Distributed Circumnavigation by Unicycles with Cyclic Repelling Strategies

(Invited Paper)

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***Abstract*—The distributed circumnavigation problem, in which the task is to circumnavigate a target of interest by a network of autonomous agents, has many applications in security and surveillance, orbit maintenance, source seeking, etc. This paper deals with the circumnavigation problem using a team of nonholonomic unicycles. A novel distributed solution is proposed based on cyclic repelling strategies to achieve a circular motion around a target in a circular formation. This new approach considers minimum number of information flow links and local measurements only, yet a uniform distribution of unicycles rotated around the target is accomplished. The asymptotic collective behavior is analyzed based on the block diagonalization of circulant matrices by a Fourier transform. Simulation results also verify the validity of the proposed control algorithm.**

I. Introduction

The circumnavigation problem, in which the task is to circumnavigate a target or an area of interest, has many applications in security and surveillance, orbit maintenance, source seeking, etc. The circumnavigation task can be performed by either a single agent or a group of agents. For the single-agent case, a control strategy is proposed for holonomic vehicles in [1]. In that paper a nonlinear periodically time-varying algorithm is devised to localize the target, which uses only the agent's own position and its distance from the target. In [2] a similar problem is considered while bearing measurements are used instead of distance measurements. For the unicycle-like vehicle, a range-only strategy is presented by using a sliding mode approach in [3].

However, due to the development of multi-agent systems, the study of circumnavigation by a team of autonomous agents has attracted a lot of attention in recent years. Compared with the single-agent case, besides circumnavigating around the target, the agents are also required to keep an optimal configuration surrounding the target in order to minimize the target's escape window. Moreover, the agents are often required to do so in a distributed way, i.e., each agent should implement the control and measurement individually. Some of the early works include [4] where a group of holonomic mobile robots are used. A cyclic pursuit-based strategy is proposed in [5] which achieves uniform distribution around the target by decoupling the target tracking and inter-agent coordination tasks. Guo et al. study a moving-target enclosing strategy in [6] by using an adaptive scheme to estimate the velocity of the target. The circumnavigation problem has also been extended to nonholonomic robots, which pose more challenges on the analysis since the kinematic constraints are taken into account. For example, Ceccarelli et al. present a control law to drive a group of unicycle-type vehicles to achieve circular motions around a static target in [7]. However, this control law does not lead to an even spacing formation. A hybrid control strategy which can guarantee that the vehicles are eventually evenly spaced on the circle is introduced in [8].

In this paper we study the problem of circumnavigating a target of interest by a group of nonholonomic unicycle-type vehicles. Rather than designing complicated control laws which can be proved to perform the task globally, we notice that even a much simpler distributed control law based on a cyclic repelling strategy can also perform the circumnavigation task well. The control law uses only local information and is simple and easily implementable, which is important from a practical viewpoint. However, although the closed-loop system looks simple, at present a rigorous stability analysis of such systems is generally an extremely hard task because of the nonlinearity of the system. In this paper, we first extend the pure cyclic pursuit strategy in [9] and [10] to a more general scheme, which comprises both the cyclic pursuit and the cyclic repelling cases. We then show that the collective behaviors of the system can be shaped through appropriate control parameters, for which we provide a sufficient condition to guarantee the boundedness of the system. The equilibrium configurations of the system under different control parameters are calculated. We show that by setting this control parameter to some specific values which make the interaction between the agents to be a cyclic repelling structure, the agents can eventually achieve collective circular motions around the target and keep a regular

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polygon formation. The local stability of these equilibrium polygons is investigated based on the block diagonalization of circulant matrices by a Fourier transform. Numerical examples are also given to demonstrate the effectiveness of the proposed distributed control strategy.

II. Problem setup

Consider a team of N autonomous unicycle-like robots traveling in the plane. It is assumed that there is a point-like static target or beacon $\mathcal{T}$ in the plane. Our objective is to drive all robots to move on a circle centered at the target, to uniformly distribute them along this circle, and to maintain this formation in circular motions, based on relative position measurements.

For each agent $i = 1, 2, \cdots, N$, we employ the following unicycle-like robot model:

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= \omega_i, \end{aligned} \tag{1}$$

where (x_i, y_i) is the Cartesian coordinate of the ith robot in the world frame and θ_i gives the orientation of robot i, also in the world frame. The longitudinal velocity v_i and angular velocity ω_i are the control inputs.

To accomplish the mission, each robot i measures the relative position $\mathbf{u}_{\mathcal{T}}^{(i)}$ of the target and the relative position $\mathbf{u}_{+}^{(i)}$ of another agent $i+1$ (see Fig. 1 for illustration). Both measurements are taken in robot i's local frame. Also, due to the circular nature of the desired formation, modulo N operation is used to identify robots, i.e., robot $N+1$ is the same as robot 1.

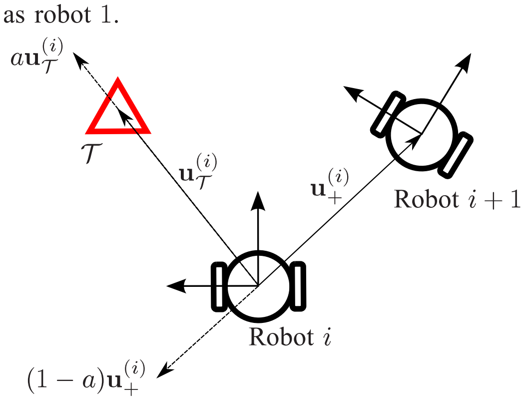

Fig. 1. An illustration of the control law for $a > 1$, in which the triangle represents the target.

The relationship between the relative coordinate and the global coordinate in the world frame is given as

$$\mathbf{u}_{\mathcal{T}}^{(i)} = R(\theta_i)\begin{bmatrix} x_{\mathcal{T}} - x_i \\ y_{\mathcal{T}} - y_i \end{bmatrix}, \mathbf{u}_{+}^{(i)} = R(\theta_i)\begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix},$$

where

$$R(\theta_i) = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$$

is the rotation matrix, and $(x_{\mathcal{T}}, y_{\mathcal{T}})$ is the Cartesian coordinate of the target.

III. Local control strategy and boundedness analysis

In this paper, we propose and examine the following simple continuous-time strategy:

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{bmatrix} k_v & 0 \\ 0 & k_\omega \end{bmatrix}\left[(1-a)\mathbf{u}_{+}^{(i)} + a\mathbf{u}_{\mathcal{T}}^{(i)}\right], \tag{2}$$

where k_v and k_ω are positive control gains.

- When $a = 0$, the control law (2) becomes the pure cyclic pursuit control which has been studied in [10].
- When $a \in (0, 1)$, robot i pursues both the target and robot $i+1$.
- When $a_i = 1$, each robot pursues the target independently and it can be shown that $(x_i, y_i) \to (x_{\mathcal{T}}, y_{\mathcal{T}})$ as $t \to \infty$.
- When $a \in (1, \bar{a})$ with $\bar{a} > 1$, robot i pursues the target while repelling itself from robot $i+1$. Without the attraction from the target, it is obvious that cyclic repelling will lead to divergence of (x_i, y_i). However, due to both the attraction effect of the target and repulsion effect between agents, a circular motion around the target can be achieved for suitable a as we will see in the paper.

Using the pseudo-linearization as in [10], we will conclude that the trajectories of all (x_i, y_i)'s are bounded under the control law (2) when a takes value in a certain set.

To show that, we define

$$\begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} := \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \frac{k_v}{k_\omega}\begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} - \begin{bmatrix} x_{\mathcal{T}} \\ y_{\mathcal{T}} \end{bmatrix},$$

which is the relative position of a point with a distance k_v/k_ω ahead of robot i with respect to the target. Differentiating $(\tilde{x}_i, \tilde{y}_i)$ and applying (2) result in

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{y}}_i \end{bmatrix} &= k_v \left\{ (1-a)\begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{y}_{i+1} \end{bmatrix} - \begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \end{bmatrix} \right\} \\ &\quad + \frac{k_v^2}{k_\omega}\left\{ -(1-a)\begin{bmatrix} \cos\theta_{i+1} \\ \sin\theta_{i+1} \end{bmatrix} + \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \right\}. \end{aligned}$$

To describe the system more concisely, let us define $\tilde{x} = [\tilde{x}_1 \cdots \tilde{x}_N]^T$, $\tilde{y} = [\tilde{y}_1 \cdots \tilde{y}_N]^T$, $\cos\theta = [\cos\theta_1 \cdots \cos\theta_N]^T$, and $\sin\theta = [\sin\theta_1 \cdots \sin\theta_N]^T$. We then have

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = -k_v \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \frac{k_v^2}{k_\omega}\begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \tag{3}$$

where $M = \text{circ}\begin{bmatrix} 1 & (a-1) & 0 & \cdots & 0 \end{bmatrix}$. Here in this paper we use $\text{circ}\begin{bmatrix} A_1 & A_2 & \cdots & A_N \end{bmatrix}$ to represent a (block) circulant matrix with the first (block) row $\begin{bmatrix} A_1 & A_2 & \cdots & A_N \end{bmatrix}$. We will meet block circulant matrix in Section V.

We first recall a simplified version of Rayleigh-Ritz theorem.

Lemma 1. *[11] Suppose that $A \in \mathbb{R}^{n\times n}$ is a real symmetric matrix with eigenvalues λ_k satisfying $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1$. Then $\lambda_1 x^T x \leq x^T A x \leq \lambda_n x^T x$.*

Let us use $\sec(\cdot)$ to denote the secant function and use $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ to denote the floor and ceiling functions, respectively.

The following result shows the boundedness of the robot trajectories.

Theorem 1. *For a group of N unicycles* (1) *under control law* (2)*, their trajectories are ultimately bounded if*

$$a \in \left[0, 1 - \sec\left(\frac{2\lfloor N/2 \rfloor \pi}{N}\right)\right).$$

Remark 1. *In the above theorem, we can see that when N is even, the right bound is 2 and when N is odd, the right bound is slightly greater than 2.*

Proof of Theorem 1: The case of $a = 0$ has been proved in [10]. Here we consider the case of $a \in \left(0, 1 - \sec\left(\frac{2\lfloor N/2 \rfloor \pi}{N}\right)\right)$.

Let $V = \frac{1}{2}(\tilde{x}^T\tilde{x} + \tilde{y}^T\tilde{y})$. Taking the time derivative of V along the solution of (3), we obtain

$$\dot{V} = -k_v \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix}^T \begin{bmatrix}\bar{M} & \mathbf{0} \\ \mathbf{0} & \bar{M}\end{bmatrix} \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix} + \frac{k_v^2}{k_\omega} \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix}^T \begin{bmatrix}M & \mathbf{0} \\ \mathbf{0} & M\end{bmatrix} \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$$

where

$$\bar{M} = \frac{M + M^T}{2} = \text{circ}\begin{bmatrix}1 & \frac{a-1}{2} & 0 & \cdots & 0 & \frac{a-1}{2}\end{bmatrix}.$$

According to [12], the eigenvalues of $\bar{M}$ are

$$\lambda_k = 1 + \frac{a-1}{2}\left(\varphi_k + \varphi_k^{N-1}\right) = 1 + (a-1)\cos\left(\frac{2k\pi}{N}\right),$$

where $\varphi_k = e^{2j\pi k/N}$. We can see that the smallest eigenvalue $\min_k(\lambda_k) = \lambda_{\lfloor N/2 \rfloor} = 1 + (a-1)\cos\left(\frac{2\lfloor N/2 \rfloor \pi}{N}\right)$. To make sure $\min_k(\lambda_k) > 0$, it requires that $a < 1 - \sec\left(\frac{2\lfloor N/2 \rfloor \pi}{N}\right)$. Hence with Lemma 1 we have

$$\dot{V} \leq -k_v \min_k(\lambda_k) \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix}^T \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix} + \frac{k_v^2}{k_\omega} \begin{bmatrix}\tilde{x} \\ \tilde{y}\end{bmatrix}^T \begin{bmatrix}M & \mathbf{0} \\ \mathbf{0} & M\end{bmatrix} \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix},$$

Then $\dot{V} < 0$ for $\|\frac{\tilde{x}}{\tilde{y}}\| > \frac{k_v}{k_\omega}\|M\|\sqrt{N}/\min_k(\lambda_k)$. By the ultimate boundedness result in [13, Section 4.8], it follows that the trajectories are bounded. ■

IV. Equilibria of Interest

In this section we analyze the collective behaviors at equilibria for the system under control law (2) and then we concentrate on finding a special class of equilibria of interest.

Without loss of generality, we let $k_v = 1$ and $k_\omega = 1$. Instead of working on the original dynamics of x_i, y_i and θ_i, we will look at the dynamics of v_i, ω_i and

$$\beta_i := \theta_{i+1} - \theta_i \tag{4}$$

to investigate the collective behaviors of system (1) under control law (2). Taking derivatives of (2) and β_i, we have

$$\begin{aligned} \dot{v}_i &= (1-a)v_{i+1}\cos\beta_i - v_i + \omega_i^2, \\ \dot{\omega}_i &= (1-a)v_{i+1}\sin\beta_i - v_i\omega_i, \\ \dot{\beta}_i &= \omega_{i+1} - \omega_i. \end{aligned} \tag{5}$$

We are interested in those formations that satisfy $\dot{v}_i \equiv 0$ and $\dot{\omega}_i \equiv 0$ for all i, which correspond to uniform circular motions.

Theorem 2. *Suppose $\dot{v}_i \equiv 0$, $\dot{\omega}_i \equiv 0$. If $a \in \left[0, 1 - \sec\left(\frac{2\lfloor N/2 \rfloor \pi}{N}\right)\right)$, then*

- *all the agents move on concentric circles with equal angular velocities or*
- *all the agents remain stationary at a single point.*

If in addition, $a \neq 0$, then

- *the center of concentric circles is the target $(x_\mathcal{T}, y_\mathcal{T})$ or*
- *the agents keep stationary at the target $(x_\mathcal{T}, y_\mathcal{T})$.* [1]

Proof: First, we show that all the angular velocities are equal. Suppose $\dot{v}_i \equiv 0$ and $\dot{\omega}_i \equiv 0$ for all i. Then according to (5), it is obtained that

i) if $a = 1$, then $v_i = 0$ and $\omega_i = 0, \forall i$. Thus, β_i's are constants.

ii) if $a \neq 1$, all β_i's should also be constants. To show that, we differentiate $\dot{v}_i$ and $\dot{\omega}_i$ in (5) and get

$$\begin{aligned} \ddot{v}_i &= -(1-a)v_{i+1}\dot{\beta}_i\sin\beta_i, \\ \ddot{\omega}_i &= (1-a)v_{i+1}\dot{\beta}_i\cos\beta_i. \end{aligned}$$

Since $\ddot{v}_i = \ddot{\omega}_i = 0$, then $(1-a)^2\dot{\beta}_i^2 v_{i+1}^2 = 0$, hence $\dot{\beta}_i = 0$ or $v_{i+1} = 0$. When $v_{i+1} = 0$, it consequently forces $v_i = 0$ and $\omega_i = 0$ and then $v_i = 0, \omega_i = 0, \forall i$. Therefore we have $\omega_{i+1} = \omega_i := \bar{\omega}$ for all i.

Secondly, we show that all the agents move on concentric circles or remain stationary.

1) If $\bar{\omega} \neq 0$, then the trajectory of each i can be written as

$$\begin{bmatrix}x_i(t) \\ y_i(t)\end{bmatrix} = \begin{bmatrix}\bar{x}_i \\ \bar{y}_i\end{bmatrix} + r_i \begin{bmatrix}\cos(\phi_{0i} + \bar{\omega}t) \\ \sin(\phi_{0i} + \bar{\omega}t)\end{bmatrix}, \tag{6}$$

$$\theta_i(t) = \phi_{0i} + \bar{\omega}t \pm \pi/2, \tag{7}$$

where $(\bar{x}_i, \bar{y}_i)$ is the agent i's center of circular motion and $r_i = v_i/\bar{\omega}$ the radius of the circle. The sign of $\pi/2$ is positive when the agent moves around the target counterclockwise and negative when the agent moves clockwise.

According to (2), we have

$$(1-a)\begin{bmatrix}x_{i+1} \\ y_{i+1}\end{bmatrix} - \begin{bmatrix}x_i \\ y_i\end{bmatrix} + a\begin{bmatrix}x_\mathcal{T} \\ y_\mathcal{T}\end{bmatrix} = R(-\theta_i)\begin{bmatrix}v_i/k_v \\ \omega_i/k_\omega\end{bmatrix}.$$

Substituting (6) into the above equality we get

$$\begin{aligned} (1-a)\begin{bmatrix}\bar{x}_{i+1} \\ \bar{y}_{i+1}\end{bmatrix} - \begin{bmatrix}\bar{x}_i \\ \bar{y}_i\end{bmatrix} + a\begin{bmatrix}x_\mathcal{T} \\ y_\mathcal{T}\end{bmatrix} = R(-\theta_i)\begin{bmatrix}v_i/k_v \\ \omega_i/k_\omega\end{bmatrix} - \\ (1-a)r_{i+1}\begin{bmatrix}\cos(\phi_{0i+1} + \bar{\omega}t) \\ \sin(\phi_{0i+1} + \bar{\omega}t)\end{bmatrix} + r_i\begin{bmatrix}\cos(\phi_{0i} + \bar{\omega}t) \\ \sin(\phi_{0i} + \bar{\omega}t)\end{bmatrix}. \end{aligned}$$

Integrating both sides of the above equality from some time instant t_0 to $t_0 + 2\pi/\bar{\omega}$, since the right hand side is zero, we get

$$(1-a)\begin{bmatrix}\bar{x}_{i+1} \\ \bar{y}_{i+1}\end{bmatrix} - \begin{bmatrix}\bar{x}_i \\ \bar{y}_i\end{bmatrix} + a\begin{bmatrix}x_\mathcal{T} \\ y_\mathcal{T}\end{bmatrix} = \mathbf{0}. \tag{8}$$

So when $a = 0$, we get $(\bar{x}_{i+1}, \bar{y}_{i+1}) = (\bar{x}_i, \bar{y}_i)$ directly, which implies the agents move on concentric circles.

[1] When $a = 0$, the term in (2) which the target involves in vanishes so there is no target.

To discuss the case of $a \neq 0$, we rewrite (8) as a recursion relation:

$$(1-a)\begin{bmatrix}\bar{x}_{i+1}-x_{\mathcal{T}}\\ \bar{y}_{i+1}-y_{\mathcal{T}}\end{bmatrix}=\begin{bmatrix}\bar{x}_{i}-x_{\mathcal{T}}\\ \bar{y}_{i}-y_{\mathcal{T}}\end{bmatrix}.$$

Applying this relation N times, we get

$$(1-a)^N\begin{bmatrix}\bar{x}_{i}-x_{\mathcal{T}}\\ \bar{y}_{i}-y_{\mathcal{T}}\end{bmatrix}=\begin{bmatrix}\bar{x}_{i}-x_{\mathcal{T}}\\ \bar{y}_{i}-y_{\mathcal{T}}\end{bmatrix}.$$

So when $a \in \Big(0, 1-\sec\big(\frac{2\lfloor N/2\rfloor\pi}{N}\big)\Big)$, then $(1-a)^N \neq 1$ and thus we can obtain that $(\bar{x}_i, \bar{y}_i) = (x_{\mathcal{T}}, y_{\mathcal{T}}), \forall i$. This means the agents move on concentric circles centered at the target.

2) If $\bar{\omega} = 0$, then we show that $v_i = 0$ for all i. Suppose by contradiction that there exists $v_i \neq 0$. Then it can be inferred that the trajectory of robot i approaches ∞ since v_i is a nonzero constant nonzero and ω_i is zero. This contradicts Theorem 1. Therefore, $v_i = 0, \omega_i = 0, \forall i$ and thus we get

$$(1-a)\begin{bmatrix}x_{i+1}-x_i\\ y_{i+1}-y_i\end{bmatrix}+a\begin{bmatrix}x_{\mathcal{T}}-x_i\\ y_{\mathcal{T}}-y_i\end{bmatrix}=\mathbf{0}$$

from (2). Thus similar to the analysis about the center of concentric motions, we can get $(x_{i+1}, y_{i+1}) = (x_i, y_i)$ when $a = 0$ and $(x_i, y_i) = (x_{\mathcal{T}}, y_{\mathcal{T}})$ for all i when $a \in \Big(0, 1-\sec\big(\frac{2\lfloor N/2\rfloor\pi}{N}\big)\Big)$. This case indicates that the agents remain stationary at a single point and the single point is exactly the location of the target if additionally $a \neq 0$. ∎

Theorem 2 shows that if $a \in \Big(0, 1-\sec\big(\frac{2\lfloor N/2\rfloor\pi}{N}\big)\Big)$, then at equilibria, the agents either form a polygon formation which rotates around the target or the agents rendezvous at the target. But if a is not in the set $\Big[0, 1-\sec\big(\frac{2\lfloor N/2\rfloor\pi}{N}\big)\Big)$, the agents may have different behaviors. For example, when $a = 2$ and N is even, the agents may run on eccentric circles.

Next we are going to look at a special class of equilibria that corresponds to the uniform circular motions about the target. Let $\xi_i = [v_i, \omega_i, \beta_i]^T$ and we can write each subsystem (5) as

$$\dot{\xi}_i = f(\xi_i, \xi_{i+1}). \tag{9}$$

Let $\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_N^T]^T$. Then the overall system can be written as

$$\dot{\xi} = \hat{f}(\xi). \tag{10}$$

The equilibria of system (10) are the solutions of a set of $3N$ nonlinear equations

$$\begin{aligned}(1-a)v_{i+1}\cos\beta_i - v_i + \omega_i^2 &= 0,\\ (1-a)v_{i+1}\sin\beta_i - v_i\omega_i &= 0,\\ \omega_{i+1}-\omega_i &= 0,\end{aligned} \tag{11}$$

where $i = 1, \ldots, N$. We have proved in Theorem 2 that $\omega_i = \bar{\omega}, \forall i$. However, it is still difficult to solve this set of equations directly, but a special set of equilibria of interest can be derived. That is,

$$\begin{aligned}v_1 = v_2 = \cdots = v_N &:= \bar{v},\\ \omega_1 = \omega_2 = \cdots = \omega_N &:= \bar{\omega},\\ \beta_1 = \beta_2 = \cdots = \beta_N &:= \bar{\beta},\end{aligned} \tag{12}$$

for some $\bar{v}, \bar{\omega}$ and $\bar{\beta}$. When (12) holds, it follows from (11) that

$$\begin{aligned}(1-a)\bar{v}\cos\bar{\beta} - \bar{v} + \bar{\omega}^2 &= 0,\\ (1-a)\bar{v}\sin\bar{\beta} - \bar{v}\bar{\omega} &= 0.\end{aligned} \tag{13}$$

If $v_i = \bar{v} = 0, \omega_i = \bar{\omega} = 0$ for all i, then from Theorem 2 we know all the agents stay at the target point, which is also an equilibrium of the system. Nevertheless, we are more interested in the case $v_i = \bar{v} \neq 0, \omega_i = \bar{\omega} \neq 0$ for all i that represents the uniform circular motion around the target.

According to (4), we have $(\sum_i \beta_i) \bmod 2\pi = 0$. If we restrict $\beta_i \in [0, 2\pi)$, then $\sum_i \beta_i = 2d\pi$ for some integer $d \in \{1, 2, \cdots, N-1\}$.

Theorem 3. *The set*

$$\{\xi | v_i = \bar{v}, \omega_i = \bar{\omega}, \beta_i = \bar{\beta}, \forall i\} \tag{14}$$

is an equilibrium set of system (10) *with* $\bar{v} = \frac{(1-a)^2\sin^2\bar{\beta}}{1-(1-a)\cos\bar{\beta}}$, $\bar{\omega} = (1-a)\sin\bar{\beta}$ *and* $\bar{\beta} = \frac{2d\pi}{N}$.

Proof: This can be obtained by solving (13) with $\bar{\beta} = \frac{2d\pi}{N}$ directly. ∎

This set of equilibria corresponds to the collective behavior that all the agents move on the same circle with even spacing between neighboring agents. In other words, it corresponds to the generalized regular polygons formed by the group of robots in the plane. We provide the formal definition below.

Definition 1. *[14] Let $d < N$ be a positive integer and R the counterclockwise rotation in the plane about the origin through angle $\frac{2d\pi}{N}$ and let $z_i \neq 0$ be a point in the plane. Then point $z_{i+1} = Rz_i, i = 1, \ldots, N-1$ and edges $z_{i+1} - z_i, i = 1, \ldots, N$ define a generalized regular polygon, which is denoted by $\{\frac{N}{d}\}$.*

The polygon $\{\frac{N}{d}\}$ is call *positively* oriented if $d \leq N/2$ or *negatively* oriented if $d > N/2$.

V. Stability analysis

Stability of equilibrium polygons defined in Theorem 3 will be investigated in this section. Let $\bar{\xi} = \mathbf{1}_N \otimes [\bar{v}, \bar{\omega}, \bar{\beta}]^T$ (where $\bar{v} \neq 0, \bar{\omega} \neq 0$) be an equilibrium point (14). The linearized model about the equilibrium point for each subsystem (9) is then obtained to have the following form

$$\dot{\tilde{\xi}}_i = A\tilde{\xi}_i + B\tilde{\xi}_{i+1},$$

where $\tilde{\xi}_i = \xi_i - [\bar{v}, \bar{\omega}, \bar{\beta}]^T$ and the Jacobian matrices A and B are calculated as

$$A = \begin{bmatrix}-1 & 2\bar{\omega} & -(1-a)\bar{v}\sin\bar{\beta}\\ -\bar{\omega} & -\bar{v} & (1-a)\bar{v}\cos\bar{\beta}\\ 0 & -1 & 0\end{bmatrix},$$

$$B = \begin{bmatrix}(1-a)\cos\bar{\beta} & 0 & 0\\ (1-a)\sin\bar{\beta} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Moreover, the linearized model of the overall system (10) has the form

$$\dot{\tilde{\xi}} = \hat{A}\tilde{\xi},$$

where $\tilde{\xi} = \xi - \bar{\xi}$ and $\hat{A}$ is the block circulant matrix

$$\hat{A} = \mathrm{circ}[A, B, 0_{3\times3}, \dots, 0_{3\times3}].$$

A block circulant matrix can be block diagonalized by means of the Fourier Matrix [12]. This is stated in the following lemma.

Lemma 2. *The eigenvalues of $\hat{A}$ are the collection of all eigenvalue of*

$$A + B, A + \varphi B, A + \varphi^2 B, \dots, A + \varphi^{N-1} B,$$

where $\varphi = e^{2j\pi/N}$.

According to Lemma 2, each diagonal block in the diagonalized form of $\hat{A}$ is

$$D_i = A + \varphi^{i-1} B,\ i \in \{1, 2, \cdots, N\}.$$

To determine the local stability of each equilibrium polygon, we need to locate the eigenvalues of all D_i's. This is not a trivial step considering that D_i is a complex matrix. However, since to locate the eigenvalues of D_i is equivalent to locate the roots of its characteristic polynomial, we propose to use the following theorem.

Theorem 4. *[15, Theorem 3.16] Consider a complex polynomial of the third degree*

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

where $a_1, a_2, a_3 \in \mathbb{C}$. Define the Hermitian matrix

$$H = \begin{bmatrix} a_1 + \bar{a}_1 & a_2 - \bar{a}_2 & a_3 + \bar{a}_3 \\ -a_2 + \bar{a}_2 & a_1\bar{a}_2 + a_2\bar{a}_1 - a_3 - \bar{a}_3 & a_3\bar{a}_1 - a_1\bar{a}_3 \\ a_3 + \bar{a}_3 & a_1\bar{a}_3 - a_3\bar{a}_1 & a_2\bar{a}_3 + a_3\bar{a}_2 \end{bmatrix},$$

where $\bar{c}$ denotes the complex conjugate of c. The polynomial $p(\lambda)$ is asymptotically stable if and only if H is positive definite.

We denote the leading principal minors of H as h_1, h_2, and h_3 and recall the fact that a Hermitian matrix H is positive definite if and only if its leading principal minors are positive. We apply Theorem 4 to the characteristic polynomial of D_i with $a = 2$ and obtain that

$$\begin{aligned} h_1 =& 2(c_r c_i - c_r + 2), \\ h_2 =& 8(c_r - 1)(c_r c_i^3 - 2c_r^2 c_i^2 + 2c_r c_i^3 + 3c_r^2 c_i \\ &- 8c_r c_i + 2c_i - c_r^2 + 3c_r - 4), \\ h_3 =& -32(c_r - 1)^2(c_i - 1)(c_i + c_r)^3 \\ &\cdot (2c_r c_i^2 - 2c_i^2 - 4c_r^2 c_i - 5c_r c_i + 8c_i + 5c_r^2 + 2c_r - 8), \end{aligned}$$

where $c_r = \cos\bar{\beta}$ and $c_i = \cos\frac{2(i-1)\pi}{N}$.

It is easy to check that $h_1 > 0$. We can further prove that

- $h_2 > 0$ and,
- the last term of h_3: $2c_r c_i^2 - 2c_i^2 - 4c_r^2 c_i - 5c_r c_i + 8c_i + 5c_r^2 + 2c_r - 8 < 0$.

Stability of a given D_i is therefore dependent on the sign of

$$\tilde{h}_3 = (c_i - 1)(c_i + c_r)^3.$$

When $i = 1$, $c_i = 1$ and $\tilde{h}_3 = 0$. Calculating the eigenvalues of D_1 we can find that this corresponds to a zero eigenvalue. However, it does not affect the stability of the system. Similar to [10, Lemma 2], this zero eigenvalue occurs because of the definition of β_i which intrinsically satisfies $\sum_i \beta_i = 2k\pi, k \in \mathbb{N}$.

When $i \in \{2, 3, \cdots, N\}$, $\tilde{h}_3 > 0$ is equivalent to $c_r < -c_i$, that is

$$\cos\frac{2d\pi}{N} < -\cos\frac{2(i-1)\pi}{N}.$$

Thus, $\tilde{h}_3 > 0$ for all $i \in \{2, 3, \cdots, N\}$ is equivalent to

$$\cos\frac{2d\pi}{N} < -\cos\frac{2\pi}{N} \text{ and } \cos\frac{2d\pi}{N} < -\cos\frac{2(N-1)\pi}{N},$$

from which we obtain

$$\frac{(N-2)\pi}{N} < \frac{2d\pi}{N} < \frac{(N+2)\pi}{N},$$

and thus

$$\frac{N}{2} - 1 < d < \frac{N}{2} + 1. \tag{15}$$

A. Case I: N is odd.

Theorem 5. *If $a = 2$ and N is odd, then among all the equilibrium polygons $\left\{\frac{N}{d}\right\}$, only $\left\{\frac{N}{\lfloor N/2 \rfloor}\right\}$ and $\left\{\frac{N}{\lceil N/2 \rceil}\right\}$ are asymptotically stable.*

Proof: This is a direct result of (15). ■

B. Case II: N is even.

When $a = 2$ and N is even, $\tilde{h}_3 > 0$ for all $i \in \{2, 3, \cdots, N\}$ requires that $d = N/2$, which is the case $\bar{v} = 0$ and $\bar{\omega} = 0$ by Theorem 3. Therefore we consider $d = \frac{N}{2} \pm 1$ instead as this is a more interesting case corresponding to the uniform circular motions. In this case, besides $i = 1$, $\tilde{h}_3 = 0$ when $i = 2$ and $i = N - 1$, i.e., $c_i = -c_r$. However, with a minor modification of the parameter a, we show in the following result that the equilibrium polygons $\left\{\frac{N}{N/2 \pm 1}\right\}$ are asymptotically stable.

Theorem 6. *If N is even and $a = 2 - \varepsilon$ where ε is a small positive real number, then among all the equilibrium polygons $\left\{\frac{N}{d}\right\}$, only $\left\{\frac{N}{N/2 \pm 1}\right\}$ are asymptotically stable.*

Proof: The Taylor series of the third leading principal minor h_3 at $a = 2$ is

$$\begin{aligned} &(192c_r^9 - 128c_r^8 - 897c_r^7 + 640c_r^6 + 1536c_r^5 - 1152c_r^4 \\ &- 1153c_r^3 + 896c_r^2 + 320c_r - 256)(a - 2) + o(a - 2). \end{aligned} \tag{16}$$

It can be proved that the term $(192c_r^9 \cdots - 256) < 0$ when $d = N/2 \pm 1$, so (16) is positive if $a = 2 - \varepsilon$ if $\varepsilon > 0$ is small enough. ■

Note that when the equilibrium polygon is positively oriented, $\bar{\omega} < 0$, so the agents move along the circle in a clockwise direction. Conversely, when the equilibrium polygon is negatively oriented, $\bar{\omega} > 0$, the agents move along the circle in a counterclockwise direction.

One problem of this control law is that when $N = 4n + 2$ with some positive integer n, overlap of vehicles will happen. There are several ways to overcome this situation. For example, the agents can decide to let one of them exit the task via a distributed negotiation mechanism. Or the vehicle can just adjust their control gains to let themselves circumnavigate the target on concentric circles with different radii. Another solution might be rearrange the interaction topology between the agents to form two cyclic repelling structures, each with $2n + 1$ agents.

VI. Simulations

In this section, we conduct some simulations to demonstrate our results. Here we set $k_v = 1$ and $k_\omega = 1$.

Due to the space restriction, we only present the simulation results when N is odd. In Fig. 2, five robots are initialized at a unstable equilibrium polygons $\{\frac{5}{1}\}$. From the simulations we can see that the vehicles escape away from the original equilibrium polygon and converge to a stable equilibrium polygons $\{\frac{5}{2}\}$. Fig. 3 shows that five vehicles whose initial postures are randomly chosen converge to an equilibrium polygon $\{\frac{5}{2}\}$.

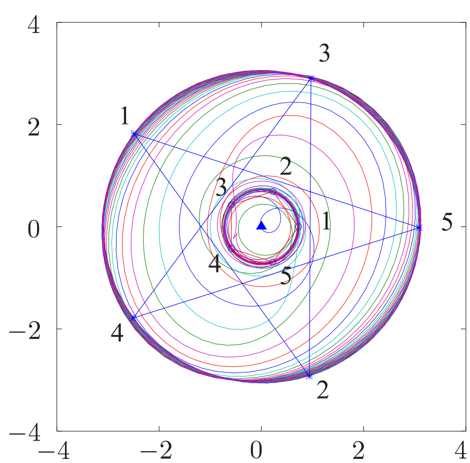

Fig. 2. $\{\frac{5}{1}\} \rightarrow \{\frac{5}{2}\}$.

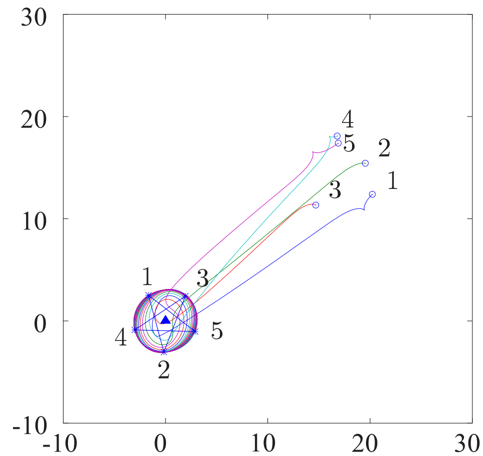

Fig. 3. Random $\rightarrow \{\frac{5}{2}\}$.

VII. Conclusion

In this paper, a distributed control law for a network of nonholonomic vehicles, which are deployed to achieve collective circular motions around a target of interest, has been proposed. We show that the collective behaviors of the multi-vehicle systems can be shaped by choosing different control weights. A sufficient condition for the boundedness of the trajectories is provided. We then show that by setting the control weight to some specific values, the agents can achieve an evenly spaced circular formation around the target.

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