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Distributed Consensus of Multi-Agent Systems with Finite-Level Quantization

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Abstract— **This paper is concerned with consensus control of undirected networks of discrete-time first-order agents under quantized communication. A distributed protocol is proposed based on dynamic encoding and decoding with finite level uniform quantizers. It is shown that under the protocol designed, for a connected network, average-consensus can be achieved with an exponential convergence rate based on a single-bit information exchange between each pair of adjacent nodes at each time step. As the number of agents increases, the explicit form of the asymptotic convergence rate is given in relation to the number of nodes, the number of the quantization levels and the ratio between the algebraic connectivity and the spectral radius of the Laplacian of the communication graph.**

I. INTRODUCTION

Recently distributed consensus and average-consensus problems have been paid much attention to by the control community ([1]-[6]). Many effective distributed control and estimation algorithms are proposed based on consensus algorithms ([7]-[8]). However, most of the works in the above literature use an ideal communication model between agents. When agents have real-valued states, this assumption is equivalent to the requirement that the communication channels between agents have unlimited capacity (bandwidth). It is well known that in real digital networks, communication channels have a finite channel capacity, and quantization plays an important role. Therefore, consensus problems under quantized communication become interesting and more meaningful.

In [9]-[11], average-consensus algorithms were designed with each agent having an integer-valued state. These algorithms can drive each agent to some interger approximation of the average of the initial states. In [12]-[14], quantized average-consensus problems were studied with real-valued states. In [12]-[13], algorithms with an uniform quantizer of infinite levels were proposed to ensure the boundness of the consensus error. Furthermore, an algorithm based on dynamic quantization was proposed in [14]. The number of quantization levels, however, will diverge as the number of agents increases. In [15], random dither was used to make the quantization error a "white" noise. Then the distributed stochastic approximation method ([16]-[18]) was applied to achieve approximate average-consensus.

It is well known that for feedback stabilization of linear time-invariant systems with communication constraints, the minimal bit rate (channel capacity) was given in [19] and [20] and the case with logarithmic quantizers was considered in [21] and [22]. Naturally, one may ask, for distributed cooperative control problems of multiagent systems with finite communication data rate, how many bits of information does each pair of adjacent agents need to exchange at each time step to achieve consensus of the whole network?

In this paper, we consider the average-consensus control for discrete-time first-order undirected networks with a finite communication date rate. Each agent has a real-valued state but can only exchange symbolic data with its neighbors. The communication between agents is based on dynamic encoding and decoding with finite-level quantization. We design a distributed protocol with error compensation. The protocol is characterized by three parameters: the control gain, the scaling function, and the number of quantization levels.

We show that if the network is connected, then for any given number of quantization levels, the control gain and the scaling function can be chosen properly such that average consensus can be asymptotically achieved. In particular, the control parameters can be properly chosen such that average-consensus can be achieved by using a single-bit quantizer. This indicates that no matter how large a network is, as long as it is connected, one can always design a distributed protocol to ensure average-consensus with merely one bit information exchange between each pair of adjacent agents at each time step. We also give the relationship between the convergence rate and the number of quantization levels. We show that under the protocol designed, the consensus error vanishes exponentially, and faster convergence requires more bits for quantization.

Our proposed consensus algorithm may be applied to the distributed estimation over large scale sensor networks, where the number of nodes is often large. This give rise to investigating the asymptotic property of the closed-loop system as the number N of nodes approaches infinity. We show that in some sense, the asymptotic optimal convergence rate is $O(\exp\{-\frac{KQ_N^2}{2\sqrt{N}}t\})$ when using a $(2K+1)$-level quantizer, where Q_N, an important physical quantity reflecting the synchronizability of a network, is the ratio between the second smallest eigenvalue (algebraic connectivity) and the

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largest eigenvalue (spectral radius) of the Laplacian matrix of the topology graph. Our result shows that when the communication data rate is limited, the convergence rate of distributed consensus not only depends on the connectivity but also the synchronizability of the communication graph.

The remainder of this paper is organized as follows. In Section II, we present the model of the network, design the distributed protocol, and formulate the problem to be investigated. In Section III, we prove that under the protocol designed and mild conditions, average-consensus can be achieved with an exponential convergence rate. Then we analyze the asymptotic performance as $N \to \infty$ and give an explicit form of the asymptotic convergence rate. In Section IV, we give a numerical example to demenstrate our results. In Section V, we give some concluding remarks.

The following notation will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones. I denotes the identity matrix with an appropriate size. For a given set $\mathcal{S}$, the number of its elements is denoted by $|\mathcal{S}|$. For a given vector or matrix A, its transpose is denoted by A^T, its Euclidean norm is denoted by $\|A\|_2$. For a given positive number x, the maximum integer less than or equal to x is denoted by $\lfloor x \rfloor$; the minimum integer greater than or equal to x is denoted by $\lceil x \rceil$.

II. Problem formulation

A. Average-consensus problem

In this paper, the dynamics of each agent is modeled as a discrete-time first-order integrator:

$$x_i(t+1) = x_i(t) + hu_i(t),\ t = 0,1,...,\ i = 1,2,...,N, \quad (1)$$

where $x_i(t) \in \mathbb{R}$ is the ith agent's state, $u_i(t) \in \mathbb{R}$ is the ith agent's control input, and h is the control gain. The information flow among agents are modeled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, ..., N\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ is the set of edges and $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{G}$. An edge denoted by the unordered pair (j,i) represents a communication channel from j to i. Note that $\mathcal{A}$ is a symmetric matrix. For any $i,j \in \mathcal{V}$, $a_{ij}=a_{ji} \geq 0$, and $a_{ij} > 0$ if and only if $j \in N_i$. Also, $\deg_i = \sum_{j=1}^N a_{ij}$ is called the degree of i, and $d^*=\max_i \deg_i$ is called the degree of $\mathcal{G}$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\deg_1, ..., \deg_N)$. A sequence of edges (i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k) is called a path from node i_1 to node i_k. The graph $\mathcal{G}$ is called a connected graph if for any $i, j \in \mathcal{V}$, there is a path from i to j.

The dynamic system (1) together with the communication graph $\mathcal{G}$ is usually called a dynamic network ([4]). A group of controls $\mathcal{U} = \{u_i, i = 1,2...,N\}$ is called a distributed protocol if for all i, $u_i(t)$ only depends on $x_i(s)$ and $x_j(s)$, $j \in N_i$, $s \leq t$. The average-consensus control is to design a distributed protocol for the dynamic network, such that for any initial values $x_1(0), ..., x_N(0)$, all the agents asymptotically reach an agreement with $\frac{1}{N}\sum_{j=1}^N x_j(0)$ when $t \to \infty$. That is, $\frac{1}{N}\sum_{j=1}^N x_j(0)$ can be computed asymptotically in a distributed manner.

B. Protocol design

In [4], a weighted-average protocol was proposed:

$$u_i(t) = \sum_{j\in N_i} a_{ij}(x_j(t) - x_i(t)),\ t = 0,1,..., \quad i = 1,2,...,N. \quad (2)$$

In (2), the ith agent needs the perfect state information of its neighbors. In this paper, we assume that perfect state information is not available, but only symbolic data can be exchanged between agents, and the communication channels are modeled as noiseless digital channels each with a pair of encoder and decoder. The encoder Φ_j of the jth agent is given by

$$\begin{cases} \xi_j(0) &= 0, \\ \xi_j(t) &= g(t-1)\Delta_j(t) + \xi_j(t-1), \\ \Delta_j(t) &= q(\frac{x_j(t)-\xi_j(t-1)}{g(t-1)}), \\ & \quad t = 1,2,... \end{cases} \quad (3)$$

where $\xi_j(t)$ is the internal state of Φ_j, and $\Delta_j(t)$, which is the output of Φ_j, is sent to the neighbors of the jth agent. Here, $q(\cdot)$ is a finite-level uniform quantizer, and $g(t) > 0$ is a scaling function.

The quantizer $q(\cdot)$: $\mathbb{R} \to \Gamma$ is a map from $\mathbb{R}$ to the set Γ of quantized levels. In this paper, we consider a finite-level uniform quantizer with

$$\Gamma = \{0, \pm i, i = 1,2,...K\}.$$

The number of quantization levels is $2K+1$. The associated quantizer $q(\cdot)$ is given by

$$q(y) = \begin{cases} 0, & -1/2 < y < 1/2, \\ i, & \frac{2i-1}{2} \leq y < \frac{2i+1}{2}, \\ & i = 1,2,...,K-1, \\ K, & y \geq \frac{2K-1}{2}, \\ -q(-y), & y \leq -1/2. \end{cases} \quad (4)$$

Remark 1: The encoder Φ_j is a scaled difference encoder, and $\xi_j(t)$ is a one-step predictor. In this difference encoding algorithm, at each time step the "prediction error", $x_j(t) - \xi_j(t-1)$ is quantized. Generally speaking, the amplitude of prediction error is smaller than that of state $x_j(t)$ itself, so it can be represented by fewer bits.

Remark 2: If consensus is achieved, then the prediction error $x_j(t) - \xi_j(t-1)$ vanishes as $t \to \infty$. Therefore, intuitively the scaling function $g(t)$ should satisfy the following properties. On one hand, $g(t)$ should converges to zero asymptotically make the quantizer persistently excited, such that the agents receive the their neighbors' information continuously. On the other hand, $g(t)$ should be large enough such that the quantizer will not be saturated.

For each communication channel $(j,i) \in \mathcal{E}$, the ith agent receives $\Delta_j(t)$, and then uses the following decoder Ψ_{ji} to estimate $x_j(t)$:

$$\begin{cases} \widehat{x}_{ji}(0) &= 0, \\ \widehat{x}_{ji}(t) &= g(t-1)\Delta_j(t) + \widehat{x}_{ji}(t-1), \\ & \quad t = 1,2,... \end{cases} \quad (5)$$

where $\widehat{x}_{ji}(t)$ is the output of Ψ_{ji}.

Remark 3: When the output $\Delta_j(t)$ of the quantizer is zero, the jth agent does not send any information, so for a $(2K+1)$-level quantizer $q(\cdot)$, the communication channel (j,i), $i \in N_j$ is required to be capable of transmitting $\lceil \log_2(2K) \rceil$ bits without error at each time step. In particular, the quantizer $q(\cdot)$ given by

$$q(y) = \begin{cases} 0, & -1/2 < y < 1/2, \\ 1, & y \geq 1/2, \\ -1, & y \leq 1/2 \end{cases} \tag{6}$$

is a one-bit quantizer.

We propose a distributed protocol as

$$u_i(t) = \sum_{j \in N_i} a_{ij}(\widehat{x}_{ji}(t) - \xi_i(t)), \ t = 0,1,..., \quad i = 1,2,...N. \tag{7}$$

Denote

$$X(t) = [x_1(t),...,x_N(t)]^T, \widehat{X}(t) = [\xi_1(t),...,\xi_N(t)]^T, \quad e(t) = X(t) - \widehat{X}(t), \delta(t) = X(t) - J_N X(t), \tag{8}$$

where $J_N = \frac{1}{N}\mathbf{1}\mathbf{1}^T$.

Remark 4: If $\xi_i(t)$ is replaced by $x_i(t)$, then the protocol (7) becomes

$$u_i(t) = \sum_{j \in N_i} a_{ij}(\widehat{x}_{ji}(t) - x_i(t)), \ t = 0,1,... \quad i = 1,2,...N. \tag{9}$$

The protocol (9) is a natural extension of the protocol (2) to the case with quantized communications and it has some computational advantage over the protocol (7). One may wonder why we use the protocol (7) rather than the protocol (9). We give some explanations below.

From (3) and (5), it follows that

$$\widehat{x}_{ji}(t) = \xi_j(t), \ t = 0,1,..., \ i \in N_j, \ j = 1,2,...,N. \tag{10}$$

Thus, the internal state $\xi_j(t)$ of encoder Φ_j is equal to the estimates of $x_j(t)$ by its neighbors. By the symmetry of $\mathcal{A}$ and (10), the protocol (7) can be rewritten as

$$\begin{aligned} u_i(t) &= \sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t) - (x_j(t) - \widehat{x}_{ji}(t)) \\ &\quad +(x_i(t) - \xi_i(t))] \\ &= \sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t)] - \sum_{j \in N_i} a_{ij}(x_j(t) - \widehat{x}_{ji}(t)) \\ &\quad + \sum_{j \in N_i} a_{ji}(x_i(t) - \widehat{x}_{ij}(t)). \end{aligned} \tag{11}$$

It can be seen that, in our protocol (7), the control input of the ith agent consists of three terms. The first term, $\sum_{j \in N_i} a_{ij}[x_j(t) - x_i(t)]$, which is just the control input of the protocol (2), plays the main role. The second term, $-\sum_{j \in N_i} a_{ij}(x_j(t) - \widehat{x}_{ji}(t))$, represents the weighted sum of estimation errors for the neighbors' states. The last term $\sum_{j \in N_i} a_{ji}(x_i(t) - \widehat{x}_{ij}(t))$ is the weighted sum of estimation errors for $x_i(t)$ by the neighbors.

The last term in (11), which we call an error-compensation term, plays an important role in our protocol. Substituting the protocol (3), (5) and (7) into the system (1) leads to

$$\begin{cases} X(t+1) &= (I - h\mathcal{L})X(t) + h\mathcal{L}e(t), \\ \widehat{X}(t+1) &= g(t)Q[\frac{X(t+1) - \widehat{X}(t)}{g(t)}] + \widehat{X}(t), \end{cases}$$

where $Q([y_1,...,y_N]^T) = [q(y_1),...,q(y_N)]^T$. From the above, noting that $J_N L_{\mathcal{G}} = 0$, we have

$$\frac{1}{N}\sum_{j=1}^{N} x_j(t+1) = \frac{1}{N}\sum_{j=1}^{N} x_j(t), \ t = 0,1,...$$

It can be seen that the closed-loop system preserves the average state under the protocol (7). If the error-compensation term is removed, then by (11), the protocol (7) reduces to the protocol (9), and the closed-loop system becomes

$$X(t+1) = (I - h\mathcal{L})X(t) - h\mathcal{A}e(t). \tag{12}$$

Generally speaking, the closed-loop system (12) does not preserve the average state, and worse still, it can be shown that the closed-loop system (12) may be divergent if $e(t)$ is a bounded white noise ([23]). That is why we use the protocol (7) rather than the protocol (9). This type of protocols which can preserve the state average have been proposed in [12] and some similar methodology of error compensation has been addressed in [24].

III. Finite-level Quantized Consensus

For the protocol designed and the resulting closed-loop system (12), We are concerned about whether the network can achieve consensus with finite-level quantized communication. If so, how many bits are necessary for each pair of adjacent agents to exchange at each time step? Can we give a quantitative description of the relationship between the convergence rate and the control parameters? In this section, we will answer the above questions.

To get the main results, we need the following assumptions:

A1) $\mathcal{G}$ is connected.

A2) $\max_i |x_i(0)| \leq C_x$, $\max_i |\delta_i(0)| \leq C_\delta$, where C_x and C_δ are known nonnegative constants.

Below is a basic result on Laplacian matrices:

Lemma 3.1: ([25]) If $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is an undirected graph, then the Laplacian $\mathcal{L}$ is a symmetric matrix, and has N real eigenvalues, in an ascending order:

$$0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq ... \leq \lambda_N(\mathcal{L}) \leq 2d^*,$$

and

$$\min_{x \neq 0, \mathbf{1}^T x = 0} \frac{x^T \mathcal{L} x}{\|x\|^2} = \lambda_2(\mathcal{L}),$$

where $\lambda_2(\mathcal{L})$ is called the algebraic connectivity of $\mathcal{G}$. In particular, if $\mathcal{G}$ is connected, then $\lambda_2(\mathcal{L}) > 0$.

We need the following lemmas.

Lemma 3.2: If Assumption A1) holds and $h < \frac{2}{\lambda_N(\mathcal{L})}$, then $\rho_h < 1$, where

$$\rho_h = \max_{2 \leq i \leq N} |1 - h\lambda_i(\mathcal{L})|. \tag{13}$$

Furthermore, if $h < \frac{2}{\lambda_2(\mathcal{L})+\lambda_N(\mathcal{L})}$, then $\rho_h = 1 - h\lambda_2(\mathcal{L})$.

Lemma 3.3: Suppose Assumptions A1)-A2) hold. For any given $h \in (0, \frac{2}{\lambda_N(\mathcal{L})})$ and $\gamma \in (\rho_h, 1)$, let

$$K_1(h,\gamma) = \lfloor M_1(h,\gamma) - \frac{1}{2} \rfloor + 1, \quad (14)$$

$$M_1(h,\gamma) = \frac{\sqrt{N} h^2 \lambda_N^2(\mathcal{L})}{2\gamma(\gamma - \rho_h)} + \frac{1 + h\lambda_N(\mathcal{L})}{2\gamma}, \quad (15)$$

and

$$g_0 > \max\{\frac{C_x}{K + \frac{1}{2}}, \frac{2(\gamma - \rho_h)(C_\delta \gamma + hC_x \lambda_N(\mathcal{L}))}{h\lambda_N(\mathcal{L})}\}. \quad (16)$$

Then for any given $K \geq K_1(h,\gamma)$, under the protocol (3), (5) and (7) with the $(2K+1)$-level uniform quantizer (4) and the scaling function $g(t) = g_0\gamma^t$, the closed-loop system (12) satisfies

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \ i = 1, 2..., N.$$

The *convergence rate of average-consensus* ([26]) is defined as

$$r_{\text{asym}} = \sup_{X(0)\neq J_N X(0)} \lim_{t\to\infty} \left(\frac{\|X(t) - J_N X(0)\|_2}{\|X(0) - J_N X(0)\|_2}\right)^{1/t}.$$

For the convergence rate of our algorithm, we have the following theorem.

Theorem 3.1: Suppose the conditions of Lemma 3.3 hold, then under the protocol (3), (5) and (7) with the $(2K+1)$-level uniform quantizer (4), the closed-loop system (12) satisfies

$$\|\delta(t)\|_2 = O(\gamma^t), \ t \to \infty,$$

and $r_{\text{asym}} \leq \gamma$, where $\delta(t)$ is the consensus error defined by (8).

Due to space limit, the proofs of Lemmas 3.2, 3.3 and Theorem 3.1 are omitted here.

Remark 5: Lemma 3.3 says that by using a scaling function decaying exponentially and a $\lceil \log_2(2K_1(h,\gamma)) \rceil$-bit uniform quantizer, the protocol (3), (5) and (7) can ensure average-consensus to be achieved asymptotically. It is worth pointing out that for any given h and γ, the bit number $\lceil \log_2(2K_1(h,\gamma)) \rceil$ is a conservative estimate, and in practice, fewer bits may be required. However, the number $K_1(h,\gamma)$ gives us some intuitive clues on the relationship between the number of bits required and the control gain h and the scaling factor γ.

Remark 6: Theorem 3.1 gives an estimate for the convergence rate of the consensus. The smaller the γ, the faster the consensus error converges to zero. Note that γ can be made arbitrarily close to ρ_h, which is the convergence rate for the case with perfect communication ([26]). From Lemma 3.3, it is shown that a smaller γ, namely a faster convergence rate, requires more bits to be communicated, and when $\gamma \to \rho_h$, the required number of bits goes to infinity.

From Lemma 3.3 and Theorem 3.1, it can be seen that if the convergence rate γ is fixed (i.e., independnent of N of agents), then the number of quantization levels, $2K_1(h,\gamma)+1$, will increase to infinity as $N \to \infty$. However, in many cases, we do not expect that the number of bits is too large. To satisfy this requirement, we can use a fixed number of quantization levels at the cost of slower convergence. We have the following result.

Theorem 3.2: Suppose Assumptions A1)-A2) hold. For any given $K \geq 1$, let

$$\Omega_K = \{(\alpha,\beta) | \ \alpha \in (0, \frac{2}{\lambda_2(\mathcal{L}) + \lambda_N(\mathcal{L})}), \beta \in (\rho_\alpha, 1), M_1(\alpha,\beta) < K + \frac{1}{2}\}, \quad (17)$$

where ρ_α is defined by (13) and $M_1(\alpha,\beta)$ is defined by (15). Then, (i) Ω_K is nonempty. (ii) For any $(h,\gamma) \in \Omega_K$, under the protocol (3), (5) and (7) with $g(t) = g_0\gamma^t$ and the $(2K+1)$-level uniform quantizer (4), the closed-loop system (12) satisfies

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \ i = 1, 2..., N,$$

where g_0 is a constant satisfying (16).

Proof: (i) Noting that

$$\lim_{\alpha\to 0}\left[\frac{\sqrt{N}\alpha\lambda_N^2(\mathcal{L})}{2\lambda_2(\mathcal{L})} + \frac{1+\alpha\lambda_N(\mathcal{L})}{2}\right] = \frac{1}{2},$$

we know that for any given $K \geq 1$, there exists $\alpha^* \in (0, \frac{2}{\lambda_2(\mathcal{L})+\lambda_N(\mathcal{L})})$ such that

$$\frac{\sqrt{N}\alpha^*\lambda_N^2(\mathcal{L})}{2\lambda_2(\mathcal{L})} + \frac{1+\alpha^*\lambda_N(\mathcal{L})}{2} < K + \frac{1}{2}. \quad (18)$$

By Lemma 3.2, it is known that $\rho_{\alpha^*} = 1 - \alpha^*\lambda_2(\mathcal{L}) < 1$. By this and (15), we get

$$\lim_{\gamma\to 1} M_1(\alpha^*,\gamma) = \frac{\sqrt{N}\alpha^*\lambda_N^2(\mathcal{L})}{2\lambda_2(\mathcal{L})} + \frac{1+\alpha^*\lambda_N(\mathcal{L})}{2}.$$

Then by (18), we know that there exists $\gamma^* \in (\rho_{\alpha^*}, 1)$, such that

$$M_1(\alpha^*,\gamma^*) < K + \frac{1}{2}.$$

Therefore $(\alpha^*,\gamma^*) \in \Omega_K$, that is, Ω_K is nonempty.

(ii) For any $(h,\gamma) \in \Omega_K$, by (17), we know that $h \in (0, \frac{2}{\lambda_N(\mathcal{L})})$, $\gamma \in (\rho_h, 1)$, and $\frac{1}{2} < M_1(h,\gamma) < K + \frac{1}{2}$. Thus, by (14), one gets $K_1(h,\gamma) \leq K$, which together with Lemma 3.3 leads to the conclusion of the theorem.

Remark 7: From Theorem 3.2, it is shown that as long as the network is connected, we can always design a distributed protocol to ensure average-consensus with each agent sending merely one bit of information to its neighbors at each time step.

The set Ω_K is a plane point set described by three nonlinear inequalities. Generally speaking, it is difficult to get an explicit solution of these inequalities. However, by introducing a free parameter $\epsilon_0 \in (0,1)$, we can get a simple algorithm to choose (h, γ) from Ω_K for any given $K \geq 1$.

Algorithm 1:

(i) Choose a constant $\epsilon_0 \in (0,1)$.

(ii) Choose the control gain $h \in (0, h_K^*(\epsilon_0))$, where

$$h_K^*(\epsilon_0) = \min\{\frac{2}{\lambda_2(\mathcal{L})+\lambda_N(\mathcal{L})}, \frac{2K\epsilon_0\lambda_2(\mathcal{L})}{\sqrt{N}\lambda_N^2(\mathcal{L})+\lambda_2(\mathcal{L})\lambda_N(\mathcal{L})\epsilon_0+(2K+1)\lambda_2^2(\mathcal{L})\epsilon_0(1-\epsilon_0)}\},$$

(iii) Let $\gamma = 1-(1-\epsilon_0)h\lambda_2(\mathcal{L})$.

The result below show that any pair (h, γ) generated by Algorithm 1 belongs to Ω_K and any point in Ω_K can be generated by Algorithm 1.

Theorem 3.3: For any given $K \geq 1$, and $\epsilon_0 \in (0,1)$, let

$$\Omega_{K,\epsilon_0} = \{(\alpha,\beta)|\ \alpha \in (0, h_K^*(\epsilon_0)), \beta = 1-(1-\epsilon_0)\alpha\lambda_2(\mathcal{L})\}.$$

Then we have $\Omega_K = \bigcup_{\epsilon_0\in(0,1)} \Omega_{K,\epsilon_0}$.

In many cases, the number N of the network nodes is large and we are concerned about the asymptotic property as N approaches infinity. In the following, we investigate the asymptotic performance of the closed-loop system. It can be seen that the asymptotic value of γ has a very compendious expression.

Theorem 3.4: Suppose Assumption A1) holds. Then for any given $K \geq 1$,

$$\lim_{N\to\infty} \frac{\inf_{(h,\gamma)\in\Omega_K} \gamma}{\exp\{-\frac{KQ_N^2}{2\sqrt{N}}\}} = 1,$$

where $Q_N = \frac{\lambda_2(\mathcal{L})}{\lambda_N(\mathcal{L})}$.

Remark 8: Q_N is an important physical factor. It is shown that a network exhibits better synchronizability if Q_N is large ([27]). Theorem 3.4 shows that in some sense, the best convergence rate we can achieve is $O(\exp\{-\frac{KQ_N^2}{2\sqrt{N}}t\})$. Therefore, the best convergence rate is closely related to the number of the quantization levels, the scale and the synchronizability of the network.

Due to space limit, the proofs of Theorem 3.3 and 3.4 are omitted here.

IV. Numerical Example

In this section, we use an example to demonstrate the validity of the proposed consensus protocol and Theorem 3.1.

Example 1: We consider a network with 30 nodes and $0-1$ weights [1] shown in Fig. 1. The initial states are chosen as $x_i(0) = i$, $i = 1, ..., 30$. The one-bit quantizer is used. The curves of states with $h = 0.1$, $g(t) = 20(0.95)^t$ are shown in Fig. 2. Then the 5-level quantizer is used. The curves of states with $h = 0.1$, $g(t) = 20(0.975)^t$ are shown in Fig. 3. It can be seen that average-consensus is achieved with an exponential rate in both cases and a smaller γ leads to a faster convergence.

[1] $0-1$ weights means that $a_{ij} = 1$, if $(i,j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$.

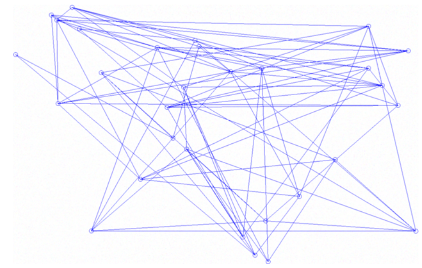

Fig. 1. Network topology of Example 1. The edges of the graph are randomly generated according to $P\{(i,j) \in \mathcal{E}_\mathcal{G}\} = 0.2$, for any unordered pair (i,j).

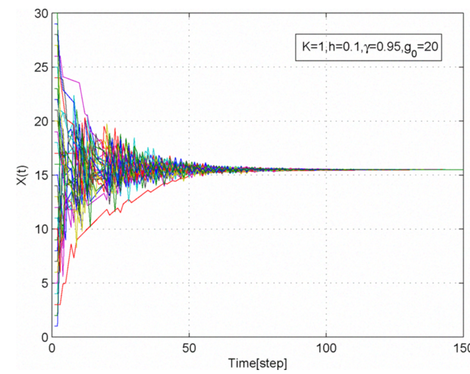

Fig. 2. Curves of states with $K = 1, \gamma = 0.95$.

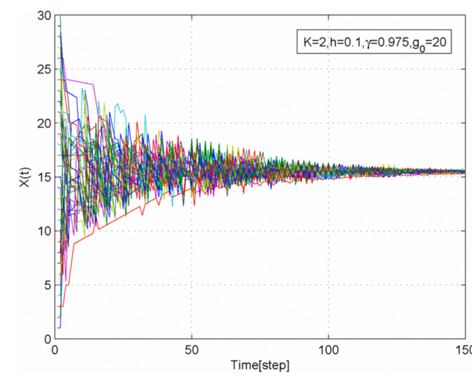

Fig. 3. Curves of states with $K = 2, \gamma = 0.975$.

V. Concluding Remarks

In this paper, the average-consensus control problem has been considered for undirected networks of discrete-time first-order agents under finite bit-rate communication. Based on scaled uniform quantization, a dynamic difference encoding and decoding scheme is used for the communication between each pair of agents. A distributed protocol has been proposed, where the control input of each agent is a weighted sum of the difference between the estimate of its neighbor's state and the internal state of its own encoder. This type of protocol is equivalent to adding an error compensation term to the original weighted average type protocol. It is shown that for a connected undirected dynamic network with first-order agents, no matter how many agents there are, we can always design a distributed protocol to ensure that average-consensus is achieved asymptotically with as few as one bit information exchange between each pair of adjacent agents at each time step. It is shown that the convergence rate is closely related to the number of network nodes, the number of quantization levels and the synchronizability of the network.

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