R_1) \leq g_k(P, R_2),\ \forall R_1 \leq R_2. \tag{8}$$

For convenience, we show that the sensor noise levels do not affect the stability of the Kalman filter under two switching sensors. To this purpose, denote $R_M = R_1 + R_2$ and $R_m = \min\left\{\lambda_{\min}\left(R_1\right), \lambda_{\min}\left(R_2\right)\right\} \cdot I$, where $\lambda_{\min}(R_i) > 0$ is the minimum eigenvalue of R_i. Then, it follows from the monotonicity of $g_k(\cdot,\cdot)$ and $R_m \leq R_{\sigma_k} \leq R_M$ that

$$g_k\left(P_k, R_m\right) \leq g_k\left(P_k, R_{\sigma_k}\right) \leq g_k\left(P_k, R_M\right), \forall k \in \mathbb{N}. \tag{9}$$

This essentially implies that the time-varying property of R_{σ_k} does not affect the stability of P_k. It is also known from [8] that the stability conditions of both $P_{k+1} = g_k\left(P_k, R_m\right)$ and $P_{k+1} = g_k\left(P_k, R_M\right)$ are the same. Thus, there is no loss of generality to assume that $R_1 = R_2 = R$. This implies that the new challenge solely lies in the time-varying nature of the observation matrix C_{σ_k}.

It should be noted that the stability analysis of a time-varying system is usually much more involved than that of a time-invariant system. Since the focus of this work is on quantifying the effect of the lossy network on the stability of the Kalman filter, we first derive the stability condition of the Kalman filter without packet losses, which corresponds to $\gamma_k = 1$ for all $k \in \mathbb{N}$.

By virtue of [14], a necessary and sufficient condition for the stability of the Kalman filter without packet losses is that $\left(A, C_{\sigma_k}\right)$ is uniformly detectable. This requires the unstable modes of the system uniformly observable since all the state variables associated with the stable modes of the system will be exponentially stable in the mean square sense. For this purpose, we only need to consider the state subspace corresponding to unstable modes. Hence, it is sensible to make the following assumption.

Assumption 1. All the eigenvalues of A lie outside or on the unit circle.

Then, $\left(A, C_{\sigma_k}\right)$ is required to be uniformly observable under Assumption 1 for the stability of the Kalman filter without packet losses, *i.e.*, there exist a positive integer h, and positive numbers $\beta_0 > \alpha_0 > 0$ such that

$$\beta_0 I \geq \sum_{i=k}^{k+h} \left(A^{i-k}\right)^T C_{\sigma_i}^T C_{\sigma_i} A^{i-k} \geq \alpha_0 I > 0, \forall k \in \mathbb{N}.$$

The above uniformly observability condition can be further simplified as stated in the following result.

Lemma 2. The system $\left(A, C_{\sigma_k}\right)$ with σ_k given in (3) is *uniformly observable* if and only if both

$$\left(A^2, \begin{bmatrix} C_1 \\ C_2 A \end{bmatrix}\right) \text{ and } \left(A^2, \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix}\right) \tag{10}$$

are observable.

Moreover, if A is nonsingular, the observability property of the following systems

$$\left(A^2, \begin{bmatrix} C_1 \\ C_2 A \end{bmatrix}\right) \text{ and } \left(A^2, \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix}\right) \tag{11}$$

are equivalent.

Proof. The first part directly follows from the definition of observability [15]. We only need to elaborate the second part. For notational simplicity, denote $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ as $[C_1; C_2]$. Let the observability test matrices be $\mathcal{C}_1 = \left[C_1; C_2 A; \ldots; C_1 A^{2(n-1)}; C_2 A^{2n-1}\right]$ and $\mathcal{C}_2 = \left[C_1 A; C_2; \ldots; C_1 A^{2n-1}; C_2 A^{2(n-1)}\right]$. Consider $\mathcal{C}_1$ and $\mathcal{C}_2 A$, it is clear that the rows of both matrices associated with C_2 are the same. By the Cayley-Hamilton theorem, there exist $a_i \in \mathbb{R}$ such that $A^{2n} = a_0 I + a_1 A^2 + \ldots + a_{n-1} A^{2(n-1)}$. Pre-multiply both sides of the equality by C_1, it follows that the last row of $\mathcal{C}_2 A$ associated with C_1 can be linearly represented by the rows of $\mathcal{C}_1$. This further implies that each row of $\mathcal{C}_2 A$ can be represented by the rows of $\mathcal{C}_1$. Hence, rank $\left(\mathcal{C}_2 A\right) \leq$ rank $\left(\mathcal{C}_1\right)$. Similarly, one can argue that rank $\left(\mathcal{C}_1 A\right) \leq$ rank $\left(\mathcal{C}_2\right)$. Since A is nonsingular, it obviously holds that rank $\left(\mathcal{C}_1\right)$ = rank $\left(\mathcal{C}_1 A\right)$ and rank $\left(\mathcal{C}_2\right)$ = rank $\left(\mathcal{C}_2 A\right)$. Combing the above, we obtain that rank $\left(\mathcal{C}_1\right)$ = rank $\left(\mathcal{C}_2\right)$, which completes the proof.

Thus, the uniform observability property of the periodically switching system is converted into that of two time-invariant systems, each of which are observed by two sensors at each time.

In general, the non-singularity assumption on A is mild, *e.g.* it holds for all systems satisfying Assumption 1. By Lemma 2, we focus on the system with the following observability property in this paper since it is the basic requirement for the stability of the Kalman filter without packet losses under Assumption 1.

Assumption 2. Let $C = \left[C_1 A; C_2\right]$, the system $\left(A^2, C\right)$ is observable.

Remark 2. By the PBH test [15], the observability of $\left(A^2, C\right)$ implies that (A, C) is observable while the observability of (A, C) usually does not imply that $\left(A^2, C\right)$ is observable. For instance, $A = \text{diag}(1, -1)$ and $C = [1, 1]$. This, together with Lemma 2, essentially implies that using two sensors to observe the same system at each time requires a weaker condition for the stability of the Kalman filter than that of periodically switching sensor at each time, which certainly is consistent with our intuition as the former case supplies more information than the later one. We also mention that the observability of $(A^2, [C_1; C_2])$ does not imply that of $\left(A^2, C\right)$. For example, $A = \text{diag}(1, -1)$, $C_1 = [1\ 1]$ and $C_2 = [1\ -1]$. It should be noted that both $\left(A, C_1\right)$ and $\left(A, C_2\right)$ are observable.

IV. STABILITY OF THE INTERMITTENT KALMAN FILTER

In this section, we establish the network condition on the packet loss process γ_k for the stability of the intermittent Kalman filter under two periodic sensors.

Denote the packet receival rate $p = \mathbb{P}\{\gamma_k = 1\}$. By Theorem 4 in [1], one can easily establish the following necessary condition.

Theorem 1. Consider the networked system in Fig. 1, a necessary condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that $|\lambda_{\max}|^2(1-p) < 1$, where $\lambda_{\max}$ is the maximum eigenvalue in magnitude of A.

In fact, the above necessary condition has been derived by many authors [1,2,6,7] under a single sensor case, and has been shown to be sufficient as well for some special cases. It is interesting to investigate whether this condition is sufficient under the present framework. For a time-invariant observation matrix, *i.e.*, $C_1 = C_2$, it is shown that the condition in Theorem 1 is also sufficient if C_1 is invertible on the observable subspace [7] or (A, C_1) is a non-degenerate system [1]. Note that the periodic switching between two stable subsystems may lead to an unstable system due to the destabilizing effect of the switching. For example, one can verify that the system

$x_{k+1} = A_k x_k$ is internally unstable where

$$A_k = \frac{1}{8} \cdot \begin{bmatrix} 0 & 9 + 7 \cdot (-1)^k \\ 9 - 7 \cdot (-1)^k & 0 \end{bmatrix},$$

although A_k has all eigenvalues inside the unit circle for each k. This intuitively implies that the derivation of sufficient condition for the filter stability is more involved under the time-varying observation matrices.

In the previous section, the stability condition of the Kalman filter using two periodically switching sensors can be lifted into that of a time-invariant system with two measurement sensors if there is no packet loss (cf. Lemma 2). This motivates us to check whether, under i.i.d. packet losses, the problem under consideration can be converted into the stability analysis of the Kalman filter for a time-invariant system using two measurement sensors simultaneously over two independent lossy channels, each of which is subject to an i.i.d. packet loss process. It turns out to be positive. To elaborate it, we recall a result in [1].

Lemma 3 [1]. Let $\mathcal{O} = \sum_{i=1}^{\infty} \gamma_i \left(A^{-i}\right)^T C_{\sigma_i}^T C_{\sigma_i} A^{-i}$. Under Assumption 1, there exists two positive numbers α and β such that

$$\beta \cdot \mathbb{E}\left[\mathcal{O}^{-1}\right] \geq \sup_{k \in \mathbb{N}} \mathbb{E}\left[P_{k|k}\right] \geq \alpha \cdot \mathbb{E}\left[\mathcal{O}^{-1}\right]. \quad (12)$$

Then, we obtain an interesting result on the equivalent stability property of the networked systems.

Theorem 2. Consider the networked systems in Fig. 1 and Fig. 2. If A is nonsingular, the necessary and sufficient conditions for the stability of the corresponding MMSE estimators are the same.

Proof. Note that $P_{k|k-1} \geq P_{k|k}$ and $P_{k+1|k} = AP_{k|k}A^T + Q$, it is obvious that $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] < \infty$ is equivalent to $\sup_{k \in \mathbb{N}} \mathbb{E}[P_{k|k}] < \infty$. By Lemma 3, the filter stability of the networked system in Fig. 1 is equivalent to that

$$\mathbb{E}\left[\mathcal{O}^{-1}\right] < \infty. \quad (13)$$

Since γ_k is an i.i.d. process, $\mathcal{O}$ can be rewritten by

$$\begin{aligned} \mathcal{O} &= \sum_{i=1}^{\infty} \left(A^{-2i}\right)^T \left[\gamma_{2i-1} A^T C_1^T \ \gamma_{2i} C_2^T\right] \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i} \\ &\overset{d}{=} \sum_{i=1}^{\infty} \left(A^{-2i}\right)^T \left[\alpha_i A^T C_1^T \ \beta_i C_2^T\right] \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i}, \quad (14) \end{aligned}$$

where $\overset{d}{=}$ means the equivalence in distribution on both sides, and α_i, β_i are two i.i.d. Bernoulli processes with the same statistics with γ_i, *i.e.*, $\mathbb{E}[\alpha_i] = \mathbb{E}[\beta_i] = p$. Thus, the filter stability of the networked systems in Fig. 1 is equivalent to that of the first networked system in Fig. 2. The rest of the proof is similarly established.

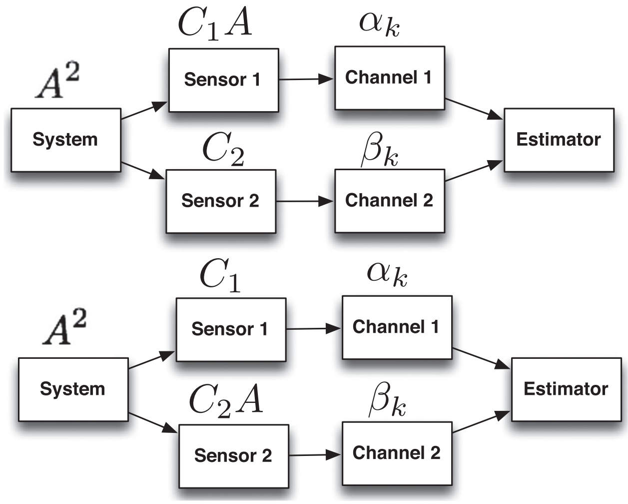

Fig. 2. Networked systems over lossy channels. The open-loop system and the sensor measurement matrices are accordingly denoted above the blocks of systems and sensors. All the lossy channels are subject to the i.i.d. packet loss with the same statistical properties and mutually independent.

The above result can be immediately used to derive a sufficient condition for filter stability.

Theorem 3. Consider the networked system in Fig. 1 satisfying Assumption 1, a sufficient condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] < \infty$ is that

$$\mathbb{E}\left(\sum_{i=0}^{\infty} \zeta_i \left(A^{-2i}\right)^T \left[A^T C_1^T \ C_2^T\right] \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} A^{-2i}\right)^{-1} < \infty, \quad (15)$$

where ζ_i is an i.i.d. process with $\mathbb{P}\left\{\zeta_i = 1\right\} = p^2$.

Proof. By (14), define $\zeta_i = \min\left\{\alpha_i, \beta_i\right\}$, which is again an i.i.d. process with $\mathbb{P}\left\{\zeta_i = 1\right\} = \mathbb{P}\left\{\alpha_i = 1\right\} \mathbb{P}\left\{\beta_i = 1\right\} = p^2$. Then, it follows that

$$\left[\alpha_i A^T C_1^T \ \beta_i C_2^T\right] \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix} \geq \zeta_i \cdot \left[A^T C_1^T \ C_2^T\right] \begin{bmatrix} C_1 A \\ C_2 \end{bmatrix}. \quad (16)$$

Combing Lemma 3 and (14), the proof is completed.

By Theorem 1 and 3, a simple sufficient condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] < \infty$ of a certain class of systems is obtained.

Theorem 4. Consider the networked system in Fig. 1 satisfying Assumption 1 and 2. If $C = [C_1; C_2]$ is of full row rank, a sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$|\lambda_{\max}|^4 \left(1 - p^2\right) < 1. \tag{17}$$

Proof. Since $|\lambda_{\max}|^4 \left(1 - p^2\right) < 1$, there exists a sufficiently small $\epsilon > 0$ such that $\left(|\lambda_{\max}| + \epsilon\|A\|\right)^4 \left(1 - p^2\right) < 1$. Let $\rho = |\lambda_{\max}| + \epsilon\|A\|$, it follows from Lemma 15 [1] that $\|A\|^k \le M\rho^k$ for any $k \in \mathbb{N}$, where $M = \sqrt{n}(1 + 2/\epsilon)^{n-1}$.

If C is of full rank, it holds that $C^TC \succ \lambda_{\min}\left(C^TC\right)\cdot I$, where $\lambda_{\min}\left(C^TC\right) > 0$ is the minimum eigenvalue of C^TC. This implies that

$$\begin{aligned}&\mathbb{E}\left(\sum_{i=0}^{\infty}\zeta_i\left(A^{-2i}\right)^T C^TCA^{-2i}\right)^{-1}\\&\prec \frac{1}{\lambda_{\min}\left(C^TC\right)}\mathbb{E}\left(\sum_{i=0}^{\infty}\zeta_i\left(A^{-2i}\right)^T A^{-2i}\right)^{-1}\end{aligned} \tag{18}$$

Note that $\mathbb{P}\left\{\zeta_1=0,\dots,\zeta_k=0,\dots\right\} = \lim_{k\to\infty}\left(1-p^2\right)^k = 0$. Then, the sum $\sum_{i=0}^{\infty}\zeta_i\left(A^{-2i}\right)^T A^{-2i}$ is positive definite with probability one.

Define a stopping time τ as follows, *i.e.*,

$$\tau := \inf\{k \in \mathbb{N} | \zeta_k = 1\}, \tag{19}$$

whose probability mass distribution is given by $\mathbb{P}\{\tau = k+1\} = p^2\left(1-p^2\right)^k$. Hence,

$$\begin{aligned}\mathbb{E}\left(\sum_{i=0}^{\infty}\zeta_i\left(A^{-2i}\right)^T A^{-2i}\right)^{-1} &\le \mathbb{E}\left[\zeta_\tau A^{2\tau}\left(A^{2\tau}\right)^T\right]\\ &\le \left(\mathbb{E}\left[\|A\|^{4\tau}\right]\right) I \le \left(M\cdot\mathbb{E}\left[\rho^{4\tau}\right]\right) I\\ &= M\rho^4\left(1-p^2\right)\sum_{k=0}^{\infty}\rho^{4k}\left(1-p^2\right)^k\cdot I,\end{aligned}$$

which is finite since $\rho^4\left(1-p^2\right) < 1$. The rest of the proof follows from Theorem 3.

Remark 3. The main conservativeness of the sufficient condition lies in the use of Theorem 3. We use a simple example to illustrate the conservativeness, where $A = \operatorname{diag}\left(\lambda_1, -\lambda_1\right)$, and $C_1 = C_2 = [1, 1]$. By [1], the necessary and sufficient condition is that $|\lambda_1|^2(1-p) < 1$, which is still weaker than $|\lambda_1|^4\left(1-p^2\right) < 1$. Note that this approach does not fully exploit the system structure.

Similarly, the following result is straightforward.

Theorem 5. Consider the networked system in Fig. 1 satisfying Assumption 1 and 2. If either C_1 or C_2 is of full row rank, a sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] \prec \infty$ is that

$$|\lambda_{\max}|^4 \left(1 - p^2\right) < 1. \tag{20}$$

Remark 4. It should be noted that if C_1 is of full row rank and C_2 is a zero matrix, it follows from [1] the condition in Theorem 5 is also sufficient.

By Theorem 2, one obtains that $\mathbb{E}\left[\mathcal{O}^{-1}\right] \prec \infty$ is equivalent to the stability of the Kalman filter of the following networked system

$$x_{k+1} = A^2x_k + w_k, \tag{21}$$

which is observed by two sensors at each time with measurement equations

$$\begin{aligned}y_{k,1} &= C_1Ax_k + v_{k,1},\\ y_{k,2} &= C_2x_k + v_{k,2}.\end{aligned} \tag{22}$$

In the above, $\left(A^2, [C_1A; C_2]\right)$ is observable. $v_{k,1}$ and $v_{k,1}$ are two independent white Gaussian noises. The sensor measurements $y_{k,1}$ and $y_{k,2}$ are sent via two independent lossy channels to the estimator. See Fig. 2 where packet loss processes are modeled by two independent process α_k and β_k, respectively. Then, the corresponding Kalman filters of the networked systems in Fig. 2 require the same network condition for filter stability if A is nonsingular. Thus, it is sufficient to establish the network condition for the stability of the Kalman filter of the first networked system in Fig. 2.

In general, it is challenging to establish the necessary and sufficient condition for a general vector system. Nonetheless, the following procedures can help to reduce the complexity of the problem. Motivated by [1], we will exploit the system structure under Assumption 2, which is classified as follows.

1. Both $\left(A^2, C_1A\right)$ and $\left(A^2, C_2\right)$ are observable.
2. Only one of $\left(A^2, C_1A\right)$ and $\left(A^2, C_2\right)$ is observable.
3. Neither $\left(A^2, C_1A\right)$ or $\left(A^2, C_2\right)$ is observable but $\left(A^2, [C_1A; C_2]\right)$ is observable.

In fact, it only needs to consider Case 1 since the other two cases can be converted into the combination of Case 1 and that in [2,8]. We shall elaborate it in detail.

For Case 2, it does not lose generality to assume that $\left(A^2, C_1A\right)$ is observable but $\left(A^2, C_2\right)$ is not observable. By the Kalman canonical decomposition [15], there exists a coordinate transformation such that $\left(A^2, C\right)$ is transformed into the following structure

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, C_1A = \begin{bmatrix} C_{1,1} & C_{1,2} \end{bmatrix}, C_2 = [0\ C_{2,2}], \quad (23)$$

where $(A_{i,i}, C_{i,i})$ and $(A_{2,2}, C_{1,2})$ are observable. This means that the state variables corresponding to $A_{1,1}$ can only be observed by the sensor associated with measurement matrix C_1A. Then, the filter stability analysis can be further reduce to the case of using only one sensor as in [1,2,8] and Case 1.

Theorem 6. Under Case 2, $\mathbb{E}\left[\mathcal{O}^{-1}\right] \prec \infty$ if and only if $\mathbb{E}\left[\mathcal{O}_1^{-1}\right] \prec \infty$ and $\mathbb{E}\left[\mathcal{O}_2^{-1}\right] \prec \infty$ where

$$\mathcal{O}_1 = \sum_{i=1}^{\infty} \alpha_i \left(A_{1,1}^{-i}\right)^T C_{1,1}^T C_{1,1} A_{1,1}^{-i}$$

and

$$\mathcal{O}_2 = \sum_{i=1}^{\infty} \left(A_{2,2}^{-i}\right)^T \left(\alpha_i C_{1,2}^T C_{1,2} + \beta_i C_{2,2}^T C_{2,2}\right) A_{2,2}^{-i}.$$

Proof. By $\mathbb{E}\left[\mathcal{O}^{-1}\right] \prec \infty$, one can easily verify that $\mathbb{E}\left[\mathcal{O}_1^{-1}\right] \prec \infty$. Partition the state vector as $x_k = \left[x_{k,1}; x_{k,2}\right]$ in conformity with A^2. It follows that

$$\begin{aligned} x_{k+1,2} &= A_{2,2}x_{k,2} + w_{k,2}; \\ y_{k,1} &= C_{1,2}x_{k,2} + C_{1,1}x_{k,1} + v_{k,1}; \\ y_{k,2} &= C_{2,2}x_{k,2} + v_{k,2}. \end{aligned}$$

Since $\mathbb{E}\left[\mathcal{O}_1^{-1}\right] \prec \infty$, the estimation error covariance matrix corresponding to the state variables $x_{k,1}$ is stable. In particular, let $\tilde{x}_{k,i} = x_{k,i} - \hat{x}_{k,i}$, then $\sup_k \mathbb{E}\left[\tilde{x}_{k,1}\tilde{x}_{k,1}^T\right] \prec \infty$. Hence, we can use the following measurement to replace $y_{k,1}$ *viz.*

$$y'_{k,1} = y_{k,1} - C_{1,1}\hat{x}_{k,1} = C_{1,2}x_{k,2} + v'_{k,1}$$

where $v'_{k,1} = C_{1,1}\tilde{x}_{k,1} + v_{k,1}$. By $\mathbb{E}\left[\mathcal{O}_1^{-1}\right] \prec \infty$, it follows that $\sup_k \mathbb{E}\left[\tilde{x}_{k,2}\tilde{x}_{k,2}^T\right] \prec \infty$. Thus, the state vector of the following subsystems can be stably estimated

$$\begin{aligned} x_{k+1,2} &= A_{2,2}x_{k,2} + w_{k,2}; \\ y'_{k,1} &= C_{1,2}x_{k,2} + v'_{k,1}; \\ y_{k,2} &= C_{2,2}x_{k,2} + v_{k,2}. \end{aligned}$$

By Lemma 3, we finally obtain that $\mathbb{E}\left[\mathcal{O}_2^{-1}\right] \prec \infty$.

The necessity can be similarly proved, and is therefore omitted.

For Case 3, it follows from proposition III.1 in [16] that there exists a coordinate transformation such that (A^2, C) has the structure either

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}, C_1A = \begin{bmatrix} 0 & C_{1,2} \end{bmatrix}, C_2 = \begin{bmatrix} C_{2,1} & 0 \end{bmatrix} \quad (24)$$

or

$$A^2 = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ 0 & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{bmatrix}, \quad (25)$$
$$C_1A = [0\ C_{1,2}\ C_{1,3}], C_2 = [C_{2,1}\ 0\ C_{2,3}].$$

The first structure indicates that the measurement matrix C_1 can only be used to observe the state subspace corresponding to $A_{2,2}$ and C_2 observes the complement state subspace. While in the second structure, both sensors can observe a common subspace corresponding to $A_{3,3}$. The decomposition in the first structure is very appealing as it helps us to convert the problems under consideration into the case with only an observation matrix, which has been considered in [1,2,8]. In the second structure, the common observable subspace associated with $A_{3,3}$ is observed by both sensors, which is the same as Case 1.

Hence, we only need to derive the network condition for stability of the Kalman filter over two independent lossy channels for the system satisfying that both (A^2, C_1A) and (A^2, C_2) are observable, which jointly with Assumption 1 implies that (A^2, C_1) and (A^2, C_2A) are observable. To sum up, it is sufficient to focus on the systems satisfying that

Assumption 3. Both (A^2, C_i) and (A^2, C_iA) are observable for any $i \in \{1, 2\}$.

V. SECOND-ORDER SYSTEM

Together with [1], we are able to fully characterize the necessary and sufficient condition for the stability of the Kalman filter using two periodically switching sensors over a lossy network for the second-order system, *i.e.* $A \in \mathbb{R}^{2\times 2}$.

By [1], we only focus on the second-order system satisfying the following condition.

Assumption 4. $A = \text{diag}(\lambda_1, \lambda_2)$ where $\lambda_1 = \lambda_2 \exp(2\pi r I/d)$, $I^2 = -1$ and $d > r > 0$ are irreducible integers.

Then, the necessary and sufficient condition on the filter stability can be exactly given by single inequalities.

Theorem 7. Consider the second-order networked system in Fig. 1 satisfying Assumption 3 - 4. Then, a necessary and sufficient condition for $\sup_{k\in\mathbb{N}} \mathbb{E}[P_k] < \infty$ is that

$$|\lambda_1|^{2d/(d-c)}(1-p) < 1, \tag{26}$$

where c is determined by the number of invertible C_i, and is given by

$$c = \begin{cases} 1, & \text{if } \max\left\{\text{rank}(C_1), \text{rank}(C_2)\right\} = 1, \\ 0, & \text{if } \min\left\{\text{rank}(C_1), \text{rank}(C_2)\right\} = 2, \\ 0.5, & \text{otherwise.} \end{cases} \tag{27}$$

Sketch of Proof.

1. Case $c = 1$. If $C_1 = a \cdot C_2$ and consider the networked systems in Fig. 1, it is equivalent to the system observed by one sensor. This is because the measurements from both sensors are the same except for a scaling by a, which is equivalent to the case without switching. Then, the rest of the proof follows from [1].

   If $\text{rank}(C_1) = \text{rank}(C_2) = 1$, and consider the networked systems in Fig. 2. Let $\zeta_i = \max\{\alpha_i, \beta_i\}$, define a stopping time as follows

   $$\tau_1 = \min\{k|\zeta_k = 1, k \geq 1\}.$$

   Due to the independence of α_i and β_i, the probability mass distribution of τ is given by

   $$\mathbb{P}\{\tau_1 = k\} = \begin{cases} 1-(1-p)^2, & \text{if } k = 1; \\ (1-p)^{2(k-1)}(1-(1-p)^2), & \text{if } k > 1. \end{cases} \tag{28}$$

   By Assumption 3, it follows that $\lambda_1^2 \neq \lambda_2^2$. Together with Assumption 4, it implies that $2r/d$ is not an integer. Then, there exists a positive integer $r_1 < d$ such that $\lambda_1^2 = \lambda_2^2 \exp(2\pi r_1 I/d)$, and r_1, d are irreducible.

   In view of the proof of Theorem 7 in [1], the necessary and sufficient condition becomes that $\mathbb{E}\left[|\lambda_1|^{4\tau_1} 1_{\{\tau_1 \in S_d\}}\right] < 1$, where $S_d = \{kd|\forall k \in \mathbb{N}\}$ and 1_A is a standard indicator function for any set A. By (30), one can easily compute that

   $$\mathbb{E}\left[|\lambda_1|^{4\tau_1} 1_{\{\tau_1 \in S_d\}}\right] = \frac{1-(1-p)^2}{(1-p)^2} \times \frac{\left(|\lambda_1|^2(1-p)\right)^{2d}}{1-\left(|\lambda_1|^2(1-p)\right)^{2d}} < 1,$$

   which is equivalent to that $|\lambda_1|^{2d/(d-1)}(1-p) < 1$.
2. Case $c = 0$. This is trivial, and the proof is omitted.
3. Case $c = 0.5$. Without loss of generality, we assume that $\text{rank}(C_1) = 1$ and $\text{rank}(C_2) = 2$. Define a stopping time

   $$\tau_2 = \min\{k|\alpha_k = 1, \beta_k = 0, \zeta_i = 0, \forall i \leq k-1\} \tag{29}$$

   By the independence property of α_i and β_i, the probability mass distribution of τ_2 is given by

   $$\mathbb{P}\{\tau_2 = k\} = p(1-p)^{2k-1}. \tag{30}$$

   Similarly, the necessary and sufficient condition becomes that

   $$\mathbb{E}\left[|\lambda_1|^{4\tau_2} 1_{\{\tau_2 \in S_d\}}\right] < 1.$$

   Then, it follows that

   $$\mathbb{E}\left[|\lambda_1|^{4\tau_2} 1_{\{\tau_2 \in S_d\}}\right] = \frac{p}{1-p} \times \frac{\left(|\lambda_1|^2(1-p)\right)^{2d}}{1-\left(|\lambda_1|^2(1-p)\right)^{2d}} < 1,$$

which is equivalent to that $|\lambda_1|^{2d/(d-0.5)}(1-p) < 1$.

Remark 5. As remarked in [1], it is very difficult to establish the necessary and sufficient condition for the filter stability of the second order-system satisfying Assumption 4. While for the other cases, the condition becomes simple and is given by $|\lambda_{\max}|^2(1-p) < 1$, where $\lambda_{\max}$ is the largest open loop pole in magnitude, the proof of which can be established by the same approach as in [1].

5.1 Extension to higher-order systems

It is known that the study of general vector systems is very challenging and left to our future work. However, if A is in a certain form, the necessary and sufficient condition for the stability of the Kalman filter can be easily established.

Assumption 5. $A^{-1} = \mathrm{diag}(J_1, \ldots, J_m)$ and $\mathrm{rank}(C_1) = \mathrm{rank}(C_2) = 1$, where $J_i = \lambda_i^{-1} I_i + N_i \in \mathbb{R}^{n_i \times n_i}$ and $|\lambda_i| > |\lambda_{i+1}|$. I_i is an identity matrix with a compatible dimension and the (j, k)-th element of N_i is 1 if $k = j + 1$ and 0, otherwise.

Theorem 8. Consider the networked system in Fig. 1 satisfying Assumptions 1 - 4. Then, a necessary and sufficient condition for $\sup_{k \in \mathbb{N}} \mathbb{E}[P_k] < \infty$ is that

$$|\lambda_{\max}|^2 (1 - p) < 1. \tag{31}$$

Proof. It can be proved by following a similar line as that of Theorem 13 in [1].

VI. CONCLUSION

Motivated by the necessity of using switching sensors in the networked system, we have examined the stability of Kalman filtering with i.i.d. packet losses. Some necessary and sufficient conditions have been derived, which are able to characterize the effect of the periodically switching sensors on the stability. It is stressed that the result of this work is very preliminary and the problem deserves further investigation.

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