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TWO SCHEMES OF DATA DROPOUT COMPENSATION FOR LQG CONTROL OF NETWORKED CONTROL SYSTEMS

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ABSTRACT

This paper investigates the LQG control problem for networked control systems (NCSs) with packet losses, where the packet losses are considered to appear in both the sensor-to-controller channel and controller-to-actuator channel. Bernoulli random processes are used to describe the packet losses in the two channels. Two simple compensation schemes are explored for state estimation with missing measurements in which the input of the plant is set to zero if a packet is lost, and the hold-input strategy, in which the previous input is used with packet dropout. The optimal static controller gains and the critical loss probabilities for the two schemes are presented, and their performances are compared in terms of numerical simulations. The conclusion is that neither of the two schemes can be claimed to be superior to the other, as the stability regions of the two strategies are reversely complemented to each other whether for the scalar or vector example.

Key Words: LQG, packet dropout, networked control systems, Kalman filter, optimal control.

I. INTRODUCTION

Feedback control systems wherein the control loops are closed through a real-time network are called networked control systems (NCSs) [1]. When sensors, actuators, and controllers are connected with information over a real-time network medium, data packet loss often occurs [2], especially in a wireless NCS. The reasons for packet dropout are due to communication noise, interference, or congestion both from sensors to controllers (S/C) and from controllers to actuators (C/A).

Many researchers in past decades have analyzed state estimator and filter design under lossy links without packet loss compensator in S/C [3–5]. There are several works that studied these problems with packet loss in both S/C and C/A [3,6,8–10,19], which are modeled by Bernoulli processes [11]. Nevertheless, Sinopoli *et al.* [3] doesn't consider optimal control. The optimal controllers are presented in [6,8] using the state feedback method. Han *et al*. designed a piecewise state feedback controller for optimal H_∞ performance [7]. Zhao *et al.* [23] provides a design procedure for constructing a controller with the maximum possible H_∞ consensus performance region in multi-agent systems. Since the packet losses in the control loop often render the closed loop system unstable, Schenato uses an intermittent Kalman filter to handle packet losses to ensure the statistical convergence properties of the estimation error covariance [3]. Furthermore, Sahebsara *et al.* deal with these packet losses with H_∞ filtering to control the convergence of the estimation error covariance [7] leading to an optimal H_∞ performance [12,13]. There are some other methods, for example, Zhao *et al.* [14] presents distributed finite-time tracking control for multi-agent systems under a time-invariant communication topology via an observer-based approach. Wang *et al.* [15] proposes a one-step prediction-based packet dropout compensation method, and the NCS is modeled as a discrete switched system with parametric uncertainties [16]. Overall, most works in the literature consider two different schemes for handling the problem of packet loss: one is zero-input strategy, where the actuator input to the plant is set to zero when the control packet from the controller to the actuator is lost [8,16,17] with no computational resources in actuators, as shown in Fig. 1. The other is hold-input strategy, where the latest control input stored in the actuator buffer is used when a packet loss occurs [1,4,18], as shown in Fig. 2.

However, there are few studies in the literature that simultaneously discuss the zero-input scheme and hold-input scheme except [6] and [22], where Schenato only considers the LQ-like performance in [6]. And Zhang *et al.* present a generalized compensation scheme which involves the above two schemes to design the optimal linear filters for networked systems with communication constraints [22], where controller design is not considered. Motivated by these considerations, the purpose of this work is to design the LQG controller for NCSs with packet losses compensation by using a state estimator. The packet dropouts are considered for the case with both S/C and C/A channels, which are

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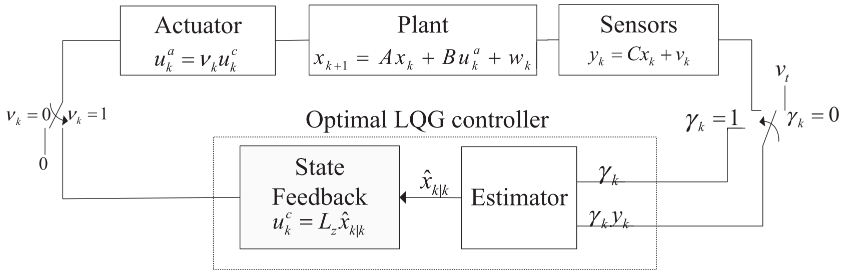

Fig. 1. Diagram of NCS with packet loss: zero-input strategy.

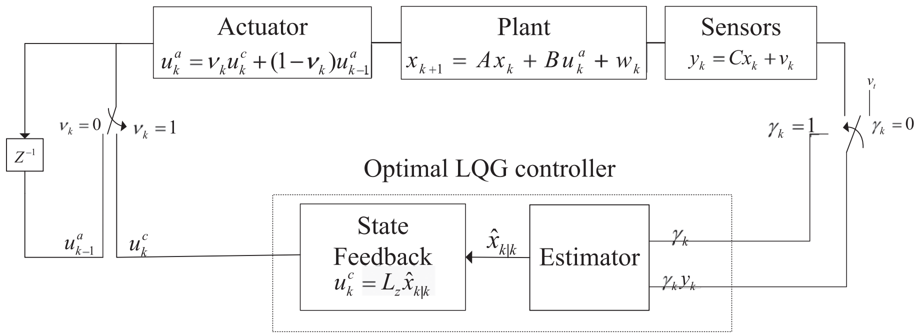

Fig. 2. Diagram of NCS with packet loss: hold-input strategy.

modeled as two independent Bernoulli processes. We will explore two simple compensation schemes: zero-input strategy, where the input to the plant is set to zero if a packet is lost, and hold-input strategy, where the previous stored value in the buffer is used if the packet fails to transmit. The optimal static gains and the critical loss probabilities are calculated using linear matrix inequalities (LMIs) for the two compensation schemes. Moreover, the performances of these two schemes are compared using numerical cases. Coincidentally, the stability regions of the two strategies relatively complement each other for both scalar feedback and vector feedback examples.

Notations. Throughout this paper, R denotes the set of real numbers, R^n denotes the n-dimensional Euclidean space, and $A^{n\times m}$ refers to the set of all $n \times m$ real matrices. A^T represents the transpose of the matrix A, while A^{-1} denotes the inverse of A. For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite, (respectively, positive-definite). I is the identity matrix with appropriate dimensions.

II. PROBLEM FORMULATION

Consider the following discrete-time linear dynamic system

$$x_{k+1} = Ax_k + Bu_k^a + w_k \tag{1}$$

$$u_k^a = v_k u_k^c \tag{2}$$

$$y_k = \gamma_k C x_k + v_k \tag{3}$$

where u_k^a is the control input, u_k^c is the controller output, (x_0, w_k, v_k) are Gaussian, uncorrelated, white noises, with means $(\bar{x}_0, 0, 0)$ and covariances (P_0, Q, R), respectively, and $Q \geq 0, R \geq 0$. w_k is independent of w_s for $s \neq t$. (γ_k, v_k) are i.i.d Bernoulli random variables with Prob$[\gamma_k = 0] = \gamma$ and Prob $[v_k = 0] = v$. We assume that the full state estimator $\hat{x}_{k|k}$ is available to a remote controller that adopts the linear feedback rule

$$u_k^c = L\hat{x}_{k|k} = \begin{cases} L_z \hat{x}_{k|k}, & \text{when zero-input strategy applied} \\ L_h \hat{x}_{k|k}, & \text{when hold-input strategy applied} \end{cases}$$

where L is the controller gain matrix. The subscripts z and h in gains L_z and L_h indicate the zero-input and the hold-input strategy, respectively. The links between in both S/C and C/A are lossy, and the stochastic binary variables $(\gamma_k, v_k) \in \{0, 1\}$ describe the packet dropouts in both S/C and C/A, respectively. We consider two control compensation strategies. In the zero-input strategy, the closed loop system is described by (4)

$$\begin{aligned} x_{k+1} &= Ax_k + Bv_k u_k^c + w_k \\ y_k &= \gamma_k C x_k + v_k \end{aligned} \tag{4}$$

In the case of the hold-input strategy shown in Fig. 2, the closed loop dynamics system is expressed as follows

$$\begin{aligned} x_{k+1} &= Ax_k + B\left(v_k u_k^c + (1-v_k)u_{k-1}^a\right) + w_k \\ y_k &= \gamma_k C x_k + v_k \end{aligned} \tag{5}$$

Define the following information sets $\boldsymbol{\Gamma}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k, \mathbf{v}^{k-1}\}$, where $\mathbf{y}^k = (y_k, y_{k-1}, \ldots, y_1)$, $\boldsymbol{\gamma}^k = (\gamma_k, \gamma_{k-1}, \ldots, \gamma_1)$, and $\mathbf{v}^k = (v_k, v_{k-1}, \ldots, v_1)$. The following cost function is considered

$$\begin{aligned} &J_N\left(\mathbf{u}^{N-1}\bar{x}_0, P_0\right) \\ &= \mathbf{E}\left[x_N^T W_N x_N + \sum_{k=0}^{N-1}\left(x_k^T W_k x_k + v_k u_k^T U_k u_k\right) | \mathbf{u}^{N-1}, \bar{x}_0, P_0\right] \end{aligned} \tag{6}$$

where $\mathbf{u}^{N-1} = (u_{N-1}, u_{N-2}, \ldots, u_1)$. By applying the Kalman filter, the state estimate is given by

$$\begin{aligned} \hat{x}_{k|k} &\triangleq E[x_k | \boldsymbol{\Gamma}_k] \\ e_{k|k} &\triangleq x_k - \hat{x}_{k|k} \\ P_{k|k} &\triangleq E\left[e_{k|k} e_{k|k}^T | \boldsymbol{\Gamma}_k\right] \end{aligned} \tag{7}$$

The following facts are required in the derivation of the estimator.

Lemma 2.1 [8]. The following facts are true

(a) $E\left[(x_k - \hat{x}_k)\hat{x}_k^T | \boldsymbol{\Gamma}_k\right] = E\left[e_{k|k}\hat{x}_k^T | \boldsymbol{\Gamma}_k\right] = 0,$

(b) $E\left[\hat{x}_k^T S x_k | \boldsymbol{\Gamma}_k\right] = \hat{x}_k^T S x_k + \mathrm{trace}\left(SP_{k|k}\right), \forall S \geq 0,$

(c) $\mathbf{E}[\mathbf{E}[g(x_{k+1})|\boldsymbol{\Gamma}_{k+1}]|\boldsymbol{\Gamma}_k] = E[g(x_{k+1})|\boldsymbol{\Gamma}_k], \forall g(\cdot).$

III. ESTIMATOR DESIGN

According to system (4), equations for the optimal estimator are derived by using arguments similar to those used in Standard Kalman filtering, and it follows that

$$\hat{x}_{k+1|k} \triangleq AE[x_k | \boldsymbol{\Gamma}_k] + v_k B u_k^c \tag{8}$$

$$e_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k} = Ae_{k|k} + w_k \tag{9}$$

$$P_{k+1|k} \triangleq E\left[e_{k+1|k} e_{k+1|k}^T | \boldsymbol{\Gamma}_k\right] = AP_{k|k}A^T + Q \tag{10}$$

where the independence of the w_k and $\boldsymbol{\Gamma}_k$, and the requirement that u_k is a deterministic function of $\boldsymbol{\Gamma}_k$, are used. As $y_{k+1}, \gamma_{k+1}, w_k$ and $\boldsymbol{\Gamma}_k$ are independent, the correction step is given by

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right) \tag{11}$$

$$e_{k+1|k+1} = \left(I - \gamma_{k+1} K_{k+1} C\right) e_{k+1|k} - \gamma_{k+1} K_{k+1} v_{k+1} \tag{12}$$

$$P_{k+1|k+1} = P_{k+1|k} - \gamma_{k+1} K_{k+1} C P_{k+1|k} \tag{13}$$

$$K_{k+1} = P_{k+1|k} C^T \left(CP_{k+1|k} C^T + R\right)^{-1} \tag{14}$$

For hold-input strategy, we derive the equations for the optimal estimator using similar arguments to zero-input strategy. The innovation step of state is given by

$$\hat{x}_{k+1|k} \triangleq (A + v_k BL)\hat{x}_{k|k} + (1 - v_k)Bu_{k-1}^a \tag{15}$$

Thus, the innovation step of hold-input strategy is obtained by (15), (9), and (10). The correction step is the same as for the zero-input strategy (11)–(14). And we get

$$\begin{aligned} \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right) \\ &= \left(A - v_k BL - \gamma_{k+1} K_{k+1} CA\right)\hat{x}_{k|k} \\ &\quad + \gamma_{k+1} K_{k+1} CAx_k + (1 - v_k)Bu_{k-1}^a \\ &\quad + \gamma_{k+1} K_{k+1} v_{k+1} + \gamma_{k+1} K_{k+1} Cw_k \end{aligned}$$

By using a modified Kalman filter formulation, it is easy to infer

$$\begin{aligned} P_{k+1|k} &= AP_{k|k-1}A^T \\ &\quad + Q - \gamma_k AP_{k|k-1}C^T\left(CP_{k|k-1}C^T + R\right)^{-1} CP_{k|k-1}A^T \end{aligned} \tag{16}$$

The error covariance matrices $P_{k+1|k}$ are the same through the two strategies. Note that (13) indicates that the error covariance matrices $\left\{P_{k|k}\right\}_{k=0}^N$ are stochastic since they depend on the sequence $\{\gamma_k\}$. Moreover, as the matrix $P_{k+1|k+1}$ is a nonlinear function of the previous covariance $P_{k|k}$, the accurate forecast of these matrices cannot be computed directly. Nevertheless, they can be bounded by computable deterministic quantities, from which we can derive the following lemma.

Lemma 3.1. The expected error covariance matrix $\mathbf{E}[P_{k|k}]$ satisfied the following bounds [3]

$$\widetilde{P}_{k|k} \leq \mathbf{E}\left[P_{k|k}\right] \leq \hat{P}_{k|k}, \forall k \geq 0 \tag{17}$$

where the matrices $\hat{P}_{k|k}$ and $\widetilde{P}_{k|k}$ can be computed as follows

$$\hat{P}_{k+1|k} = A\hat{P}_{k|k-1}A^T + Q - \gamma A\hat{P}_{k|k-1}C^T\left(C\hat{P}_{k|k-1}C^T + R\right)^{-1}C\hat{P}_{k|k-1}A^T \tag{18}$$

$$\hat{P}_{k|k} = \hat{P}_{k|k-1} - \gamma\hat{P}_{k|k-1}C^T\left(C\hat{P}_{k|k-1}C^T + R\right)^{-1}C\hat{P}_{k|k-1} \tag{19}$$

$$\widetilde{P}_{k+1|k} = (1-\gamma)A\widetilde{P}_{k|k-1}A^T + Q \tag{20}$$

$$\widetilde{P}_{k|k} = (1-\gamma)\widetilde{P}_{k|k-1} \tag{21}$$

where the initial conditions are $\hat{P}_{0|0} = \widetilde{P}_{0|0} = P_0$.

Proof. The argument is based on the observation that the matrices $P_{k+1|k}$ and $P_{k|k}$ are concave and monotonic functions of $P_{k|k-1}$. The proof is the same as [3] and is thus omitted. The above results can be summarized as follows.

Theorem 3.2. Consider the system (1)–(3) and the problem of minimizing the cost function (6) within the class of admissible policies $u_k = f(\mathbf{\Gamma}_k)$, where $\mathbf{\Gamma}_k$ is the information available under two strategies as shown in Figs 1 and 2. Thus the following results hold.

(a) The optimal estimator, given by (8)–(15), is independent of the control input u_k.
(b) The optimal estimator gain K_k is time-varying and stochastic since it depends on the past observation loss sequence $\{\gamma_i\}_{i=1}^{k}$.

While the standard LQG optimal regulator always stabilizes the original system, in the case of observation losses, the stability can be lost if the arrival probabilities γ are below a certain threshold. This observation comes from the Modified Riccati Algebraic Equation (MARE), $P_{k+1} = \Pi(P_k, A, C, Q, R, \gamma)$, as described in (16). The results about the MARE are summarized in the following theorem.

Theorem 3.3. Assume that $\left(A, Q^{1/2}\right)$ is controllable, (A, C) is detectable, and A is unstable. Consider the MARE as defined in (16), then the following results hold.

(a) The MARE has a unique strictly positive definite solution P_∞ when $\gamma > \gamma_c$, where γ_c is the critical loss probability.
(b) The critical probability γ_c satisfies the following analytical bounds

$$\gamma_m \le \gamma_c \le \gamma_M,\ \ \gamma_m \triangleq 1 - \frac{1}{\max_i\left|\lambda_i^u(A)\right|^2},\ \ \gamma_M \triangleq 1 - \frac{1}{\Pi_i\left|\lambda_i^u(A)\right|^2}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. In particular, $\gamma_c = \gamma_m$ if C is square and invertible, and $\gamma_c = \gamma_M$ if C is rank one.

The proof of this theorem can be found in [3]. The proof $\gamma_c = \gamma_m$ when C is square and invertible can be found in [20], and the proof $\gamma_c = \gamma_M$ if C is rank one in [21].

IV. OPTIMAL CONTROL

Derivation of the optimal feedback control law and the corresponding value for the objective function will follow the dynamic programming approach based on the cost-to-go iterative procedure. Define the optimal value function $V_k(x_k)$ as follows

$$\begin{aligned} V_N(x_N) &\triangleq \mathbf{E}\left[x_N^T W_N x_N|\mathbf{\Gamma}_k\right], \\ V_k(x_k) &\triangleq \min_{u_k} \mathbf{E}[x_k^T W_k x_k + v_k u_k^T U_k u_k \\ &\quad + V_{k+1}(x_{k+1})|\mathbf{\Gamma}_k], k = N-1, \ldots, 1 \end{aligned} \tag{22}$$

Now make the following computations, which we use to derive the optimal LQG controller

$$\begin{aligned} \mathbf{E}\left[x_{k+1}^T S x_{k+1}|\mathbf{\Gamma}_k\right] &= \mathbf{E}\left[x_k^T A^T S A x_k|\mathbf{\Gamma}_k\right] + v u_k^T B^T S B u_k \\ &\quad + 2v u_k^T B^T S A \hat{x}_{k|k} + \mathrm{trace}(SQ) \\ \mathbf{E}\left[e_{k|k}^T T e_{k|k}|\mathbf{\Gamma}_k\right] &= \mathrm{trace}\left(T\mathbf{E}\left[e_{k|k}e_{k|k}^T|\mathbf{\Gamma}_k\right]\right) \\ &= \mathrm{trace}\left(TP_{k|k}\right), \forall T \ge 0 \end{aligned} \tag{23}$$

where both the independence of v_k, w_k, x_k, and the zero mean property of w_k are exploited.

4.1 Zero-input strategy

Under zero-input strategy the following theorem holds.

Theorem 4.1. The value function $V_k(x_k)$ defined in (22) for the system dynamics of (1)–(3) can be written as

$$V_k(x_k) = \mathbf{E}\left[x_k^T S_k x_k|\mathbf{\Gamma}_k\right] + c_k, k = N, \ldots, 0 \tag{24}$$

where the matrix S_k and the scalar c_k can be computed recursively as follows

$$S_k = A^T S_{k+1} A + W_k - vA^T S_{k+1} B\left(B^T S_{k+1} B + U_k\right) B^T S_{k+1} A \tag{25}$$

$$c_k = \mathrm{trace}\left(\left(A^T S_{k+1} A + W_k - S_k\right) P_{k|k}\right) + \mathrm{trace}(S_{k+1} Q) + \mathbf{E}[c_{k+1}|\mathbf{\Gamma}_k] \tag{26}$$

with initial values $S_N = W_N, c_N = 0$. Moreover, the optimal control input is given by

$$u_k = -\left(B^T S_{k+1} B + U_k\right)^{-1} B^T S_{k+1} A \hat{x}_{k|k} = L_Z \hat{x}_{k|k} \tag{27}$$

Proof. The proof employs an induction argument. The claim is clearly true for $k = N$ with the choice of parameters $S_N = W_N, c_N = 0$. Suppose now that the claim is true for $k+1$, *i.e.* $V_{k+1}(x_{k+1}) = \mathbf{E}\left[x_{k+1}^T S_{k+1} x_{k+1} | \boldsymbol{\Gamma}_{k+1}\right] + c_{k+1}$. The value function at time step k is the following

$$\begin{aligned}
V_k(x_k) &= \min_{u_k} \mathbf{E}\left[x_k^T W_k x_k + v_k u_k^T U_k u_k + V_{k+1}(x_{k+1}) | \boldsymbol{\Gamma}_k\right] \\
&= \min_{u_k} \mathbf{E}\left[x_k^T W_k x_k + v_k u_k^T U_k u_k | \boldsymbol{\Gamma}_k\right] \\
&\quad + \mathbf{E}\left[\mathbf{E}\left[x_{k+1}^T S_{k+1} x_{k+1} + c_{k+1} | \boldsymbol{\Gamma}_{k+1}\right] | \boldsymbol{\Gamma}_k\right] \\
&= \min_{u_k} \mathbf{E}[x_k^T W_k x_k + v_k u_k^T U_k u_k + x_{k+1}^T S_{k+1} x_{k+1} \\
&\quad + c_{k+1} | \boldsymbol{\Gamma}_k] \\
&= \mathbf{E}\left[x_k^T W_k x_k + x_k^T A^T W_{k+1} A x_k | \boldsymbol{\Gamma}_k\right] + \mathrm{trace}(S_{k+1} Q) \\
&\quad + \mathbf{E}[c_{k+1} | \boldsymbol{\Gamma}_k] \\
&\quad + v \min_{u_k} \left(u_k^T (U_k + B^T S_{k+1} B) u_k + 2 u_k^T B^T S_{k+1} A \hat{x}_{k|k}\right)
\end{aligned} \tag{28}$$

$$v_c = \mathrm{argmin}_v \Psi_v(Y, Z) > 0, 0 \le Y \le I.$$

$$\Psi_v(Y,Z) = \begin{bmatrix} Y & Y & \sqrt{v} Z U^{\frac{1}{2}} & \sqrt{v}\left(YA^T + ZB^T\right) & \sqrt{1-v} Y A^T \\ Y & W^{-1} & 0 & 0 & 0 \\ \sqrt{v} U^{\frac{1}{2}} Z H^T & 0 & I & 0 & 0 \\ \sqrt{v}(AY + BZ^T) & 0 & 0 & Y & 0 \\ \sqrt{1-v} A Y & 0 & 0 & 0 & Y \end{bmatrix}$$

where we use Lemma 2.1(c) to get the third equality, and (23) to obtain the last equality. The value function is a quadratic function of the input, therefore, the minimizer can be simply obtained by solving $\partial V_k / \partial u_k = 0$, which gives (27). The optimal feedback controller is, thus, a simple linear function of the estimated state. If we substitute the minimizer back into (21) and use Lemma 2.1(b) we get

$$\begin{aligned}
V_k(x_k) &= \mathbf{E}\left[x_k^T W_k x_k + x_k^T A^T W_{k+1} A x_k \right. \\
&\quad \left. - v x_k^T A^T S_{k+1} B (U_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A x_k | \boldsymbol{\Gamma}_k\right] \\
&\quad + \mathrm{trace}(S_{k+1} Q) + \mathbf{E}[c_{k+1} | \boldsymbol{\Gamma}_k] \\
&\quad + v\,\mathrm{trace}\left(A^T S_{k+1} B (U_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A \hat{x}_{k|k}\right)
\end{aligned}$$

Therefore, the claim given by (24) is also satisfied for time step k for all x_k if and only if (25) and (26) are satisfied.

Theorem 4.2. Consider the modified Riccati equation, which is defined in (25). Assuming $\left(A, W^{\frac{1}{2}}\right)$ is controllable, (A, B) is detectable, and A is unstable, then the following hold.

(a) The MARE has a unique strictly positive definite solution S_∞ if and only if $v > v_c$, where v_c is the critical loss probability.
(b) The critical probability v_c satisfies the following analytical bounds

$$v_m \le v_c \le v_M,\ v_m \triangleq \frac{1}{\max_i \left|\lambda_i^u(A)\right|^2},\ v_M = \frac{1}{\Pi_i \left|\lambda_i^u(A)\right|^2}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. In particular, $v_c = v_m$ if B is invertible, and$v_c = v_M$ if B is rank one.

(c) The critical probability can be numerically computed via the solution for the following quasi-convex LMIs optimization problem

(d) If $v > v_c$, then $\lim_{k\to\infty} S_k = S_\infty$ for all initial conditions $S_0 \ge 0$,where $S_{k+1} = \Pi(S_k, A, B, W, U, v)$.

Proof of the previous theorems can be found in [8].

Using the above results, we can prove the following theorem for the infinite horizon optimal LQG under zero-input strategy.

Theorem 4.3. Consider the same system in the previous theorem, let $W_N = W_k = W$ and $U_k = U$. Moreover, let (A, B) and $(A, Q^{1/2})$ be controllable, and let (A, C) and $(A, W^{1/2})$ be observable. Assume that $v > v_c$ and $\gamma > \gamma_c$, where v_c and γ_c are defined in Theorem 4.2 and Theorem 3.3, respectively. Then the following results are gained.

(a) The infinite horizon optimal controller gain is constant

$$\lim_{k\to\infty} L_k = L_\infty = -\left(B^T S_\infty B + U\right)^{-1} B^T S_\infty A. \tag{29}$$

(b) The matrices are the positive definite solutions of the following equation

$$S_\infty = A^T S_\infty A + W - vA^T S_\infty B\left(B^T S_\infty B + U\right)^{-1} B^T S_\infty A.$$

Proof. (a) Since $v > v_c$, from Theorem 4.2(d) it follows that $\lim_{k\to\infty} S_k = S_\infty$. Therefore, (29) follows from (27). (b) (25) can be written in the terms of MARE as $S_{k+1} = \Pi(S_k, A, B, W, U, v)$, thus, as $v > v_c$ from Theorem 4.2(d) it follows that $\lim_{k\to\infty} S_{k+1} = \lim_{k\to\infty} S_k = S_\infty$.

4.2 Hold-input strategy

Under hold-input strategy the following lemma holds true:

Lemma 4.4. The value function $V_k(x_k)$ defined in (22) for the system dynamics of (1)–(3) can be written as

$$V_k(x_k) = \mathbf{E}\left[x_k^T S_k x_k | \mathbf{\Gamma}_k\right] + c_k, k = N, \ldots, 0 \tag{30}$$

where the matrix S_k and the scalar c_k can be computed recursively as follows

$$S_k = A^T S_{k+1} A + W_k - (1-v)A^T S_{k+1} B\left(B^T S_{k+1} B + U_k\right)^{-1} B^T S_{k+1} A \tag{31}$$

$$c_k = \mathrm{trace}\left(\left(A^T S_{k+1} A + W_k - S_k\right) P_{k|k}\right) + \mathrm{trace}(S_{k+1} Q) + \mathbf{E}[c_{k+1} | \mathbf{\Gamma}_k] + v\left(u_{k-1}^a\right)^T \left(U_k + B^T S_{k+1} B\right) u_{k-1}^a \tag{32}$$

with initial values $S_N = W_N, c_N = 0$. Furthermore the optimal control input is given by

$$u_k = -\left(B^T S_{k+1} B + U_k\right)^{-1} B^T S_{k+1} A \hat{x}_{k|k} = L_h \hat{x}_{k|k} \tag{33}$$

Proof. The proof uses the familiar way in (28),therefore it is omitted. The value function at time step k is the following

$$\begin{aligned} V_k(x_k) &= \min_{u_k^c} \mathbf{E}\left[x_k^T W_k x_k + v_k u_k^{aT} U_k u_k^a + V_{k+1}(x_{k+1}) | \mathbf{\Gamma}_k\right] \\ &= \mathbf{E}[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k \\ &\quad + (1-v_k)^2 u_{k-1}^{aT} (U_k + B^T S_{k+1} B) u_{k-1}^a | \mathbf{\Gamma}_k] \\ &\quad + \mathrm{trace}(S_{k+1} Q) \\ &\quad + \mathbf{E}[c_{k+1} | \mathbf{\Gamma}_k] + (1-v) \min_{u_k^c} (u_k^{cT} (U_k + B^T S_{k+1} B) u_k^c \\ &\quad + 2u_k^{cT} B^T S_{k+1} A \hat{x}_{k|k}) \end{aligned} \tag{34}$$

where u_k^c is independent of u_{k-1}^a and x_k. By solving $\partial V_k / \partial u_k^c = 0$, we get (32) and the same controller gain L_h with that of zero-input strategy. If we substitute the minimizer back into (33) we get

$$\begin{aligned} V_k(x_k) &= \mathbf{E}\left[x_k^T (W_k + A^T S_{k+1} A - (1-v) A^T S_{k+1} B (U_k \right. \\ &\quad + B^T S_{k+1} B) - 1 B^T S_{k+1} A) x_k) + \mathrm{trace}(S_{k+1} Q) \\ &\quad + \mathbf{E}[c_{k+1} | \mathbf{\Gamma}_k] + \mathrm{trace}\left(\left(W_k + A^T S_{k+1} A - S_k\right) P_{k|k}\right) \\ &\quad + v u_{k-1}^{aT} (U_k + B^T S_{k+1} B) u_{k-1}^a \end{aligned}$$

Therefore, the claim given by (30) is satisfied for time step k for all x_k when (31) and (32) are satisfied.

Theorem 4.5. Consider the modified Riccati equation defined in (31). Assuming $\left(A, W^{1/2}\right)$ is controllable, (A, B) is detectable, and A is unstable, then the following hold.

(a) The MARE has a unique strictly positive definite solution S_∞ if and only if $v < v_c$, where v_c is the critical loss probability.

(b) The critical probability v_c satisfies the following analytical bounds

$$v_m \le v_c \le v_M, v_m \triangleq \frac{1}{\Pi_i \left|\lambda_i^u(A)\right|^2}, v_M = \frac{1}{\max_i \left|\lambda_i^u(A)\right|^2}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. In particular, $v_c = v_M$ if B is invertible, and $v_c = v_m$ if B is rank one.

(c) The critical probability can be numerically computed via the solution of the following quasi-convex LMIs optimization problem

$$v_c = \mathrm{argmax}_v \Psi_v(Y, Z) > 0, 0 \le Y \le I.$$

$$\Psi_v(Y,Z) = \begin{bmatrix} Y & Y & \sqrt{1-v} Z U^{\frac{1}{2}} & \sqrt{1-v}\left(YA^T + ZB^T\right) & \sqrt{v} YA^T \\ Y & W^{-1} & 0 & 0 & 0 \\ \sqrt{1-v} U^{\frac{1}{2}} Z^T & 0 & I & 0 & 0 \\ \sqrt{1-v}(AY + BZ^T) & 0 & 0 & Y & 0 \\ \sqrt{v} AY & 0 & 0 & 0 & Y \end{bmatrix}$$

(d) If $v < v_c$, then $\lim_{k\to\infty} S_k = S_\infty$ for all initial conditions $S_0 \geq 0$,where $S_{k+1} = \Pi(S_k, A, B, W, U, v)$.

The proof is similar to Theorem 4.2.

From the results above, we can prove the following theorem for the infinite horizon optimal LQG under hold-input strategy.

Theorem 4.6. Consider the same system in the previous theorem, let $W_N = W_k = W$ and $U_k = U$. Moreover, (A, B) and $(A, Q^{1/2})$ are controllable, (A, C) and $(A, W^{1/2})$ are observable. Assume that $v < v_c$ and $\gamma > \gamma_c$, where v_c and γ_c are defined in Theorem 4.2 and Theorem 3.3, respectively. Then we have the following results.

(a) The infinite horizon optimal controller gain is constant

$$\lim_{k\to\infty} L_k = L_\infty = -\left(B^T S_\infty B + U\right)^{-1} B^T S_\infty A.$$

(b) The matrices are the positive definite solutions of the following equation

$$S_\infty = A^T S_\infty A + W - (1-v) A^T S_\infty B \left(B^T S_\infty B + U\right)^{-1} B^T S_\infty A$$

The proof is similar to Theorem 4.3, thus it is omitted.

Remark 1. From Theorem 4.1 and Lemma 4.4, the matrices S in the different compensator strategy are different. Also the critical loss probability bounds v_c are not the same. These results will be further illustrated in the following numerical examples in Section V.

V. NUMERICAL EXAMPLES

In this section, we present some cases to compare the performance under zero-input and hold-input control architectures.

Example 1. Consider a scalar unstable system (1)–(3) with parameters $A = 1.2$, $B = C = 1$, $W = U = x_0 = 1$, and no process and measurement noise, *i.e.* $R = Q = 0$.

From Theorem 3.3, we get $\gamma_c = 0.3056$, so we assume $\gamma = 0.4 > \gamma_c$ to ensure the closed loop system is stable only if the critical loss probability v_c satisfies the conditions of the two strategies, respectively. After calculating from the above theorems, the critical loss probability $v_{cz} = 0.306$ for the zero-input strategy and $v_{ch} = 0.69$ for the hold-input strategy. Suppose $v = 0.4$, which meets the stability conditions for the two strategies simultaneously, then in zero-input strategy, the matrix $S_\infty = 1.3286$ and the optimal controller gain $L_\infty = -0.6847$. However, the matrix $S_\infty = 1.5213$ and the optimal controller gain $L_\infty = -0.7241$ in hold-input strategy. And the figures of relations between loss probability v_c and weight U, and performance S are presented in Figs 3 and 4, respectively.

Remark 2. Regions in Fig. 3 indicate the best performing strategy in the space (v, U) for hold-input and zero-input control case 1. The stability region for zero-input is $v \in (v_{cz}, 1]$, where $v_{cz} = 0.306$, for hold-input $v \in [0, v_{ch})$, $v_{ch} = 0.69$. Compared with the result in Schenato's paper [6], the difference is that $U \in [0, 10]$ if $v_{cz} \in (0.306, 1]$ when the networked system stabilizes using zero-input strategy, and also $U \in [0, 10]$ if $v_{cz} \in [0, 0.69)$ using hold-input strategy. Stability regions for the two strategies are block diagrams; they have a common stability area $(0.306, 0.69)$. While in [6], the stability packet loss probability regions are the same as $[0, 0.694)$, whether in the hold-input or in the zero-input strategy and the regions are divided into the form of a curve according of U.

Remark 3. Fig. 4 shows a plot of the transition to instability in the scalar case, the dashed-dotted line shows the

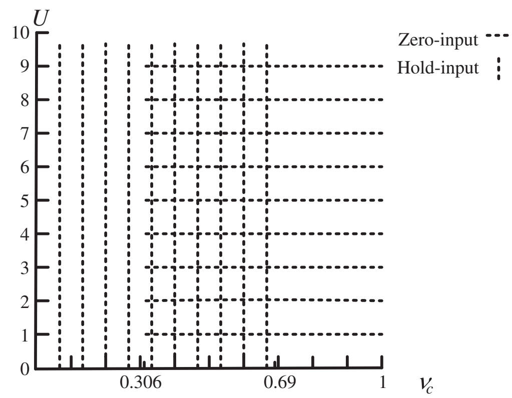

Fig. 3. Regions of loss probability v_c and weight U.

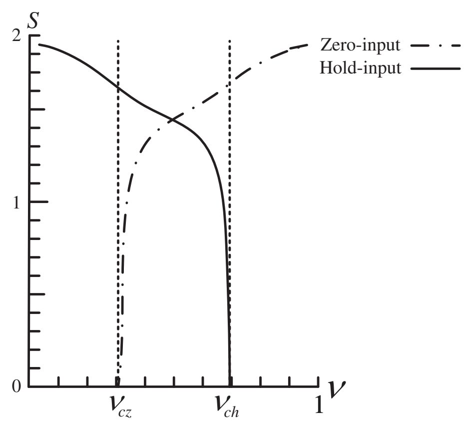

Fig. 4. Transition to instability in the scalar case.

asymptotic value of the zero-input strategy, the solid line for the asymptotic value of the hold-input strategy, and the dashed line shows the asymptote. Compared with Sinopoli's result [4], in which the transition curve is concave, in this paper it is convex.

Example 2. Consider another unstable vector system (1)–(3) with parameters $A=\begin{bmatrix}1.25 & 0\\ 1 & 1.1\end{bmatrix}, B=\begin{bmatrix}1\\1\end{bmatrix}, W=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $U=2.5$, $C=[1 \quad 1]$, $R=2.5$, $Q=20*I$.

We get $0.36 \le \gamma_c \le 0.472$ calculated by Theorem 3.3, let $\gamma=0.5>\gamma_c$ to guarantee the stability of the closed loop system when the critical loss probability v_c satisfies the conditions of the two control architectures, respectively.

In zero-input strategy, the calculations are $v_m=0.36$, $v_M=0.482$, and $v_{cz}=0.47$ by Theorem 4.2, but $0.528 \le v_c \le 0.64$ and $v_{ch}=0.528$ from Theorem 4.5 for hold-input strategy.

To meet the diverse stability conditions, we assume $v=0.5$; coincidentally, the optimal controller gain is equal, $L_\infty=[-0.7248 \quad -0.3183]$, and the matrix $S_\infty=\begin{bmatrix}2.0528 & 0.3505\\ 0.3505 & 1.6465\end{bmatrix}$.

We can intuitively compare the optimality regions from the above examples. Interestingly, the sum of the critical packet loss probabilities of the two strategies is coincidentally 1. For small packet loss probability rates v if $v \le v_{ch}$ the hold-input strategy provides a better performance than the zero-input one, but for packet loss probability $v_{cz} \le v \le 1$, zero-input strategy has a better expression. However, for $v_{ch} \le v \le v_{cz}$ the two strategies will cause the system to create overshooting or oscillations, which means it is not possible to judge which is superior by packet loss compensator using the state estimation.

VI. CONCLUSION

This paper considers LQG controller design based on zero-input strategy and hold-input strategy for NCSs where the control packets are subject to loss. The packet dropouts are modeled by Bernoulli processes, which consider both S/C and C/A channels. The packet losses compensator uses the state estimator for NCSs. We study the zero-input strategy in which the input of the actuator is set to zero if a packet is lost and the hold-input strategy in which the previous value is used with packet dropout. The optimal static gains and the critical loss probabilities for the two schemes are obtained, and their performances on numerical cases are compared. Neither of the two schemes can be claimed superior to the other using the method in this paper, while the stability regions of the two strategies are relatively complementary to each other for either a simple scalar system or vector one. Nevertheless, the algorithms in this paper can be used to compute which scheme performs better once the packet loss probability and systems parameters are known.

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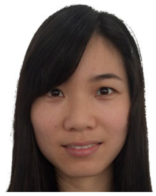

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