Stabilization for Discrete-Time Stochastic Systems with Multiple Input Delays*

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***Abstract*—This paper is concerned with the stabilization problem for discrete-time stochastic systems with multiplicative noise and multiple input delays. Our approach is based on a reduction technique, by which the system under consideration is converted into a delay-free system. The necessary and sufficient stabilization condition for stabilization and the associated feedback controller are characterized via a generalized Riccati equation.**

I. Introduction

Delay phenomenon arise naturally in many areas such as chemistry, economics, communication, networked control and so on (For more details, see, e.g., [1]). Control problems for time-delay systems have attracted much attention in the past decades [2]-[14]. [3] utilizes receding horizon control to stabilize input-delay systems and a sufficient condition for the asymptotical stability of the closed-loop system is presented in terms of a linear matrix inequality. [4] first proposes the Smith predictor method which can deal with the stabilization problem for systems with single input delay. [5] employs linear matrix inequality (LMI) approach to design H_∞ feedback control for systems with delayed state. [6] makes use of the reduction approach, which can transform delay system into delay-free system, to investigate the stabilization problem for linear systems with both state delay and input delay.

It should be noted that the aforementioned literatures focus on deterministic delay systems. The stochastic control for time delay systems are much involved due to its features of stochastic uncertainty and infinite dimension [15]-[17]. Recently, the stochastic control problem of time delay systems has received some progresses. [18] studies the linear quadratic (LQ) problem for stochastic system with delay in both state and control variables. By means of a generalized forward-backward stochastic differential equations, the optimal feedback control is given via a new type of Riccati equations. Besides, [19] and [20] resolve the LQ problem and the stabilization problem for stochastic systems with single input delay in continuous time and discrete time respectively. The necessary and sufficient stabilization condition is proposed via coupled Riccati-type equation.

This paper is concerned with input-delay systems with state dependent and control dependent noise. The stabilization problem is first converted into an equivalent one for a delay-free system by using the reduction technique, and a $\mathcal{F}_{k-1}$-measurable stabilizing controller is then presented via a generalized Riccati equation. It is noted that the historical noises are assumed to be known a priori in the controller design. This assumption is natural since the deigned controller is $\mathcal{F}_{k-1}$-measurable, see also other results on the stochastic control for systems with multiplicative noises and stochastic coefficient matrices [21].

The rest of the paper is organized as follows: Section II states the problem to be settled in this paper. In Section III, the delay system under consideration is reduced into a delay-free system. In Section IV, the main results on the stabilization of the original system are derived with the help of the stabilization of the transformed system. Conclusions are drawn in Section V. Details of proof are provided in Appendix.

Notation. R^n denotes the usual n-dimensional Euclidean space; $R^{m\times n}$ stands for the space of $m\times n$ matrices with real elements; I_n is the unit matrix of order n; The superscript $'$ is the matrix transpose; For any matrix X, $rank(X)$ represents its rank; For a real symmetric matrix X, $X>0$ (resp. $X\geq 0$) means that X is positive definite (resp. positive semi-definite); $E(\cdot)$ means the expectation with respect to the noise $\omega_k, k\geq 0$.

II. Problem Formulation

Consider the discrete-time system with multiplicative noise and input delays:

$$x_{k+1}=Ax_k+(B_0+\omega_k\bar{B}_0)u_k+(B_d+\omega_{k-d}\bar{B}_d)u_{k-d} \quad (1)$$

Here, $x_k\in R^n$ and $u_k\in R^m$ are the state and input respectively; d is the maximal delay; $\{\omega_k\}_{k\geq 0}$ is a scalar white noise process defined on a complete probability space $\{\Omega,\mathcal{P},\mathcal{F}\}$ with variance $\sigma>0$; For $k<0$, ω_k is thought to be zero; $A, B_0, \bar{B}_0, B_d$ and $\bar{B}_d$, are constant matrices with compatible dimensions.

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The object of this paper is to construct controller to stabilize system (1). It is important to make it clear that what information the control u_k has access to. Let $\mathcal{F}_k, k \geq 0$, represent the natural filtration generated by ω_k and $\mathcal{F}_{-1}$ stand for $\{\emptyset, \Omega\}$ with $\emptyset$ being the empty set. Then we make the following basic hypothesis.

Assumption 1: For any $k \geq 0$, u_k is $\mathcal{F}_{k-1}$-measurable, i.e., $\{\omega_0, \ldots, \omega_{k-1}\}$ is available in the design of u_k.

Then the problem to be addressed can be described as: finding $\mathcal{F}_{k-1}$-measurable u_k to stabilize system (1).

III. Reduction of Delay System into Delay-free System

A. The Reduction Technique

Throughout the paper, it will be assumed that A is nonsingular. Define a new state by

$$y_k \doteq x_k + \sum_{i=1}^{d} (A^{i-d-1}B_d + \omega_{k-i}A^{i-d-1}\bar{B}_d)u_{k-i}, \ k \geq 0. \quad (2)$$

Then we have the following Lemma.

Lemma 1: y_k defined by (2) obeys the following dynamic system

$$y_{k+1} = Ay_k + (C + \omega_k \bar{C})u_k, \quad (3)$$

with

$$C = B_0 + A^{-d}B_d, \ \bar{C} = \bar{B}_0 + A^{-d}\bar{B}_d.$$

Proof: By applying (1) and (2), it yields

$$\begin{aligned} y_{k+1} &= x_{k+1} + (A^{-d}B_d + \omega_k A^{-d}\bar{B}_d)u_k \\ &\quad + \sum_{i=1}^{d-1}(A^{i-d}B_d + \omega_{k-i}A^{i-d}\bar{B}_d)u_{k-i} \\ &= Ax_k + [B_0 + A^{-d}B_d + \omega_k(\bar{B}_0 + A^{-d}\bar{B}_d)]u_k \\ &\quad + (B_d + \omega_{k-d}\bar{B}_d)u_{k-d} \\ &\quad + \sum_{i=1}^{d-1}(A^{i-d}B_d + \omega_{k-i}A^{i-d}\bar{B}_d)u_{k-i} \\ &= Ax_k + [B_0 + A^{-d}B_d + \omega_k(\bar{B}_0 + A^{-d}\bar{B}_d)]u_k \\ &\quad + \sum_{i=1}^{d}(A^{i-d}B_d + \omega_{k-i}A^{i-d}\bar{B}_d)u_{k-i} \\ &= A[x_k + \sum_{i=1}^{d}(A^{i-d-1}B_d + \omega_{k-i}A^{i-d-1}\bar{B}_d)u_{k-i}] \\ &\quad + [B_0 + A^{-d}B_d + \omega_k(\bar{B}_0 + A^{-d}\bar{B}_d)]u_k \\ &= Ay_k + [B_0 + A^{-d}B_d + \omega_k(\bar{B}_0 + A^{-d}\bar{B}_d)]u_k. \end{aligned}$$

This ends the proof. ∎

B. Stabilization of the Delay-free System

Now the delay system (1) has been transformed into the delay-free system (3). According to the previous works, we have the following definition [22].

Definition 1: System (3) is called stabilizable in the mean-square sense if there exists a feedback control

$$u_k = Ky_k$$

with K a constant matrix, such that for any $y_0 \in R^n$, the closed-loop system

$$y_{k+1} = (A + CK + \omega_k \bar{C}K)y_k$$

is asymptotically mean square stable, i.e.,

$$\lim_{k \to \infty} E(y_k' y_k) = 0.$$

By [22], the following results are derived immediately.

Lemma 2: System (3) is stabilizable in the mean-square sense iff the generalized Riccati equation

$$\begin{cases} P = A'PA - (C'PA)'M^{-1}C'PA + I_n \\ M = C'PC + \sigma\bar{C}'P\bar{C} + I_m > 0 \end{cases} \quad (4)$$

admits a unique solution satisfying $P > 0$. In this case, the stabilizing control can be selected as

$$u_k = Ky_k,$$

with

$$K = -(C'PC + \sigma\bar{C}'P\bar{C} + I_m)^{-1}C'PA. \quad (5)$$

IV. Main Results

Now the stabilization of system (1) is to be considered by means of the stabilization of system (3). The main results are stated in the Theorem below.

Theorem 1: For system (1), there exists a control in the form of

$$u_k = Lx_k + \sum_{i=1}^{d}(L_i + \omega_{k-i}\bar{L}_i)u_{k-i}, \ k \geq d, \quad (6)$$

where L, L_i, and $\bar{L}_i$, $i = 1, \ldots, d$, are constant matrices, such that the corresponding closed-loop system satisfies

$$\lim_{k \to \infty} E(u_k' u_k) = 0, \ \lim_{k \to \infty} E(x_k' x_k) = 0, \quad (7)$$

for any deterministic x_0, u_{-i}, $i = 1, \ldots, d$ and any $\mathcal{F}_{k-1}-$measurable u_k, $k = 0, \ldots, d-1$, iff the generalized Riccati equation (4) has a unique solution $P > 0$. In this context, the stabilizing control is given by

$$u_k = Kx_k + \sum_{i=1}^{d}(KA^{i-d-1}B_d + \omega_{k-i}KA^{i-d-1}\bar{B}_d)u_{k-i} \quad (8)$$

for $k \geq d$, where the matrix K is defined via (5).

Proof: See Appendix A. ∎

V. Conclusion

The mean-square stabilization problem for stochastic systems with multiplicative noise and multiple input delays is studied in this paper. The system is stabilized iff there exists a unique positive definite solution to a generalized Riccati equation. In this case, the feedback gains of the stabilizing control can be directly calculated using the solution to the generalized Riccati equation.

Appendix A
Proof of Theorem 1

In this section, the necessity and sufficiency of Theorem 1 will be shown respectively. Notice that the notations y_k, C and $\bar{C}$ which will be involved in this section have been defined in subsection III-A.

A. Proof of the Necessity of Theorem 1

Lemma 3: Suppose there is a control (6), such that the closed-loop system of (1) satisfies (7). Then the following augmented system

$$z_{k+1} = Fz_k + (G + \omega_k \bar{G})u_k, \ k \geq 0, \tag{9}$$

where

$$z_k = \begin{pmatrix} z^0_{k\,n\times 1} \\ z^1_{k\,m\times 1} \\ \vdots \\ z^{2d}_{k\ \ m\times 1} \end{pmatrix} \in R^{n+2dm},$$

$$F = \begin{pmatrix} A & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I_m & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} C \\ I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \bar{G} = \begin{pmatrix} \bar{C} \\ 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with $z^0_0 = y_0$, z^i_0, $i = 1, \ldots, 2d$, being arbitrary vector in R^m and the partition of F, G and $\bar{G}$ being consistent with that of z_k, is stabilizable in the sense of Definition 1.

Proof: Let u_k be a control which stabilizes system (1):

$$u_k = Lx_k + \sum_{i=1}^{d}(L_i + \omega_{k-i}\bar{L}_i)u_{k-i}, \ k \geq d. \tag{10}$$

Then the closed-loop system of (1) under the control (10) satisfies

$$\lim_{k\to\infty} E(u'_k u_k) = 0, \ \lim_{k\to\infty} E(x'_k x_k) = 0, \tag{11}$$

for any deterministic x_0, u_{-i}, $i = 1, \ldots, d$ and any $\mathcal{F}_{k-1}-$measurable u_k, $k = 0, \ldots, d-1$.

Consider the augmented system (9). Direct computation yields

$$z^0_{k+1} = Az^0_k + (C + \omega_k \bar{C})u_k, \tag{12}$$
$$z^1_{k+1} = u_k, \tag{13}$$
$$z^2_{k+1} = \omega_k u_k, \tag{14}$$
$$z^i_{k+1} = z^{i-2}_k, \ i = 3, \ldots, 2d, \ k \geq 0. \tag{15}$$

(3), (12) and $z^0_0 = y_0$ imply that

$$z^0_k = y_k, \ k \geq 0. \tag{16}$$

From (13)-(15), it follows that

$$z^{2i-1}_k = u_{k-i}, \ z^{2i}_k = \omega_{k-i}u_{k-i}, \ i = 1, \ldots, d, \ k \geq i. \tag{17}$$

Next we will represent the control (10) as a feedback of z_k. In view of the relation between x_k and y_k, i.e., (2), (10) can be rewritten as

$$u_k = Ly_k + \sum_{i=1}^{d} K_i u_{k-i} + \sum_{i=1}^{d} \omega_{k-i}\bar{K}_i u_{k-i}, \ k \geq d, \tag{18}$$

with

$$K_i = L_i - LA^{i-d-1}B_d, \ \bar{K}_i = \bar{L}_i - LA^{i-d-1}\bar{B}_d, \ 1 \leq i \leq d.$$

By employing (16) and (17), (18) can be further expressed as

$$u_k = Lz^0_k + \sum_{i=1}^{d} K_i z^{2i-1}_k + \sum_{i=1}^{d} \bar{K}_i z^{2i}_k, \ k \geq d. \tag{19}$$

With the notation

$$\hat{K} \doteq \begin{pmatrix} L & K_1 & \bar{K}_1 & \cdots & K_d & \bar{K}_d \end{pmatrix},$$

(19) becomes

$$u_k = \hat{K}z_k, \ k \geq d. \tag{20}$$

To keep consistent with (20), we can set

$$u_k = \hat{K}z_k, \ k = 0, \ldots, d-1, \tag{21}$$

since u_k, $k = 0, \ldots, d-1$, can be any $\mathcal{F}_{k-1}$-measurable function. Thus we get the closed-loop system

$$z_{k+1} = (F + G_k\hat{K})z_k, \ k \geq 0. \tag{22}$$

Next it will be shown that system (22) is asymptotically stable. Denote

$$\Phi_{k,i} = A^{i-d-1}B_d + \omega_{k-i}A^{i-d-1}\bar{B}_d;$$

$$\Omega_{k,i} = B'_d A'^{i-d-1}A^{i-d-1}B_d + \sigma\bar{B}'_d A'^{i-d-1}A^{i-d-1}\bar{B}_d.$$

By applying the Minkowski's inequality on (2), we obtain

$$\begin{aligned} [E(y'_k y_k)]^{\frac{1}{2}} &\leq [E(x'_k x_k)]^{\frac{1}{2}} + \sum_{i=1}^{d}\{E[u'_{k-i}\Phi'_{k,i}\Phi_{k,i}u_{k-i}]\}^{\frac{1}{2}} \\ &= [E(x'_k x_k)]^{\frac{1}{2}} + \sum_{i=1}^{d}\{E[u'_{k-i}\Omega_{k,i}u_{k-i}]\}^{\frac{1}{2}} \\ &\leq [E(x'_k x_k)]^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\sum_{i=1}^{d}[E(u'_{k-i}u_{k-i})]^{\frac{1}{2}} \end{aligned}$$

where $\lambda > 0$ is a scalar such that $B'_d A'^{i-d-1}A^{i-d-1}B_d + \sigma\bar{B}'_d A'^{i-d-1}A^{i-d-1}\bar{B}_d \leq \lambda I_m$ for $i = 1, \ldots, d$. In terms of (11) and the above inequality, it can be deduced that

$$\lim_{k\to\infty} E(y'_k y_k) = 0. \tag{23}$$

Moreover, from (16)-(17), one has, for all $k \geq d$,

$$\begin{aligned} & E(z_k' z_k) \\ = \quad & E(y_k' y_k) + \sum_{i=1}^{d} [E(u_{k-i}' u_{k-i}) + E(\omega_{k-i}^2 u_{k-i}' u_{k-i})] \\ = \quad & E(y_k' y_k) + \sum_{i=1}^{d} [E(u_{k-i}' u_{k-i}) + \sigma E(u_{k-i}' u_{k-i})]. \end{aligned}$$

Together with (11) and (23), it yields

$$\lim_{k \to \infty} E(z_k' z_k) = 0.$$

So system (22) is asymptotically stable and thus system (9) is stabilized by $u_k = \hat{K} z_k$. This completes the proof. ■

Similar to Lemma 2, the stabilizable condition for system (9) can be derived from [22].

Lemma 4: System (9) is stabilizable iff the generalized Riccti equation

$$\begin{cases} X = F'XF - (G'XF)'\Upsilon^{-1}G'XF + diag\{S_1, \cdots, S_{2d+1}\} \\ \Upsilon = G'XG + \sigma \bar{G}'X\bar{G} + (2d+1)^{-1} I_m > 0 \end{cases} \tag{24}$$

where $diag$ represents the diagonal block matrix and $S_1 = I_n,\ S_{2j} = (2d+1)^{-1} I_m,\ S_{2j+1} = \sigma^{-1}(2d+1)^{-1} I_m,\ j = 1, \ldots, d$, admits a unique solution $X > 0$.

Lemma 5: Equation (24) has a unique solution $X > 0$ iff equation (4) admits a unique solution $P > 0$.

Proof: The Lemma will be shown by constructing a bijection between the positive definite solution to (24) and that to (4).

Suppose the positive definite matrix $X \in R^{n+2dm}$ is any solution to (24). Let X be partitioned as F, i.e.,

$$X = \begin{pmatrix} X_{1,1} & \cdots & X_{1,2d+1} \\ \vdots & \vdots & \vdots \\ X_{2d+1,1} & \cdots & X_{2d+1,2d+1} \end{pmatrix}$$

satisfying $X_{i,j} = X_{j,i}',\ i, j = 1, \ldots, 2d+1$.

By straightforward computation, (24) can be equivalently written as

$$\begin{aligned} X_{1,1} &= A'X_{1,1}A + S_1 \\ & \quad -(C'X_{1,1}A + X_{2,1}A)'\Upsilon^{-1}(C'X_{1,1}A + X_{2,1}A), \quad (25) \\ X_{1,j} &= A'X_{1,j+2} \\ & \quad -(C'X_{1,1}A + X_{2,1}A)'\Upsilon^{-1}(C'X_{1,j+2} + X_{2,j+2}), \\ & \qquad j = 2, \ldots, 2d-1, \quad (26) \\ X_{i,j} &= X_{i+2,j+2} \\ & \quad -(C'X_{1,i+2} + X_{2,i+2})'\Upsilon^{-1}(C'X_{1,j+2} + X_{2,j+2}), \\ & \qquad i, j = 2, \ldots, 2d-1, i \neq j, \quad (27) \\ X_{i,i} &= X_{i+2,i+2} + S_i \\ & \quad -(C'X_{1,i+2} + X_{2,i+2})'\Upsilon^{-1}(C'X_{1,i+2} + X_{2,i+2}), \\ & \qquad i = 2, \ldots, 2d-1, \quad (28) \\ X_{i,2d} &= 0,\ i = 1, \ldots, 2d-1, \\ X_{i,2d+1} &= 0,\ i = 1, \ldots, 2d, \quad (29) \\ X_{2d,2d} &= S_{2d},,\ X_{2d+1,2d+1} = S_{2d+1}, \quad (30) \\ \Upsilon &= C'X_{1,1}C + C'X_{12} + X_{2,1}C + X_{2,2} \\ & \quad +\sigma(\bar{C}'X_{1,1}\bar{C} + \bar{C}'X_{1,3} + X_{3,1}\bar{C} + X_{3,3}) \\ & \quad +(2d+1)^{-1} I_m > 0. \quad (31) \end{aligned}$$

From (29), (27) and (26), it is easy to check that

$$X_{i,j} = 0,\ i, j = 1, \ldots, 2d+1,\ i \neq j, \tag{32}$$

i.e., X is a block diagonal matrix. Thus (28) becomes

$$X_{i,i} = X_{i+2,i+2} + S_i,\ i = 2, \ldots, 2d-1.$$

Associated with (30), we get

$$\begin{aligned} X_{2i,2i} &= \sum_{j=i}^{d} S_{2j}, \\ X_{2i+1,2i+1} &= \sum_{j=i}^{d} S_{2j+1},\ i = 1, \ldots, d. \end{aligned} \tag{33}$$

Substitution of (32) and (33) into (25) and (31) leads to

$$X_{1,1} = A'X_{1,1}A + I_n - (C'X_{1,1}A)'\Upsilon^{-1}C'X_{1,1}A$$

and

$$\begin{aligned}
\Upsilon &= C'X_{1,1}C + \sum_{j=1}^{d} S_{2j} \\
&\quad +\sigma(\bar{C}'X_{1,1}\bar{C} + \sum_{j=1}^{d} S_{2j+1}) + R_1 \\
&= C'X_{1,1}C + \sigma\bar{C}'X_{1,1}\bar{C} \\
&\quad + \sum_{j=1}^{d} S_{2j} + \sigma\sum_{j=1}^{d} S_{2j+1} + R_1 \\
&= C'X_{1,1}C + \sigma\bar{C}'X_{1,1}\bar{C} + \sum_{j=1}^{d}(2d+1)^{-1}I_m \\
&\quad +\sigma\sum_{j=1}^{d}\sigma^{-1}(2d+1)^{-1}I_m + (2d+1)^{-1}I_m \\
&= C'X_{1,1}C + \sigma\bar{C}'X_{1,1}\bar{C} + I_m.
\end{aligned}$$

Notice that $X > 0$ means $X_{1,1} > 0$. Hence, $X_{1,1}$ is a solution to equation (4).

Conversely, let P be any positive definite solution to equation (4). Define X as:

$$\begin{aligned}
&X_{1,1} = P, \\
&X_{2i,2i} = (d-i+1)(2d+1)^{-1}I_m, \\
&X_{2i+1,2i+1} = (d-i+1)\sigma^{-1}(2d+1)^{-1}I_m,\ 1 \le i \le d, \\
&X_{i,j} = 0,\ i,j = 1,\ldots,2d+1,\ i \ne j.
\end{aligned}$$

Obviously, X defined above is positive definite and block diagonal. In order to show it is a solution to (24), it is enough to verify (25)-(31). First, consider (31).

$$\begin{aligned}
\Upsilon &= C'X_{1,1}C + C'X_{12} + X_{2,1}C + X_{2,2} \\
&\quad +\sigma(\bar{C}'X_{1,1}\bar{C} + \bar{C}'X_{1,3} + X_{3,1}\bar{C} + X_{3,3}) \\
&\quad +(2d+1)^{-1}I_m \\
&= C'PC + d(2d+1)^{-1}I_m + \sigma(\bar{C}'P\bar{C} \\
&\quad +d\sigma^{-1}(2d+1)^{-1}I_m) + (2d+1)^{-1}I_m \\
&= C'PC + \sigma\bar{C}'P\bar{C} + I_m = M > 0.
\end{aligned}$$

Then

$$\begin{aligned}
&A'X_{1,1}A + S_1 \\
&-(C'X_{1,1}A + X_{2,1}A)'\Upsilon^{-1}(C'X_{1,1}A + X_{2,1}A) \\
=\ &A'PA + I_n - (C'PA)'M^{-1}C'PA \\
=\ &P = X_{1,1}
\end{aligned}$$

which is actually (25). Second, (26), (27) and (29) are true since X is block diagonal and both sides of (26), (27) and (29) are zero. Third, let's prove (28). For $i = 2,\ldots,2d-1$, if i is even,

$$\begin{aligned}
&X_{i+2,i+2} + S_i \\
&-(C'X_{1,i+2} + X_{2,i+2})'\Upsilon^{-1}(C'X_{1,i+2} + X_{2,i+2}) \\
=\ &(d - \frac{i}{2})(2d+1)^{-1}I_m + (2d+1)^{-1}I_m \\
=\ &(d - \frac{i}{2} + 1)(2d+1)^{-1}I_m = X_{i,i}.
\end{aligned}$$

If i is odd,

$$\begin{aligned}
&X_{i+2,i+2} + S_i \\
&-(C'X_{1,i+2} + X_{2,i+2})'\Upsilon^{-1}(C'X_{1,i+2} + X_{2,i+2}) \\
=\ &(d - \frac{i}{2} + \frac{1}{2})\sigma^{-1}(2d+1)^{-1}I_m + \sigma^{-1}(2d+1)^{-1}I_m \\
=\ &(d - \frac{i}{2} + \frac{3}{2})\sigma^{-1}(2d+1)^{-1}I_m = X_{ii}.
\end{aligned}$$

Finally, notice that

$$\begin{aligned}
&X_{2d,2d} = (2d+1)^{-1}I_m = S_{2d}, \\
&X_{2d+1,2d+1} = \sigma^{-1}(2d+1)^{-1}I_m = S_{2d+1},
\end{aligned}$$

which is indeed (30). Thus it has been shown that X satisfies (24).

Hence, we have established the bijection between the positive definite solution to (4) and that to (24). This ends the proof. ■

According to Lemmas 3, 4 and 5, the necessity of Theorem 1 follows immediately.

B. Proof of the Sufficiency of Theorem 1

Proof: Suppose equation (4) has a unique solution satisfying $P > 0$. Let's show that system (1) is stabilized by the control (8). Lemma 2 implies that system (3) is stabilized by the control

$$u_k = Ky_k,\ k \ge d, \tag{34}$$

where K is determined by (5). Then the closed-loop system of (3) satisfies

$$\lim_{k\to\infty} E(y_k'y_k) = 0, \tag{35}$$

for any y_0 and any $\mathcal{F}_{i-1}$-measurable u_i, $i = 0,\ldots,d-1$. By making use of the relation (2), (34) is rewritten as

$$u_k = Kx_k + \sum_{i=1}^{d}(KA^{i-d-1}B_d + \omega_{k-i}KA^{i-d-1}\bar{B}_d)u_{k-i},$$

for $k \ge d$, which is actually (8). Next it remains to show that

$$\lim_{k\to\infty} E(u_k'u_k) = 0, \tag{36}$$

$$\lim_{k\to\infty} E(x_k'x_k) = 0. \tag{37}$$

From (34) and (35), it follows that

$$\lim_{k\to\infty} E(u_k'u_k) = \lim_{k\to\infty} E(y_k'K'Ky_k) = 0.$$

The verification of (37) is similar to that of (23) in the proof of Lemma 3. So the details are omitted here. Thus the proof is completed. ■

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