Automatica 49 (2013) 3097–3104

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Optimal filtering for networked systems with Markovian communication delays ☆

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ARTICLE INFO

Article history:
Received 11 May 2012
Received in revised form
20 June 2013
Accepted 27 June 2013
Available online 6 August 2013

Keywords:
Stochastic linear system
Optimal filter
Asymptotic convergence
Stability
Markovian transmission delay

ABSTRACT

This paper is concerned with optimal filter problems for networked systems with random transmission delays, while the delay process is modeled as a multi-state Markov chain. By defining a delay-free observation sequence, the optimal filter problems are transformed into ones of the Markov jumping parameter system. We first present an optimal Kalman filter, which is with time-varying, path-dependent filter gains, and the number of the paths grows exponentially in time delay. Thus an alternative optimal Markov jump linear filter is presented, in which the filter gains just depend on the present value of the Markov chain. Further, an optimal filter with constant-gains is developed, the existence condition for the stabilizing solutions to the filter is given, and it can be shown that the proposed Markov jump linear filter converges to the constant-gain filter under appropriate assumptions.

1. Introduction

In networked systems, data travels through different networks. It may suffer a time delay or may be lost completely due to some reasons such as data collisions, transmission errors and network congestion (Zhang & Yu, 2008). These phenomena will deteriorate the performance of the controllers and filters if they are not considered in the design procedure. Many researchers have studied the filter design for systems with packet losses. Recently, Kalman filtering for systems with intermittent observations was studied in Sinopoli et al. (2004). The stability of the Kalman filter in relation to the data arrival rate was investigated. It was shown that there existed a critical data arrival rate for an unstable system so that the mean filtering error covariance would be bounded for any initial condition. Further, less restrictive conditions were presented in Kar, Sinopoli, and Moura (2012). In You, Fu, and Xie (2011), the stability of Kalman filtering with Markovian packet losses was studied. The stability criteria were expressed by simple inequalities in terms of the largest eigenvalue of the open loop matrix and transition probabilities of the Markov process. In Sun, Xie, Xiao, and Chai Soh (2008), a multiple packet dropout modeling method has been presented, and an optimal linear estimator was computed recursively in terms of a Riccati difference equation.

For the state estimation of systems with independent random sensor delays, several results have been developed in Shi, Xie, and Murray (2009), Shen, Wang, Shu, and Wei (2009) and Wang, Ho, and Liu (2004) under different criteria, where the random delays were described as a set of distributed Bernoulli variables. Further, the relevant estimation results for systems with correlated transmission delays can be found in Han and Zhang (2009) where the process was modeled as a two-state Markov chain. In a very recent study, optimal filtering problems Sahebsara, Chen, and Shah (2007), Schenato (2008) and the Robust H_∞ filtering problem (Dong, Wang, & Gao, 2010) associated respectively with possible delay, uncertain observations and multiple packet dropouts were studied under a unified framework, respectively. It should be pointed out that there exist few results on multiple Markovian delayed systems, and few results considered the convergence and stability issues. This motivates us to study this interesting and challenging problem.

This paper studies the optimal filtering problems for networked systems with random observation delays. The delay process is

☆ This work was supported by the National Natural Science Foundation of China (Nos. 61104050, 61203029, 61120106011, 61034007), the National Natural Science Foundation for Distinguished Young Scholars of China (No. 60825304), and the Natural Science Foundation of Shandong Province (No. ZR2011FQ020). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Huijun Gao under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2013.07.018

modeled as a multi-state Markov chain which incorporates the packet losses naturally. A new delay-free observation sequence is first defined to denote the received observations, and thus the filtering problems can be converted into ones of a delay-free system with jumping parameters. An optimal linear mean square filter is presented based on the innovation analysis method, in which the filter gains are time-varying and sample path dependent. However, the convergence analysis to this filter is difficult. Alternatively, an optimal Markov jump linear filter is presented, in which the filter gains are just dependent on the present value of the Markov chains, but not the entire mode history. And at each time, just $\bar{r}$ filter gains are derived (where $\bar{r}$ is the maximum delay step). As a low-complexity solution, we further design a stationary Markov jump linear filter. The existence conditions for the stabilizing solutions to the filter are presented, and by a product, we show that the optimal Markov jump linear filter converges to the stationary filter.

Notations: Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the norm bounded linear space of all $m \times n$ matrices with $\mathbb{B}(\mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. For $L \in \mathbb{B}(\mathbb{R}^n)$, L' stands for the transpose of L. As usual, $L \geq 0 (L > 0)$ will mean that the symmetric matrix $L \in \mathbb{B}(\mathbb{R}^n)$ is positive semi-definite (positive definite), respectively. We set $\mathbb{B}(\mathbb{R}^n)^+ = \{L \in \mathbb{B}(\mathbb{R}^n); L = L' \geq 0\}$. In addition, we denote $\mathcal{H}^{m,n} = \{V = (V_0, \ldots, V_{\bar{r}+1}), V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)\}$, and define $\mathcal{H}^n = \mathcal{H}^{n,n}$, $\mathcal{H}^{n+} = \{V = (V_0, \ldots, V_{\bar{r}+1}) \in \mathcal{H}^n; V_i \in \mathbb{B}(\mathbb{R}^n)^+, i = 0, \ldots, \bar{r}+1\}$. For any Banach space $\mathbb{X}$, we denote $\mathbb{B}(\mathbb{X})$ as the Banach space of all bounded linear operators of $\mathbb{X}$ into $\mathbb{X}$, and set $r_\sigma(\mathcal{T})$ the spectral radius of $\mathcal{T} \in \mathbb{B}(\mathbb{X})$. Moreover, $1_{\{.\}}$ stands for the Dirac measure.

2. Problem formulations and preliminaries

2.1. Problem formulations

Consider the following discrete-time systems

$$x(k+1) = Ax(k) + Cw(k), \quad x(0) = x_0, \tag{1}$$

$$z(k) = Hx(k) + Gv(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state sequence, $z(k) \in \mathbb{R}^m$ is the output sequence, $w(k) \in \mathbb{R}^p$ is the system noise, and $v(k) \in \mathbb{R}^q$ is the output noise. The initial state x_0, $w(k)$ and $v(k)$ are null mean second-order independent wide sense stationary sequences with covariance matrices V, I_p and I_q, respectively. x_0, $w(k)$ and $v(k)$ are mutually independent, and $GG' > 0$.

The measurement $z(k)$ is time-stamped, and transmitted through a communication network. Let $r(k)$ denote the transmission delay of the measurement $z(k)$, where $r(k)$ is of a Markov process, and takes values in a finite state space $\{0, 1, \ldots, \bar{r}, \infty\}$. When $r(k) = i\ (i = 0, \ldots, \bar{r})$, it means that $z(k)$ will be received within $\bar{r}$ time steps. If the measurement is transmitted to the receiver with a delay larger than $\bar{r}$, it will be considered as one lost completely. And for this case, the random delay is set to be ∞. For the convenience of discussions, we introduce an index η to denote the delay steps, and define $\eta_i = i$ for $i = 0, 1, \ldots, \bar{r}$ and $\eta_{\bar{r}+1} = \infty$. Denote the transition probability matrix of $r(k)$ as $\Lambda = [(\lambda_{ij})]$, where $\lambda_{ij} \triangleq \text{Prob}(r(k+1) = \eta_j | r(k) = \eta_i)\ (i, j = 0, \ldots, \bar{r}, \bar{r}+1)$, and set $\pi(k) = [\pi_0(k) \ldots \pi_{\bar{r}}(k), \pi_{\bar{r}+1}(k)]'$ with $\pi_i(k) \triangleq \text{Prob}(r(k) = \eta_i)\ (i = 0, \ldots, \bar{r}, \bar{r}+1)$, then $\pi(k)$ and Λ satisfy the Kolmogorov difference equation $\pi(k+1) = \Lambda'\pi(k)$. We assume that $r(k)$ is independent of x_0, $w(k)$ and $v(k)$.

Given the above statement, we know that the possible received observations at time k are

$$\phi_{k,0}z(k-0) = \phi_{k,0}Hx(k-0) + \phi_{k,0}v(k-0), \tag{3}$$

$$\phi_{k,1}z(k-1) = \phi_{k,1}Hx(k-1) + \phi_{k,1}v(k-1), \tag{4}$$

$$\vdots$$

$$\phi_{k,\bar{r}}z(k-\bar{r}) = \phi_{k,\bar{r}}Hx(k-\bar{r}) + \phi_{k,\bar{r}}v(k-\bar{r}), \tag{5}$$

where

$$\phi_{k,i} \triangleq \begin{cases} 1, & \text{if } z(k-i) \text{ is received at the time } k; \\ 0, & \text{if } z(k-i) \text{ is not received at the time } k, \end{cases} \tag{6}$$

for $i = 0, 1, \ldots, \bar{r}$. Actually, $\phi_{k+i,i}\ (i = 0, 1, \ldots, \bar{r})$ is the indicator function of $r(k)$, and $\phi_{k+i,i} = 1$ represents the fact that the output $z(k)$ will be observed at the time $k+i$ subject to random delay i. As is well known, in a real-time control system, the output $z(k)$ can only be observed at most one time, and thus $\phi_{k+i,i}\ (i = 0, 1, \ldots, \bar{r})$ must satisfy the following relation

$$\phi_{k+i,i}\,\phi_{k+j,j} = 0, \quad i \neq j. \tag{7}$$

Let

$$y(k) = \text{col}\{\phi_{k,0}z(k-0), \ldots, \phi_{k,\bar{r}}z(k-\bar{r})\}, \tag{8}$$

$$\phi_k = \text{col}\{\phi_{k,0} \ldots \phi_{k,\bar{r}}\}, \tag{9}$$

then the filtering problems considered in this paper can be stated as:

Problem 1 (*Optimal Kalman Filter*)**.** Given the observations $\{y(s)|_{0 \leq s \leq k}\}$ and the information $\{\phi_s|_{0 \leq s \leq k}\}$, find an optimal linear minimum mean square error (LMMSE) filter $\hat{x}_o(k|k)$ of the state $x(k)$, such that

$$\text{E}\{\|x(k) - \hat{x}_o(k|k)\| | y(0), \ldots, y(k); \phi_0, \ldots, \phi_k\} \tag{10}$$

is minimized, while the filter gain is stochastic.

Problem 2 (*Optimal Markov Jump Filter*)**.** Given the observations $\{y(s)|_{0 \leq s \leq k}\}$ and the present-time value of ϕ_k, find an optimal recursive Markov jump linear (MJL) filter $\hat{x}_e(k|k)$ of the state $x(k)$, such that

$$\text{E}\{\|x(k) - \hat{x}_e(k|k)\| | y(0), \ldots, y(k); \phi_k\} \tag{11}$$

is minimized, while the filter gain is deterministic.

Problem 3 (*Stationary Markov Jump Filter*)**.** Given the observations $\{y(s)|_{0 \leq s \leq k}\}$ and the present-time value of ϕ_k, find a stationary MJL filter $\hat{x}(k|k)$ of the state $x(k)$, such that

$$\text{E}\{\|x(k) - \hat{x}(k|k)\| | y(0), \ldots, y(k); \phi_k\} \tag{12}$$

is minimized, while the filter gain is constant.

Remark 1. As for the three problems, the filter developed in Problem 1 has the smallest linear minimum mean square error compared to the latter two filters, but the filter gain to this filter is stochastic, and thus the performance analysis for this filter is difficult. The filter developed in Problem 3 is with constant gains, and thus less on-line computation is required. Compared to the proceeding two filters, this filter has the largest estimation error covariance. The filter developed in Problem 2 is in between. The precise definitions to the three problems will be given below.

2.2. Preliminaries

Now we rearrange the observations received up to time k, and define a new sequence

$$y(s, \bar{r}) \triangleq \text{col}\{\phi_{s+0,0}z(s), \ldots, \phi_{s+\bar{r},\bar{r}}z(s)\}, \quad 0 \leq s \leq k-\bar{r}, \tag{13}$$

$$y(s, k-s) \triangleq \text{col}\{\phi_{s+0,0}z(s), \ldots, \phi_{k,k-s}z(s), 0, \ldots, 0\},$$
$$k-\bar{r} < s \leq k. \tag{14}$$

It can be shown that $\{\{y(s, \bar{r})\}_{s=0}^{k-\bar{r}}; \{y(s, k-s)\}_{s=k-\bar{r}+1}^{k}\}$ is a delay-free sequence, and contains the same information as that of $\{\{y(s)\}_{s=0}^{k}\}$ (Han & Zhang, 2009).

For $0 \le s \le k-\bar{r}$, define

$$\theta(s,\bar{r}) \triangleq \mathrm{col}\{\phi_{s+0,0},\ldots,\phi_{s+\bar{r},\bar{r}}\}. \tag{15}$$

We know from (7) that only one element of $\phi_{s+i,i}(i=0,1,\ldots,\bar{r})$ is equal to 1 at most, so $\theta(s,\bar{r})$ will take values in the finite set

$$S(\bar{r}) = \{e_0(\bar{r}),\ldots,e_i(\bar{r}),\ldots,e_{\bar{r}}(\bar{r}),e_{\bar{r}+1}(\bar{r})\},$$

where $e_i(\bar{r})$ $(i=0,1,\ldots,\bar{r})$ is an $(\bar{r}+1)\times 1$-vector with all its components null except for the $(i+1)$th which is equal to one, and $e_{\bar{r}+1}(\bar{r})$ is an $(\bar{r}+1)\times 1$-vector with all elements being zeros. It can be seen that $\theta(s,\bar{r}) = e_i(\bar{r})$ represents the fact that the output $z(s)$ transmitted through the communication network with random delay $r(s)=\eta_i$. That means $\theta(s,\bar{r})$ is a Markov chain and has the same probability distribution and transition probability matrix as $r(s)$.

For $k-\bar{r} < s \le k$, define

$$\theta(s,k-s) \triangleq \mathrm{col}\{\phi_{s+0,0},\ldots,\phi_{k,k-s},0,\ldots,0\}. \tag{16}$$

Similarly, at most one element of $\phi_{s+i,i}$ $(i=0,1,\ldots,k-s)$ is equal to 1, so $\theta(s,k-s)$ will take values in the finite set

$$S(k-s) = \{e_0(k-s),\ldots,e_{k-s}(k-s),\ldots,e_{\bar{r}+1}(k-s)\},$$

where $e_i(k-s)$ $(i=0,1,\ldots,k-s)$ is an $(\bar{r}+1)\times 1$-vector with all its components null except for the $(i+1)$th which is equal to one, and $e_i(k-s)$ $(i=k-s+1,\ldots,\bar{r}+1)$ is an $(\bar{r}+1)\times 1$-vector with all elements being zeros. Obviously, $\theta(s,k-s)=e_i(k-s)$ represents the fact that the output $z(s)$ transmitted through the communication network with random delay $r(s)=\eta_i$, so $\theta(s,k-s)$ has the same probability distribution and transition probability matrix as $r(s)$.

In view of (15) and (16), the observation equations for (13) and (14) can be written as

$$y(s,\bar{r}) = H(\theta(s,\bar{r}))x(s) + G(\theta(s,\bar{r}))v(s), \quad 0\le s\le k-\bar{r}, \tag{17}$$

$$\begin{aligned} y(s,k-s) &= H(\theta(s,k-s))x(s) \\ &\quad + G(\theta(s,k-s))v(s), \quad k-\bar{r} < s \le k, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &H(\theta(s,\bar{r})) = \theta(s,\bar{r})\otimes H, \\ &G(\theta(s,\bar{r})) = \theta(s,\bar{r})\otimes G, \\ &H(\theta(s,k-s)) = \theta(s,k-s)\otimes H, \\ &G(\theta(s,k-s)) = \theta(s,k-s)\otimes G. \end{aligned}$$

For the convenience of discussions, we set $\iota = \min\{\bar{r}, k-s\}$. Note that $H(\theta(s,\iota))$ and $G(\theta(s,\iota))$ are block matrices partitioned according to the elements of $\theta(s,\iota)$ respectively, and will take values in the following set

$$\begin{aligned} &H(\theta(s,\iota)) \in \{H_0(\iota),H_1(\iota),\ldots,H_{\bar{r}+1}(\iota)\}, \\ &G(\theta(s,\iota)) \in \{G_0(\iota),G_1(\iota),\ldots,G_{\bar{r}+1}(\iota)\}, \end{aligned}$$

where $H_i(\iota)$ (or $G_i(\iota)$, $i=0,1,\ldots,\iota$) is a block matrix with dimension $m(\bar{r}+1)\times n$ (or $m(\bar{r}+1)\times q$) and all entries being zeros except for the $(i+1)$th block-row being H (or G), i.e.

$$H_i(\iota) = \begin{bmatrix} \underbrace{\overbrace{0,\ldots,0}^{i\text{ blocks}},H',0\ldots,0}_{\bar{r}+1\text{ blocks}} \end{bmatrix}', \quad i=0,1,\ldots,\iota,$$

$$G_i(\iota) = \begin{bmatrix} \underbrace{\overbrace{0,\ldots,0}^{i\text{ blocks}},G',0\ldots,0}_{\bar{r}+1\text{ blocks}} \end{bmatrix}', \quad i=0,1,\ldots,\iota,$$

and $H_i(\iota)$ (or $G_i(\iota)$, $i=\iota+1,\ldots,\bar{r}+1$) is with the dimension of $m(\bar{r}+1)\times n$ (or $m(\bar{r}+1)\times q$) and has all entries being zeros.

In view of (17) and (18), we know that the estimation problems for random delayed systems can be converted into ones of MJL systems without delays.

3. Optimal Kalman filter

In this section, we will design an optimal LMMSE filter $\hat{x}_o(k|k)$ of the state $x(k)$ via the projection formula, where

$$\hat{x}_o(k|k) \triangleq \mathrm{E}\{x(k)|y(0,\bar{r}),\ldots,y(k,0);\theta(0,\bar{r}),\ldots,\theta(k,0)\}.$$

According to Problem 1, we now present the following definition.

Definition 2. Consider the given time k. For $0\le s\le k-\bar{r}$, the LMMSE estimator of $x(s)$ is defined as

$$\begin{aligned} \hat{x}_o(s,\bar{r}) \triangleq E\{&x(s)|y(0,\bar{r}),\ldots,y(s-1,\bar{r}); \\ &\theta(0,\bar{r}),\ldots,\theta(s-1,\bar{r})\}, \end{aligned}$$

while its error covariance matrix $P(s,\bar{r})$ is defined as

$$\begin{aligned} P(s,\bar{r}) \triangleq E\{&(x(s)-\hat{x}_o(s,\bar{r}))(x(s)-\hat{x}_o(s,\bar{r}))' \\ &|y(0,\bar{r})\ldots,y(s-1,\bar{r});\theta(0,\bar{r}),\ldots,\theta(s-1,\bar{r})\}. \end{aligned}$$

For $k-\bar{r} < s\le k$, the LMMSE estimator of $x(s)$ is defined as

$$\begin{aligned} \hat{x}_o(s,k-s+1) \triangleq E\{&x(s)|y(0,\bar{r}),\ldots,y(s-1,k-s+1); \\ &\theta(0,\bar{r}),\ldots,\theta(s-1,k-s+1)\}, \end{aligned}$$

while its error covariance matrix $P(s,k-s+1)$ is defined as

$$\begin{aligned} P(s,k-s+1) \triangleq E\{&(x(s)-\hat{x}(s,k-s+1))(x(s) \\ &-\hat{x}(s,k-s+1))'|y(0,\bar{r}),\ldots,y(s-1,k \\ &-s+1);\theta(0,\bar{r}),\ldots,\theta(s-1,k-s+1)\}. \end{aligned}$$

Then the LMMSE estimation can be derived as:

Theorem 3. *Consider the system (1), (17) and (18), the LMMSE estimation $\hat{x}_o(k|k)$ is given by*

$$\hat{x}_o(k|k) = \hat{x}_o(k,1) + \phi_{k,0}K(k)(z(k)-H\hat{x}_o(k,1)), \tag{19}$$

where $K(k)$ is the solution to the following equation

$$K(k) = P(k,1)H'(HP(k,1)H'+GG')^{-1}, \tag{20}$$

and $\hat{x}_o(k,1)$ is computed by the following steps.

- *Step 1 For $0\le s\le k-\bar{r}$, calculate $\hat{x}_o(s,\bar{r})$*

$$\begin{aligned} \hat{x}_o(s+1,\bar{r}) &= A\hat{x}_o(s,\bar{r}) + \sum_{i=0}^{\bar{r}}\phi_{s+i,i}K(s)(z(s) \\ &\quad - H\hat{x}_o(s,\bar{r})), \hat{x}_o(0,\bar{r}) = 0, \end{aligned} \tag{21}$$

where

$$K(s) = AP(s,\bar{r})H'(HP(s,\bar{r})H'+GG')^{-1}, \tag{22}$$

$$\begin{aligned} &P(s+1,\bar{r}) = AP(s,\bar{r})A' - \sum_{i=0}^{\bar{r}}\phi_{s+i,i}K(s)HP(s,\bar{r})A' + CC', \\ &P(0,\bar{r}) = V. \end{aligned} \tag{23}$$

- *Step 2 For $k-\bar{r} < s\le k$, calculate $\hat{x}_o(s+1,k-s)$*

$$\begin{aligned} \hat{x}_o(s+1,k-s) &= A\hat{x}_o(s,k-s+1) + \sum_{i=0}^{k-s}\phi_{s+i,i}K(s) \\ &\quad \times (z(s)-H\hat{x}_o(s,k-s+1)), \end{aligned} \tag{24}$$

where

$$K(s) = AP(s,k-s+1)H'(HP(s,k-s+1)H'+GG')^{-1}, \tag{25}$$

$$\begin{aligned} P(s+1,k-s) &= AP(s,k-s+1)A' - \sum_{i=0}^{k-s}\phi_{s+i,i}K(s) \\ &\quad \times HP(s,k-s+1)A' + CC'. \end{aligned} \tag{26}$$

- *Step 3 Set $s+1=k$ in (24), then $\hat{x}_o(k,1)$ is obtained.*

Proof. From Definition 2, we know that the LMMSE filter subject to Problem 1 is indeed a Kalman filter. Based on the standard Kalman filter theory and the observations in (17), (18), we will obtain the results. ■

4. Optimal Markov jump linear filter

We consider in this section the optimal MJL filter in the recursive form for systems (1), (17) and (18). First, we present the following definition.

Definition 4. Consider the given time k, the optimal MJL filter $\hat{x}_e(k|k)$ of $x(k)$ is defined as

$$\hat{x}_e(k|k) \triangleq E\{x(k)|y(0,\bar{r}),\ldots,y(k,0);\theta(k,0)\}, \tag{27}$$

and for $0 \le s \le k-\bar{r}$, the optimal MJL filter $\hat{x}_e(s|s-1)$ of $x(s)$ is defined as

$$\hat{x}_e(s,\bar{r}) \triangleq E\{x(s)|y(0,\bar{r}),\ldots,y(s-1,\bar{r});\theta(s-1,\bar{r})\}, \tag{28}$$

while for $k-\bar{r} < s \le k$, the optimal MJL filter $\hat{x}_e(s|s-1)$ is defined as

$$\begin{aligned}\hat{x}_e(s,k-s+1) \triangleq E\{&x(s)|y(0,\bar{r}),\ldots,y(s-1,k-s+1);\\ &\theta(s-1,k-s+1)\}.\end{aligned} \tag{29}$$

Denote

$$\begin{aligned}
&\tilde{x}_e(k|k) = x(k) - \hat{x}_e(k|k),\\
&\tilde{x}_e(s,\bar{r}) = x(s) - \hat{x}_e(s,\bar{r}), \quad 0 \le s \le k-\bar{r},\\
&\tilde{x}_e(s,k-s+1) = x(s) - \hat{x}_e(s,k-s+1), \quad k-\bar{r} < s \le k,\\
&Y_i(k|k) = E\{\tilde{x}_e(k|k)\tilde{x}_e(k|k)' 1_{\theta(k,0)=e_i(0)}\}, \quad i = 0,1,\ldots,\bar{r}+1,\\
&Y_i(s,\bar{r}) = E\{\tilde{x}_e(s,\bar{r})\tilde{x}_e(s,\bar{r})' 1_{\theta(s,\bar{r})=e_i(\bar{r})}\}, \quad i = 0,1,\ldots,\bar{r}+1,\\
&Y_i(s,k-s+1) = E\{\tilde{x}_e(s,k-s+1)\tilde{x}_e(s,k-s+1)'\\
&\qquad \times 1_{\theta(s,k-s)=e_i(k-s)}\}, \quad i = 0,1,\ldots,\bar{r}+1,
\end{aligned}$$

then the filter $\hat{x}_e(k|k)$ is obtained in the following result.

Lemma 5. *Consider the system (1), (17) and (18) and given the time $k > \bar{r}$, the minimum mean square error solution to the MJL filter $\hat{x}_e(k|k)$ is given by*

$$\hat{x}_e(k|k) = \hat{x}_e(k,1) - F(k,\theta(k,0))(y(k) - H(\theta(k,0))\hat{x}_e(k,1)), \tag{30}$$

where the filter gain $F(k,\theta(k,0))$ is denoted as $F_i(k,0)$ for $\theta(k,0) = e_i(0)$ $(i = 0,1,\ldots,\bar{r}+1)$, which is determined by

$$\begin{aligned}&F_i(k,0)(H_i(0)Y_i(k,1)H_i(0)' + \pi_i(k)G_i(0)G_i(0)')\\ &\quad = -Y_i(k,1)H_i(0)',\end{aligned} \tag{31}$$

and $\hat{x}_e(k,1)$ and $Y_i(k,1)$ are computed as follows:

- *Step 1 Calculate $\hat{x}_e(s,\bar{r})$ and $Y_i(s,\bar{r})$ for $0 \le s \le k-\bar{r}$*

$$\begin{aligned}\hat{x}_e(s+1,\bar{r}) = A\hat{x}_e(s,\bar{r}) &- F(s,\theta(s,\bar{r}))(y(s,\bar{r})\\ &- H(\theta(s,\bar{r}))\hat{x}_e(s,\bar{r})), \quad \hat{x}_e(0,\bar{r}) = 0,\end{aligned} \tag{32}$$

  where the filter gain $F(s,\theta(s,\bar{r}))$ denoted by $F_i(s,\bar{r})$ for $\theta(s,\bar{r}) = e_i(\bar{r})$ $(i = 0,1,\ldots,\bar{r}+1)$ is calculated by

$$\begin{aligned}&F_i(s,\bar{r})(H_i(\bar{r})Y_i(s,\bar{r})H_i(\bar{r})' + \pi_i(s)G_i(\bar{r})G_i(\bar{r})')\\ &\quad = -AY_i(s,\bar{r})H_i(\bar{r})',\end{aligned} \tag{33}$$

  and $Y_i(s,\bar{r})$ $(i = 0,1,\ldots,\bar{r}+1)$ satisfies the following coupled Riccati difference equation

$$\begin{aligned}Y_j(s+1,\bar{r}) = \sum_{i=0}^{\bar{r}+1}\lambda_{ij}\{&AY_i(s,\bar{r})A' + \pi_i(s)CC'\\ &+ F_i(s,\bar{r})H_i(\bar{r})Y_i(s,\bar{r})A'\},\end{aligned} \tag{34}$$

  with the initial value $Y_j(0,\bar{r}) = \pi_j(0)V$.

- *Step 2 Calculating $\hat{x}_e(s+1,k-s)$ and $Y_i(s+1,k-s)$ for $k-\bar{r} < s \le k$*

$$\begin{aligned}\hat{x}_e(s+1,k-s) = A&\hat{x}_e(s,k-s+1)\\ &- F(s,\theta(s,k-s))(y(s,k-s)\\ &- H(\theta(s,k-s))\hat{x}_e(s,k-s+1)),\end{aligned} \tag{35}$$

  where the filter gain $F(s,\theta(s,k-s))$ is denoted as $F_i(s,k-s)$ for $\theta(s,k-s) = e_i(k-s)$ $(i = 0,1,\ldots,\bar{r}+1)$, which is calculated by

$$\begin{aligned}&F_i(s,k-s)(H_i(k-s)Y_i(s,k-s+1)H_i(k-s)'\\ &\quad + \pi_i(s)G_i(k-s)G_i(k-s)')\\ &\quad = -AY_i(s,k-s+1)H_i(k-s)',\end{aligned} \tag{36}$$

  and $Y_i(s,k-s+1)$ $(i = 0,1,\ldots,\bar{r}+1)$ satisfies the following coupled Riccati difference equation

$$\begin{aligned}Y_j(s+1,k-s) = \sum_{i=0}^{\bar{r}+1}\lambda_{ij}\{&AY_i(s,k-s+1)A' + \pi_i(s)CC'\\ &+ F_i(s,k-s)H_i(k-s)Y_i(s,k-s+1)A'\}.\end{aligned} \tag{37}$$

- *Step 3 Set $s+1 = k$ in Step 2, then $\hat{x}_e(k,1)$ and $Y_i(k,1)$ $(i = 0,1,\ldots,\bar{r}+1)$ are obtained.*

Proof. First, for $0 \le s \le k-\bar{r}$, we have from (1), (32) and note the definition of $\tilde{x}_e(s,\bar{r})$ that

$$\begin{aligned}\tilde{x}_e(s+1,\bar{r}) = (A + F(s,\theta(s,\bar{r}))H(\theta(s,\bar{r})))\tilde{x}_e(s,\bar{r})\\ + Cw(s) + F(s,\theta(s,\bar{r}))G(\theta(s,\bar{r}))v(s).\end{aligned} \tag{38}$$

In view of (38), we get

$$\begin{aligned}Y_j(s+1,\bar{r}) = \sum_{i=0}^{\bar{r}+1}\lambda_{ij}\{&AY_i(s,\bar{r})A' + \pi_i(s)CC' + AY_i(s,\bar{r})H_i(\bar{r})'\\ &\times F_i(s,\bar{r})' + F_i(s,\bar{r})H_i(\bar{r})Y_i(s,\bar{r})A' + F_i(s,\bar{r})(H_i(\bar{r})\\ &\times Y_i(s,\bar{r})H_i(\bar{r})' + \pi_i(s)G_i(\bar{r})G_i(\bar{r})')F_i(s,\bar{r})'\}.\end{aligned} \tag{39}$$

It follows from $\frac{\partial Y_i(s,\bar{r})}{\partial F_i(s,\bar{r})} = 0$ that

$$F_i(s,\bar{r})(H_i(\bar{r})Y_i(s,\bar{r})H_i(\bar{r})' + \pi_i(s)G_i(\bar{r})G_i(\bar{r})') = -AY_i(s,\bar{r})H_i(\bar{r})'.$$

The minimum mean square jump filter gain (33) is obtained, and thus (39) becomes like (34).

Next, following a procedure similar to the determination of $F_i(s,\bar{r})$ and $Y_i(s,\bar{r})$ for $0 \le s \le k-\bar{r}$, we will obtain (36), (37) and (31) immediately. ■

We can show that the filter presented in Lemma 5 is the optimal realization of the general recursive MJL filters (Han & Zhang, 2009). However, it is somewhat complicated for the filter implementation. Note that $\theta(s,\iota) = e_i(\iota)$ $(0 \le s \le k,\ i = 0,1,\ldots,\iota)$ represents the same fact that $\phi_{s+i,i} = 1$ $(0 \le s \le k,\ i = 0,1,\ldots,\iota)$, and $\theta(s,\iota) = e_i(\iota)$ $(0 \le s \le k, i = \iota+1,\ldots,\bar{r}+1)$ is equivalent to $\phi_{s+i,i} = 0$ $(0 \le s \le k, i = \iota+1,\ldots,\bar{r}+1)$. Then the filter gain $F(s,\theta(s,\iota))$ for $\theta(s,\iota) = e_i(\iota)$ $(0 \le s \le k, i = \iota+1,\ldots,\bar{r}+1)$ does not need to be determined, since there is no observation received for this case. Further, most entries of $H(\theta(s,\iota))$, $G(\theta(s,\iota))$ $(0 \le s \le k,\ i = 0,1,\ldots,\iota)$ are zeros, thus the result of Lemma 5 can be simplified.

Theorem 6. *The minimum mean square error solution to the MJL filter $\hat{x}_e(k|k)$ can be rewritten as*

$$\hat{x}_e(k|k) = \hat{x}_e(k,1) - F_0(k,0)[\phi_{k,0}z(k) - \phi_{k,0}H\hat{x}_e(k,1)], \tag{40}$$

where the filter gain $F_0(k,0)$ is determined by

$$F_0(k,0) = -Y_0(k,1)H'(HY_0(k,1)H' + \pi_0(k)GG')^{-1}, \tag{41}$$

and $\hat{x}_e(k,1)$ and $Y_0(k,1)$ are computed by the following steps:

- *Step* 1 *When* $0 \le s \le k - \bar{r}$, *calculate* $\hat{x}_e(s, \bar{r})$ *and* $Y_i(s, \bar{r})$

$$\hat{x}_e(s+1, \bar{r}) = A\hat{x}_e(s, \bar{r}) - F_i(s, \bar{r})[\phi_{s+i,i}z(s) - \phi_{s+i,i}H\hat{x}_e(s, \bar{r})],\quad i = 0, 1, \ldots, \bar{r},$$
$$\hat{x}_e(0, \bar{r}) = 0, \tag{42}$$

where the filter gain $F_i(s, \bar{r})$ $(i = 0, 1, \ldots, \bar{r})$ *is calculated by*

$$F_i(s, \bar{r}) = -AY_i(s, \bar{r})H'(HY_i(s, \bar{r})H' + \pi_i(s)GG')^{-1}, \tag{43}$$

and $Y_i(s, \bar{r})$ $(i = 0, 1, \ldots, \bar{r}+1)$ *satisfies the coupled Riccati difference equation*

$$Y_j(s+1, \bar{r}) = \sum_{i=0}^{\bar{r}} \lambda_{ij}\{AY_i(s, \bar{r})A' + \pi_i(s)CC' + F_i(s, \bar{r})HY_i(s, \bar{r})A'\} + \lambda_{\bar{r}+1,j}\{AY_{\bar{r}+1}(s, \bar{r})A' + \pi_{\bar{r}+1}(s)CC'\}, \tag{44}$$

with $Y_j(0, \bar{r}) = \pi_j(0)V$.

- *Step* 2 *When* $k - \bar{r} < s \le k$, *calculate* $\hat{x}_e(s+1, k-s)$ *and* $Y_i(s+1, k-s)$

$$\hat{x}_e(s+1, k-s) = A\hat{x}_e(s, k-s+1) - F_i(s, k-s) \times [\phi_{s+i,i}z(s) - \phi_{s+i,i}H\hat{x}_e(s, k-s+1)],\quad i = 0, \ldots, k-s, \tag{45}$$

where the filter gain $F_i(s, k-s)$ $(i = 0, 1, \ldots, k-s)$ *is calculated by*

$$F_i(s, k-s) = -AY_i(s, k-s+1)H'(HY_i(s, k-s+1) \times H' + \pi_i(s)GG')^{-1}, \tag{46}$$

and $Y_i(s, k-s+1)$ $(i = 0, 1, \ldots, \bar{r}+1)$ *satisfies the coupled Riccati difference equation*

$$Y_j(s+1, k-s) = \sum_{i=0}^{k-s} \lambda_{ij}\{AY_i(s, k-s+1)A' + \pi_i(s)CC' + F_i(s, k-s)HY_i(s, k-s+1)A'\} + \sum_{i=k-s+1}^{\bar{r}+1} \lambda_{ij}\{AY_i(s, k-s+1)A' + \pi_i(s)CC'\}. \tag{47}$$

- *Step* 3 *Set* $s + 1 = k$ *in Step* 2, *then* $\hat{x}_e(k, 1)$ *and* $Y_0(k, 1)$ *are obtained from* (45) *and* (47), *respectively.*

Remark 7. In Theorem 6, we still use the notation $F_i(s, \iota)$ to denote the filter gain, which is different from that of Lemma 5. In addition, the Riccati equations (44) and (47) differ from the standard coupled Riccati equation developed in Costa, Fragoso, and Marques (2005), which involve additional terms that depend on the transition probability of the packet dropouts, and the extra terms would be zero for the case without packet losses.

Remark 8. For the case in which A, C, H, G and λ_{ij} in (1) and (2) are time invariant and $\{r(k)\}$ satisfies the ergodic assumption, so that $\pi_i(k)$ converges to $\pi_i > 0$ as k goes to infinity, the filtering coupled difference Riccati equations (CDREs) (44) and (47) lead to the following coupled algebraic Riccati equations (CAREs)

$$Y_j(\bar{r}) \triangleq \sum_{i=0}^{\bar{r}} \lambda_{ij}\{AY_i(\bar{r})A' + \pi_iCC' - AY_i(\bar{r})H' \times (HY_i(\bar{r})H' + \pi_iGG')^{-1}HY_i(\bar{r})A'\}, + \lambda_{\bar{r}+1,j}\{AY_{\bar{r}+1}(\bar{r})A' + \pi_{\bar{r}+1}CC'\}, \tag{48}$$

$$Y_j(l) \triangleq \sum_{i=0}^{l} \lambda_{ij}\{AY_i(l+1)A' + \pi_iCC' - AY_i(l+1)H' \times (HY_i(l+1)H' + \pi_iGG')^{-1}HY_i(l+1)A'\} + \sum_{i=l+1}^{\bar{r}+1} \lambda_{ij}\{AY_i(l+1)A' + \pi_iCC'\},\quad l = \bar{r}-1, \ldots, 0. \tag{49}$$

In the next section we present a sufficient condition for the existence of a unique set of solutions $Y(l) = (Y_0(l), \ldots, Y_{\bar{r}+1}(l))$ for (48) and (49), and the convergence of $Y(s, l)$ to $Y(l)$, so that a stationary filter would be obtained.

5. Stationary Markov jump linear filter

In this section, we consider all matrices in (1) and (2) time invariant, and we will develop a stationary MJL filter for system (1) and (2). Assume that the random delay $\{r(k)\}$ is ergodic, so that there exist limit probabilities $\pi_i > 0$; $i = 0, 1, \ldots, \bar{r}+1$, with $\sum_{i=0}^{\bar{r}+1} \pi_i = 1$, such that $\pi_i(k) \to \pi_i$, is exponentially fast, as $k \to \infty$. The filter considered in this section is

$$\hat{x}(k|k) \triangleq \hat{x}(k, 1) - F_0(0)[\phi_{k,0}z(k) - \phi_{k,0}H\hat{x}(k, 1)], \tag{50}$$

$$\hat{x}(s+1, \bar{r}) \triangleq A\hat{x}(s, \bar{r}) - F_i(\bar{r})[\phi_{s+i,i}z(s) - \phi_{s+i,i}H\hat{x}(s, \bar{r})],\quad \hat{x}(0, \bar{r}) = 0,\ 0 \le s \le k - \bar{r},\ i = 0, \ldots, \bar{r}, \tag{51}$$

$$\hat{x}(s+1, l) \triangleq A\hat{x}(s, l+1) - F_i(l)[\phi_{s+i,i}z(s) - \phi_{s+i,i}H\hat{x}(s, l+1)],\quad l = \bar{r}-1, \ldots, 0,\ k-l \le s < k-l+1,\ i = 0, \ldots, l. \tag{52}$$

The goal is to find the filter gains $F_i(l)(l = \bar{r}, \ldots, 0)$, such that the proposed filter is mean square stable and minimizes its corresponding estimation error. The solution of the filtering problem developed above is closely related to the stabilizing solutions for the CAREs (48) and (49). It is noted that if there exists a stabilizable solution to (48), the future finite iterations (49) will be stable as well. Thus we just need to show the stability condition of (48).

For $Y = (Y_0, \ldots, Y_{\bar{r}+1}) \in \mathcal{H}^n$, $\Gamma = (\Gamma_0, \ldots, \Gamma_{\bar{r}+1}) \in \mathcal{H}^n$, we define the linear operator $\mathcal{T}(Y) = (\mathcal{T}_0(Y), \ldots, \mathcal{T}_{\bar{r}+1}(Y))$ as

$$\mathcal{T}_j(Y) \triangleq \sum_{i=0}^{\bar{r}+1} \lambda_{ij}\Gamma_iY_i\Gamma_i^*. \tag{53}$$

Further, for ease of notation, we denote $A_i = A(i = 0, \ldots, \bar{r}+1)$, $C_i = C(i = 0, \ldots, \bar{r}+1)$, $H_i = H(i = 0, \ldots, \bar{r})$, $H_{\bar{r}+1} = 0$, $G_i = G(i = 0, \ldots, \bar{r})$, $G_{\bar{r}+1} = 0$, and define $\bar{A} = (A_0, \ldots, A_{\bar{r}+1}) \in \mathcal{H}^n$, $\bar{C} = (C_0, \ldots, C_{\bar{r}+1}) \in \mathcal{H}^{p,n}$, $\bar{H} = (H_0, \ldots, H_{\bar{r}+1}) \in \mathcal{H}^{n,m}$, $\bar{G} = (G_0, \ldots, G_{\bar{r}+1}) \in \mathcal{H}^{q,m}$. Two structural concepts turn out to be essential: mean square stabilizability and mean square detectability.

Definition 9 (*Costa & Fragoso, 1995*)**.** For $\bar{A} \in \mathcal{H}^n$, $\bar{C} \in \mathcal{H}^{p,n}$. We say that $(\bar{A}, \bar{C}, \Lambda)$ is mean square stabilizable if there is $\bar{L} = (L_0, L_1, \ldots, L_{\bar{r}+1}) \in \mathcal{H}^{n,p}$ such that $r_\sigma(\mathcal{T}) < 1$ when $\Gamma_i = A_i + C_iL_i$ in (53) for $i = 0, 1, \ldots, \bar{r}+1$. In this case, $\bar{L}$ is said to stabilize $(\bar{A}, \bar{C}, \Lambda)$.

Definition 10 (*Costa & Fragoso, 1995*)**.** For $\bar{A} \in \mathcal{H}^n$, $\bar{H} \in \mathcal{H}^{n,m}$. We say that $(\Lambda, \bar{H}, \bar{A})$ is mean square detectable if there is $\bar{L} = (L_0, L_1, \ldots, L_{\bar{r}+1}) \in \mathcal{H}^{m,n}$ such that $r_\sigma(\mathcal{T}) < 1$ when $\Gamma_i = A_i + L_iH_i$ in (53) for $i = 0, 1, \ldots, \bar{r}+1$. In this case, $\bar{L}$ is said to stabilize $(\Lambda, \bar{H}, \bar{A})$.

In addition, for $Y(\bar{r}) = (Y_0(\bar{r}), \ldots, Y_{\bar{r}+1}(\bar{r})) \in \mathcal{H}^n$, define the linear operators

$$f^{(\bar{r})}(Y(\bar{r}), \pi) \triangleq (f_0^{(\bar{r})}(Y(\bar{r}), \pi), \ldots, f_{\bar{r}+1}^{(\bar{r})}(Y(\bar{r}), \pi))$$

as

$$f_j^{(\bar{r})}(Y(\bar{r}), \pi) \triangleq \sum_{i=0}^{\bar{r}+1} \lambda_{ij}\{(A_i + L_i(Y(\bar{r}), \pi)H_i)Y_i(\bar{r}) \times (A_i + L_i(Y(\bar{r}), \pi)H_i)' + \pi_i(C_iC_i' + L_i(Y(\bar{r}), \pi)G_iG_i'L_i(Y(\bar{r}), \pi)')\}, \tag{54}$$

where

$$L_i(Y(\bar{r}), \pi) = \begin{cases} -A_iY_i(\bar{r})H_i'(H_iY_i(\bar{r})H_i' + \pi_iG_iG_i')^{-1}, \\ \quad i = 0, \ldots, \bar{r}; \\ 0, \quad i = \bar{r}+1. \end{cases} \tag{55}$$

Let $f_j^{(\bar r)}(Y(\bar r),\pi) = f_j^{(\bar r)}(Y(\bar r)), L_i(Y(\bar r),\pi) = L_i(Y(\bar r))$, the CARE (48) can be expressed as

$$Y_j(\bar r) = f_j^{(\bar r)}(Y(\bar r)), \quad j = 0, \ldots, \bar r + 1. \tag{56}$$

Several results are presented in Appendices A and B in connection to the following definition related to (56).

Definition 11 (*Costa & Fragoso, 1995*). $Y(\bar r) = (Y_0(\bar r), \ldots, Y_{\bar r+1}(\bar r)) \in \mathcal{H}^{n+}$ is the stabilizing solution for (56), if $Y(\bar r)$ satisfies (56), and $L(Y(\bar r)) \triangleq (L_0(Y(\bar r)), \ldots, L_{\bar r}(Y(\bar r)), L_{\bar r+1}(Y(\bar r)))$ stabilize $(\Lambda, \bar A, \bar H)$, where $L_i(Y(\bar r))$ $(i = 0, \ldots, \bar r + 1)$ is as in (55).

The next result will show the sufficient condition for the existence of mean square stabilizing solution.

Lemma 12. *If $(\bar A, \bar C, \Lambda)$ is mean square stabilizable and $(\bar H, \bar A, \Lambda)$ is mean square detectable, then there exists a unique solution $\hat Y(\bar r) = (\hat Y_0(\bar r), \ldots, \hat Y_{\bar r+1}(\bar r)) \in \mathcal{H}^n$ for the CARE (56), which will coincide with the mean square stabilizing solution.*

Proof. See Appendix A. ■

In the following, we will show that the CDREs (44) and (47) converge to the CAREs (48) and (49), respectively.

Lemma 13. *If $(\bar A, \bar C, \Lambda)$ is mean square stabilizable and $(\Lambda, \bar H, \bar A)$ is mean square detectable. Then $Y(s, \bar r)(0 \le s \le k - \bar r)$ and $Y(s, k-s+1)(k-\bar r+1 < s \le k)$ defined in (44) and (47) converge to a unique set of $\hat Y(l), l = \bar r, \ldots, 0$ as k goes to infinity whenever $Y(0, \bar r) \in \mathcal{H}^{n+}$. Furthermore, $\hat Y(l), l = \bar r, \ldots, 0$ are the stabilizing solutions to the CAREs (48) and (49), respectively.*

Proof. See Appendix B. ■

Now we will present the main result of this section.

Theorem 14. *If $(\bar A, \bar C, \Lambda)$ is mean square stabilizable and $(\bar H, \bar A, \Lambda)$ is mean square detectable, then an optimal solution for the stationary filter posed in (50)–(52) is given by*

$$F_i(\bar r) = -A\hat Y_i(\bar r)H'(H\hat Y_i(\bar r)H' + \pi_i GG')^{-1}, \quad i = 0, \ldots, \bar r, \tag{57}$$

$$F_i(l) = -A\hat Y_i(l+1)H'(H\hat Y_i(l+1)H' + \pi_i GG')^{-1},$$
$$i = 0, \ldots, l, \; l = \bar r - 1, \ldots, 0, \tag{58}$$

where $\hat Y(l) = (\hat Y_0(l), \ldots, \hat Y_{\bar r+1}(l)) \in \mathcal{H}^{n+}, l = \bar r, \ldots, 0$ are the mean square stabilizing solutions of (48) and (49), respectively.

Proof. From Lemma 13 and Theorem 5.8 in Costa et al. (2005), we can show that the optimal solutions for the stable filters are as in (57) and (58). ■

6. Numerical examples

In this section, we present a numerical example to illustrate the previous theoretical results. Consider a second-order dynamic system described in (1)–(2) with the following specifications:

$$A = \begin{bmatrix} 2 & 1.1 \\ -1.7 & -0.8 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$H = \begin{bmatrix} 4 & 2 \end{bmatrix}, \qquad G = 1,$$

where $\{w(k)\}$ and $\{v(k)\}$ are mutually independent zero-mean noises with covariance matrices $Q_w = 1$ and $Q_v = 1$, respectively. $r(k)$ is the random jump delay which is modeled as a discrete-state Markov chain taking values in a finite set $S = \{0, 1, 2, 3\}$ with

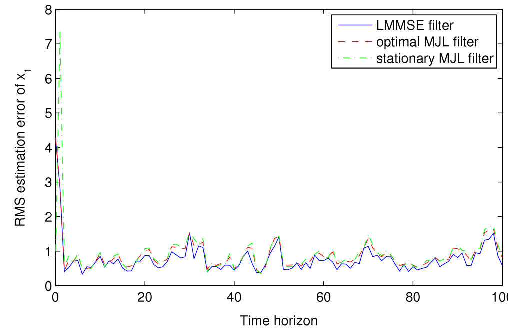

Fig. 1. The root mean square estimation errors of the first state component $x_1(k)$.

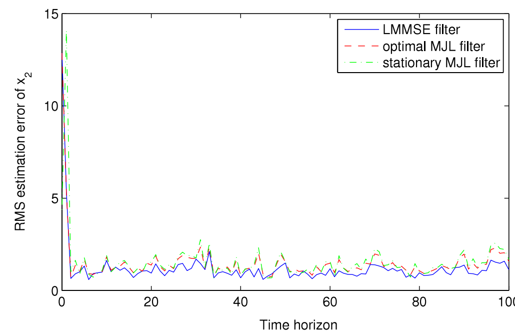

Fig. 2. The root mean square estimation errors of the second state component $x_2(k)$.

transition probability matrix

$$\Lambda = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{bmatrix}.$$

The initial distribution of $r(k)$ is $\pi(0) = [0.7\ 0.1\ 0.1\ 0.1]'$, the initial state $x(0)$ is a random variable with $E(x(0)) = 0$ and $E(x(0)x(0)') = I_2$. In the actual system we use $x(0) = [30\ 30]'$ for the simulation.

In this example, the time horizon is set to $N = 100$. Without loss of generality, we run 50 Monte Carlo simulations from $k = 0$ to 100. The simulation results are subject to the same parameters and noise sequences $\{w(k)\}, \{v(k)\}$. The LMMSE filter, the MJL filter, and the stationary MJL filter are designed via Theorems 3, 6 and 14, respectively, and the performance of the proposed three filters are compared. In Figs. 1 and 2, RMS errors in state estimation are compared for the three cases. It can be seen from the simulation results that the obtained linear estimators for systems with random delays track well and the estimation scheme proposed in this paper produces a good performance.

7. Conclusion

Three kinds of optimal filters have been developed in this paper. The first one is the optimal Kalman filtering, which requires high computation and does not converge to a steady state in general. The second one is an alternative Markov jump linear filter which just depends on the present value of the Markov chain, and thus requires less pre-computed gains. Under natural assumptions, this filter is convergent to a constant-gain filter which is viewed as the third designed filter. The existence condition on stabilizing solutions to the constant-gain filter is discussed.

Acknowledgments

The authors gratefully acknowledge the Associate Editor and the anonymous reviewers for their comments which have helped to improve the paper greatly.

Appendix A. Proof of Lemma 12

Proof. From the hypothesis $(\Lambda, \bar{H}, \bar{A})$ is detectable, we get from Theorem A.10 in Costa et al. (2005) that there exists a maximal solution $\bar{Y}(\bar{r}) \in \mathcal{H}^{n+}$ subject to (56). Denote $\bar{L}_i(\bar{r}) = L_i(\bar{Y}(\bar{r}))$ $(i = 0, \dots, \bar{r})$, $\bar{L}_{\bar{r}+1}(\bar{r}) = 0$, $\bar{A}_i(\bar{r}) = A_i + \bar{L}_i(\bar{r})H_i (i = 0, \dots, \bar{r}+1)$, and for any $Y(\bar{r}) \in \mathcal{H}^{n+}$ satisfying $f^{(\bar{r})}(Y(\bar{r})) \geq Y(\bar{r})$, denote $L_i(\bar{r}) = L_i(Y(\bar{r}))$ $(i = 0, \dots, \bar{r})$, $L_{\bar{r}+1}(\bar{r}) = 0$, $R_i(\bar{r}) = H_iY_i(\bar{r})H_i' + \pi_iG_iG_i'$ $(i = 0, \dots, \bar{r}+1)$. Then we can find $\delta > 0$ such that

$$\begin{aligned}(\bar{Y}_j(\bar{r}) - Y_j(\bar{r})) &- \sum_{i=0}^{\bar{r}+1} \lambda_{ij}\bar{A}_i(\bar{r})(\bar{Y}_i(\bar{r}) - Y_i(\bar{r}))\bar{A}_i(\bar{r})' \\ &\geq \delta((f_j^{(\bar{r})}(Y(\bar{r})) - Y_j(\bar{r})) + \sum_{i=0}^{\bar{r}+1} \lambda_{ij}(\bar{L}_i(\bar{r}) - L_i(\bar{r})) \\ &\quad \times (\bar{L}_i(\bar{r}) - L_i(\bar{r}))'). \end{aligned} \tag{A.1}$$

On the other hand, from the stabilizability of $(\bar{A}, \bar{C}, \Lambda)$, we can find $K(\bar{r}) = (K_0(\bar{r}), \dots, K_{\bar{r}+1}(\bar{r}))$ such that $r_\sigma(\mathcal{T}) < 1$, when $\Gamma_i = A_i + \pi_i^{\frac{1}{2}}C_iK_i(\bar{r}), i = 0, \dots, \bar{r}+1$.

Define

$$\Xi_i = \begin{bmatrix} \pi_i^{\frac{1}{2}}C_i & \pi_i^{\frac{1}{2}}L_i(\bar{r}) \end{bmatrix}, \quad i = 0, \dots, \bar{r}+1,$$

$$\bar{\Xi}_i = \begin{bmatrix} 0 & \pi_i^{\frac{1}{2}}\bar{L}_i(\bar{r}) \end{bmatrix}, \quad i = 0, \dots, \bar{r}+1,$$

$$\bar{K}_i(\bar{r}) = \begin{bmatrix} K_i(\bar{r}) \\ \pi_i^{-\frac{1}{2}}H_i \end{bmatrix}, \quad i = 0, 1, \dots, \bar{r},$$

$$\bar{K}_i(\bar{r}) = \begin{bmatrix} K_i(\bar{r}) \\ 0 \end{bmatrix}, \quad i = \bar{r}+1.$$

Then

$$A_i + \Xi_i\bar{K}_i(\bar{r}) = (A_i + L_i(\bar{r})H_i) + \pi_i^{\frac{1}{2}}C_iK_i(\bar{r}). \tag{A.2}$$

$$A_i + \bar{\Xi}_i\bar{K}_i(\bar{r}) = A_i + \bar{L}_i(\bar{r})H_i = \bar{A}_i(\bar{r}), \tag{A.3}$$

and

$$\begin{aligned}\sum_{i=0}^{\bar{r}+1} \lambda_{ij}(\Xi_i - \bar{\Xi}_i)(\Xi_i - \bar{\Xi}_i)' &= (f_j^{(\bar{r})}(Y(\bar{r})) - Y_j(\bar{r})) \\ &+ \sum_{i=0}^{\bar{r}+1} \lambda_{ij}(L_i(\bar{r}) - \bar{L}_i(\bar{r}))(L_i(\bar{r}) - \bar{L}_i(\bar{r}))' \end{aligned} \tag{A.4}$$

for $Y(\bar{r}) = 0$. It follows from (A.2) that $(\bar{A}, \Xi, \Lambda)$ is stabilizable when $Y(\bar{r}) = 0$. In view of (A.1) and (A.4), we have that

$$\begin{aligned}\delta &\left\{ \sum_{i=0}^{\bar{r}+1} \lambda_{ij}(\Xi_i - \bar{\Xi}_i)(\Xi_i - \bar{\Xi}_i)' \right\} \\ &\leq (\bar{Y}_j(\bar{r}) - Y_j(\bar{r})) - \sum_{i=0}^{\bar{r}+1} \lambda_{ij}(A_i + \bar{\Xi}\bar{K}_i(\bar{r})) \\ &\quad \times (\bar{Y}_i(\bar{r}) - Y_i(\bar{r}))(A_i + \bar{\Xi}\bar{K}_i(\bar{r}))' \end{aligned} \tag{A.5}$$

for $Y(\bar{r}) = 0$. Under the stabilizability of $(\bar{A}, \Xi, \Lambda)$ and the condition of (A.5), and following a similar derivation procedure as Lemma 8 in Costa et al. (2005), we can conclude that $(\bar{A}, \bar{\Xi}, \Lambda)$ is stabilizable. Further, it follows from (A.3) that $\bar{L}(\bar{r})$ stabilizes $(\bar{A}, \bar{H}, \Lambda)$, so that $\bar{Y}(\bar{r})$ is the stabilizing solution to (56). From Lemma A.14 in Costa et al. (2005), we know that $\bar{Y}(\bar{r}) \in \mathcal{H}^{(n+)}$ is the unique solution for the CARE (56). ■

Appendix B. Proof of Lemma 13

Proof. First, we know from Lemma 12 that there exists a unique set of stabilizing solutions $\hat{Y}(l), l = \bar{r}, \dots, 0$ subject to the CAREs (48) and (49), respectively.

Next, we will show the convergence of the CDREs (44) and (47). It is noted that if the Riccati equation (44) for $Y(s, \bar{r})(0 \leq s \leq k - \bar{r})$ converge, the future finite iterations (47) for $Y(k - \bar{r} + 1, \bar{r} - 1), \dots, Y(k, 1)$ are convergent as well. Thus we just need to analyze the asymptotic convergence of the Riccati equation (44). Based on the definitions of $f^{(\bar{r})}(.)$, and A_i, C_i, H_i, and G_i, the CDRE (44) can be rewritten as

$$Y_j(s+1, \bar{r}) = f_j^{(\bar{r})}(Y(s, \bar{r}), \pi(s)), \quad 0 \leq s \leq k - \bar{r}. \tag{B.1}$$

Further, we define the upper bound function $U(s, \bar{r}) \triangleq (U_0(s, \bar{r}), \dots, U_{\bar{r}+1}(s, \bar{r}))$ as

$$\begin{aligned}U_j(s+1, \bar{r}) = \sum_{i=0}^{\bar{r}+1} &\lambda_{ij}\{(A_i + L_i(\hat{Y}(\bar{r}))H_i)U_i(s, \bar{r}) \\ &\times (A_i + L_i(\hat{Y}(\bar{r}))H_i)' + \pi_i(s)(C_iC_i' + L_i(\hat{Y}(\bar{r})) \\ &\times G_iG_i'L_i(\hat{Y}(\bar{r}))')\}, \quad 0 \leq s \leq k - \bar{r}, \end{aligned} \tag{B.2}$$

where $U(0, \bar{r}) = Y(0, \bar{r})$, $L_i(\hat{Y}(\bar{r}))$ is as in (55).

And for $i = 0, 1, \dots, \bar{r}+1$, $\alpha_i(k) = \inf_{s \in R^+}\{\pi_i(k+s)\} > 0$, we define the lower bound $V(s, \bar{r}) \triangleq (V_0(s, \bar{r}), \dots, V_{\bar{r}+1}(s, \bar{r}))$ as

$$V(s+1, \bar{r}) = f^{(\bar{r})}(V(s, \bar{r}), \alpha(s)), \quad 0 \leq s \leq \bar{r}. \tag{B.3}$$

with $V(0, \bar{r}) = 0$. It can be shown by induction on s that

$$0 \leq V(s, \bar{r}) \leq V(s+1, \bar{r}), \quad 0 \leq s \leq k - \bar{r}, \tag{B.4}$$

$$V(s, \bar{r}) \leq Y(s, \bar{r}) \leq U(s, \bar{r}), \quad 0 \leq s \leq k - \bar{r}. \tag{B.5}$$

From Proposition 3.36 in Costa et al. (2005), and the fact that $L(\hat{Y}(\bar{r}))$ stabilizes $(\Lambda, \bar{H}, \bar{A})$, we have that $U(s, \bar{r})$ converges to a $U(\bar{r}) \in \mathcal{H}^{n+}$, which is the unique solution of the matrical equations in $Y(\bar{r}) = (Y_0(\bar{r}), \dots, Y_{\bar{r}+1}(\bar{r})) \in \mathcal{H}^{n+}$

$$\begin{aligned}Y_j(\bar{r}) = \sum_{i=0}^{\bar{r}+1} &\lambda_{ij}\{(A_i + L_i(\hat{Y}(\bar{r}))H_i)Y_i(\bar{r})(A_i + L_i(\hat{Y}(\bar{r}))H_i)' \\ &+ \pi_i(C_iC_i' + L_i(\hat{Y}(\bar{r}))G_iG_i'L_i(\hat{Y}(\bar{r}))')\}. \end{aligned} \tag{B.6}$$

Since $\hat{Y}(\bar{r})$ also satisfies the above equation, we have from the uniqueness of (B.6) that $\hat{Y}(\bar{r}) = U(\bar{r})$.

From (B.4), (B.5), the sequence $V(s, \bar{r})(0 \leq s \leq k - \bar{r})$ is monotine increasing, and bounded above by $\hat{Y}(\bar{r})$. Thus there exists $V(\bar{r}) = (V_0(\bar{r}), \dots, V_{\bar{r}}(\bar{r})) \in \mathcal{H}^{n+}$ such that $V(s, \bar{r})$ converges to $V(\bar{r})$ and $V(\bar{r}) = f^{(\bar{r})}(V(\bar{r}))$. From the uniqueness of the solutions of (48), it follows that $V(\bar{r}) = \hat{Y}(\bar{r})$. This and (B.5) show that $Y(s, \bar{r})$ converges to $\hat{Y}(\bar{r})$. ■

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