Automatica 89 (2018) 144–150

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Consensus control for a network of high order continuous-time agents with communication delays☆

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ARTICLE INFO

Article history:
Received 14 April 2016
Received in revised form 26 October 2017
Accepted 15 November 2017
Available online 26 December 2017

Keywords:
Multi-agent systems
Consensus control
Communication delays
Delay bound
Convergence rate

ABSTRACT

This paper is concerned with the consensus control problem for multi-agent systems with agents characterized by high-order linear continuous-time systems subject to communication delays between neighbouring nodes in the network. A new consensus protocol is proposed. It requires communication between neighbouring agents only at certain sampling points, rather than at all times. It is also unique in the sense that it is nonlinear in the continuous-time domain but linear when the agents are viewed in the sampled-data domain. Under the proposed consensus protocol, marginally stable multi-agent systems can reach consensus for any large delay. Unstable multi-agent systems achieve consensus when the time delay is within a certain range. Moreover, in the single-input case, we give an optimal control gain which yields the fastest consensus speed. The proposed technique is expected to pave a new way for new theoretical studies on network properties required for consensus control.

1. Introduction

Consensus is a process that a group of agents with different initial states reach an agreement by local communication between agents. As a distributed cooperative control of multi-agent systems, consensus control is closely related to problems such as flocking (Tanner, Jadbabaie, & Pappas, 2007), formation control (Fax & Murray, 2004), and network congestion control (Paganini, Doyle, & Low, 2001). Consensus algorithms also find wide applications in many disciplines, including smart grid (Mou, Xing, Lin, & Fu, 2015), sensor networks (Kar & Moura, 2010) and distributed parameter estimation (Kar, Moura, & Ramanan, 2012).

Consensus control problems have attracted a lot of attention, see, e.g., Ma and Zhang (2010) and You and Xie (2011a, b). Reference Ma & Zhang (2010) considers the consensus control problem for the following multi-agent system

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, \dots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the state and the control input of the ith agent, respectively; $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices. The consensus protocol is given by

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)),\ \ i = 1, \dots, N, \tag{2}$$

where $\{a_{ij},\ i,\ j = 1, \dots, N\}$ are elements of the adjacency matrix and $K \in \mathbb{R}^{m\times n}$ is a gain matrix. It is shown in Ma and Zhang (2010) that there exists a gain K such that the multi-agent system (1) reaches consensus under the protocol (2) if and only if (A, B) is stabilizable and the network topology has a spanning tree. In this case, such a K can be constructed by a standard Riccati equation. It is also pointed out in Ma and Zhang (2010) that the above results fail to have counterparts in discrete-time linear multi-agent systems. A necessary and sufficient consensusability condition for discrete-time multi-agent systems with a single input is presented in You and Xie (2011b). Besides a controllability requirement, this condition contains an inequality involving unstable eigenvalues of A and the ratio λ_2/λ_N (where λ_2 and λ_N are the smallest and the largest non-zero eigenvalues of the Laplacian matrix for the

☆ This work was supported by the Taishan Scholar Construction Engineering by Shandong Government (Grant No. 61573221), the National Natural Science Foundation of China (Grant Nos. 61120106011, 61633014, 61403235, 61703250), and the Natural Science Foundation of Shandong Government (Grant Nos. ZR2017BF002, ZR2017BA029, ZR2015FM016). The material in this paper was partially presented at 2016 American Control Conference,July 6–8, 2016, Boston, MA, USA. This paper was recommended for publication in revised form by Associate Editor Wei Ren under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2017.12.006
0005-1098/

network topology, respectively). The control gain solving consensus is given by modified Riccati inequalities.

Aforementioned works all deal with consensus problems without delay. When delays happen in the information transmission between neighbours, a commonly used consensus protocol is

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}(x_j(t-\tau) - x_i(t-\tau)). \quad (3)$$

Most works in the literature study consensus control problems with time delay in the following framework: for a fixed K, seek an upper bound $\bar{\tau}$ for the delay such that consensus can always be achieved under protocol (3) for any $\tau \in [0, \bar{\tau})$, see Cepeda-Gomez (2015), Munz, Papachristodoulou, and Allgower (2010), Olfati-Saber and Murray (2004) and Xu, Zhang, and Xie (2013). For example, Olfati-Saber and Murray (2004) considers integrator dynamics and obtains an exact delay bound ('exact' means that the bound is necessary and sufficient) for the protocol (3) with $K = 1$ by analysing the roots of certain characteristic equation. Cepeda-Gomez (2015) investigates high-order multi-agent systems and characterizes the exact delay bound for general gains by using the cluster treatment of characteristic roots paradigm. Departing from these works, Li and Fu (2016), Wang, Zhang, and Fu (2015), and Zhou and Lin (2014) discuss consensus control problems with time delay in another framework. They design K to be a function of delay τ, denoted by $K(\tau)$, such that protocol (3) with $K = K(\tau)$ renders system consensus when the delay is equal to τ. It is not a concern whether this control gain works for other values of delay. Zhou and Lin (2014) focuses on system (1) where all the eigenvalues of A lie on the imaginary axis. It is shown that consensus can be achieved for arbitrarily large delay. Allowing A to have eigenvalues on the open right-half plane, Wang et al. (2015) give a delay bound below which consensus can be achieved. However, this bound is presented using the maximal value of a function, which cannot be solved analytically.

This paper is concerned with the consensus problem for the multi-agent system (1) with communication delays. Different from (3), a new consensus protocol is proposed. It requires relative state and input signals between neighbouring agents only at certain sampling points, rather than all the time. This means that only a limited amount of communication is needed between neighbouring agents. Our consensus control gain is delay dependent like Wang et al. (2015) and Zhou and Lin (2014). The motivation for designing such a gain is that we hope to deal with larger delay than using a delay-independent gain as in Xu et al. (2013). The method of constructing consensus control gains is using modified Riccati inequalities and is from reference You & Xie (2011b). Our approach is as follows. First, the consensus problem for discrete-time multi-agent systems with multi-step communication delay is studied. It is transformed to a delay-free consensus problem by the reduction technique (Artstein, 1982). Then, this result is applied to the problem under consideration via the sampled-data models. The contribution of this paper includes two aspects. First, for marginally stable agents (here, "marginally stable" means that all the eigenvalues of the system are located on the closed left-half plane), consensus is guaranteed for any large delay. For unstable agents, consensus is achieved when the delay is below a bound which depends on the network topology and the agent dynamics. This bound is shown to be larger than that in Wang et al. (2015) and Xu et al. (2013) in some cases. Secondly, the influence of consensus control gains on the consensus speed is investigated and an optimal gain yielding the fastest consensus speed is provided in the single-input case.

The rest of the paper is organized as follows. The problem formulation is given in Section 2. The consensus control problem for discrete-time multi-agent systems with multi-step delay is discussed in Section 3. The problem under consideration is solved in Section 4. Performance analysis of the proposed consensus protocol is given in Section 5. Numerical examples are provided in Section 6. Conclusions are presented in Section 7. A useful proposition is given in the Appendix.

Notations: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are the sets of n-order column vectors and $n \times m$-order matrices with real elements, respectively. For a complex number c, $\mathrm{Re}(c)$, $\mathrm{Im}(c)$, $|c|$, and $\bar{c}$ stand for its real part, imaginary part, modular, and conjugate, respectively. For a matrix $X \in \mathbb{R}^{n\times m}$, X' is its transpose. For a matrix $X \in \mathbb{R}^{n\times n}$, $\rho(X)$, $\mathrm{tr}(X)$, and $\lambda_j(X)$, $j = 1, \ldots, n$, denote its spectral radius, trace and eigenvalues, respectively. For a symmetric matrix X, $X > 0$ means that it is positive definite. For a positive integer N, $\bar{N}$ represents the set $\{1, \ldots, N\}$; e^X represents the exponential of a matrix.

2. Problem formulation

Let the directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ denote the communication topology between multi-agents with the set of vertices $\mathcal{V} = \{1, 2, \ldots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The ith vertex represents the ith agent and the edge $(i, j) \in \mathcal{E}$ denotes that the agent j receives information from the agent i. Self-edges are not allowed. The set of neighbours of the ith agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements and $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. The in-degree of the ith vertex and the in-degree matrix are denoted by $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $\mathcal{D} = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$, respectively. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Note that $a_{ij} = a_{ji}$, $\forall i, j \in \mathcal{V}$, if and only if $\mathcal{G}$ is an undirected graph (You & Xie, 2011b). Obviously, for an undirected graph, $\mathcal{L}$ is a symmetric, positive semi-definite matrix and all its eigenvalues λ_i, $i \in \bar{N}$, are non-negative. For a connected graph having a spanning tree, we have $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$.

The dynamics of each agent is given by (1). Suppose the communication delay from agent j to agent i is $s_{ij}\tau$ where s_{ij} is a positive integer and τ is positive and constant. The maximal value of s_{ij} is $\bar{s}$, i.e., $\max_{i,j\in\bar{N}}\{s_{ij}\} = \bar{s}$. In this context, the available information for the controller $u_i(t)$ is $\{x_j(s), u_j(s) : s \leq t - s_{ij}\tau,\ j \in \mathcal{N}_i\}$ and $\{x_i(s), u_i(s) : s \leq t - s_{ii}\tau\}$. The aim is to design the controller $u_i(t)$ for each agent i using the above available information such that the multi-agent system (1) achieves consensus.

Definition 1. The agents in the network achieve consensus if $\lim_{t\to\infty} x_j(t) - x_i(t) = 0$, $\forall i, j \in \bar{N}$, for any initial value $x_i(0)$.

The following assumptions are made in this paper.

Assumption 1. The network topology $\mathcal{G}$ is an undirected connected graph.

Assumption 2. All the eigenvalues of A lie in the closed right-half plane.

Assumption 3. (A, B) is controllable and B has full column rank.

Remark 1. If some eigenvalues of A lie in the open left-half plane, it is a standard practice to decompose the system (1) into two subsystems, one asymptotically stable which requires no consensus control action, and one with eigenvalues in the closed right-half plane, which is considered under Assumption 2. Thus, Assumption 2 does not lose generality.

Remark 2. The assumption that the communication delays are multiples of a positive number τ and known by involved agents can be justified for commonly used medium access control (MAC) protocols where time stamping is used for data transmission. Namely, time stamping allows each agent to easily determine the amount of transmission delay for each received signal, and if the delay is not an integer multiple of τ, the received signal can be held for a fractional amount of extra delay to make the delay an integer multiple.

3. Consensus for discrete-time agents with multi-step delay

In this section, we consider the discrete-time multi-agent system

$$x_i(k+1) = \tilde{A}x_i(k) + \tilde{B}u_i(k),\ i \in \bar{N}, \tag{4}$$

where $x_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}^m$ are the state and the control input of agent i, respectively; $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{B} \in \mathbb{R}^{n\times m}$ are constant matrices. The network topology is given by the undirected connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. The communication between neighbours is with delay $d \geq 1$. The problem is to design the controller $u_i(k)$ for each agent i using the information of $\{x_j(s)-x_i(s), u_j(s), u_i(s)\}$ where $s \leq k-d$ and $j \in \mathcal{N}_i$ such that the system (4) achieves consensus. This problem will be converted into a consensus control problem without delay using the reduction technique (Artstein, 1982).

Lemma 1. *Suppose that*

(1) Every eigenvalue of $\tilde{A}$ lies on or outside the unit circle and it is not a root of the polynomial $\sum_{i=0}^{d-1} z^i$;

(2) $(\tilde{A}^d, \tilde{B})$ is controllable and $\tilde{B}$ has full column rank;

(3) $\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} < \frac{1}{\prod_{j=1}^n |\lambda_j(\tilde{A})|^d}$.

Then the control protocol

$$\begin{aligned} &u_i(kd+d) = \ldots = u_i(kd+2d-1) \\ &= K\sum_{j=1}^N a_{ij}[x_j(kd) - x_i(kd) \\ &\quad + \sum_{m=1}^d \tilde{A}^{-m}\tilde{B}u_j(kd) - \sum_{m=1}^d \tilde{A}^{-m}\tilde{B}u_i(kd)] \end{aligned} \tag{5}$$

renders system (4) consensus if the gain matrix K satisfies $\rho(\tilde{A}^d - \lambda_i\sum_{m=1}^d \tilde{A}^{-m}\tilde{B}K) < 1, i = 2, \ldots, N$. One of such K is designed in the following way: Select a δ such that $\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} \leq \delta < \frac{1}{\prod_i |\lambda_i(\tilde{A})|^d}$; find a positive definite solution Q to the inequality

$$Q - (\tilde{A}^d)'Q\tilde{A}^d + (1-\delta^2)(\tilde{A}^d)'Q\tilde{B}(\tilde{B}'Q\tilde{B})^{-1}\tilde{B}'Q\tilde{A}^d > 0;$$

set $K = 2(\lambda_2+\lambda_N)^{-1}(\tilde{B}'Q\tilde{B})^{-1}\tilde{B}'Q(\sum_{i=0}^{d-1}\tilde{A}^i)^{-1}\tilde{A}^{2d}$.

Proof. Consider the system (4) with the control protocol (5). Define new states $z_i(k),\ i \in \bar{N}$, as $z_i(k) \doteq x_i(kd) + \sum_{m=1}^d \tilde{A}^{-m}\tilde{B}u_i(kd+m-1)$. Due to $u_i(kd+d) = u_i(kd+d+1) = \cdots = u_i(kd+2d-1)$, the above equation can be rewritten as

$$z_i(k) = x_i(kd) + \sum_{m=1}^d \tilde{A}^{-m}\tilde{B}u_i(kd), \tag{6}$$

which means that the control protocol (5) is

$$u_i(kd+d) = K\sum_{j=1}^N a_{ij}[z_j(k) - z_i(k)]. \tag{7}$$

Direct computation yields

$$z_i(k+1) = \tilde{A}^d z_i(k) + \sum_{m=1}^d \tilde{A}^{-m}\tilde{B}u_i(kd+d). \tag{8}$$

Note that the above multi-agent system with the protocol (7) is delay free and its consensus problem is solved in You and Xie (2011a, b). By applying the results in these references, this lemma can be proven. This ends the proof. □

4. Consensus for continuous-time agents with multiple time delays

This section will present one of the main results of this paper, i.e., the designing of control protocols for the consensus control problem stated in Section 2. Note that notations $\tilde{A}$, $\tilde{B}$, and d in the previous section will be specified as $\tilde{A} = e^{A\tau}$, $\tilde{B} = \int_0^\tau e^{As}dsB$, and $d = \bar{s}$, respectively. Before giving the result, we take an extra assumption about the maximal communication delay.

Assumption 4. Any two distinct eigenvalues of A, denoted by μ_1 and μ_2, satisfy $\mathrm{Im}(\mu_1\bar{s}\tau - \mu_2\bar{s}\tau) \neq 2q\pi, \forall q = \pm1, \pm2, \ldots,$ whenever $\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2)$.

Theorem 1. *Under Assumptions 1–4, if the maximal time delay $\bar{s}\tau$ satisfies*

$$\bar{s}\tau < \frac{1}{tr(A)}\ln(\frac{\lambda_N+\lambda_2}{\lambda_N-\lambda_2}), \tag{9}$$

then the multi-agent system (1) can reach consensus under the following control protocol

$$\begin{aligned} &u_i(t),\ t \in [k\bar{s}\tau, k\bar{s}\tau+\bar{s}\tau), \\ &= K\sum_{j=1}^N a_{ij}[e^{A\tau(s_{ij}-\bar{s})}x_j(k\bar{s}\tau - s_{ij}\tau) - e^{A\tau(s_{ii}-\bar{s})} \\ &\times x_i(k\bar{s}\tau - s_{ii}\tau) + \sum_{m=\bar{s}-s_{ij}+1}^{\bar{s}} e^{-A\tau m}\tilde{B}u_j(k\bar{s}\tau \\ &- s_{ij}\tau) - \sum_{m=\bar{s}-s_{ii}+1}^{\bar{s}} e^{-A\tau m}\tilde{B}u_i(k\bar{s}\tau - s_{ii}\tau)], \end{aligned} \tag{10}$$

whenever K satisfies $\rho(\tilde{A}^d - \lambda_i\sum_{m=1}^d \tilde{A}^{-m}\tilde{B}K) < 1, i = 2, \ldots, N$. One of such K can be designed as follows: choose a δ such that $\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} \leq \delta < e^{-tr(A)\bar{s}\tau}$; find a positive definite solution $Q > 0$ to the following modified algebraic Riccati inequality

$$Q - e^{A'\bar{s}\tau}Qe^{A\bar{s}\tau} + (1-\delta^2)e^{A'\bar{s}\tau}QB(B'QB)^{-1}B'Qe^{A\bar{s}\tau} > 0;$$

set $K = 2(\lambda_2+\lambda_N)^{-1}(B'QB)^{-1}B'QV^{-1}e^{A\bar{s}\tau}$ with $V = \sum_{m=1}^{\bar{s}} e^{-Am\tau}\int_0^\tau e^{As}ds$.

Proof. Step 1: The conditions of Lemma 1 will be verified.

(1) Any eigenvalue of $\tilde{A}$ lies on or outside the unit circle and it is not a root of the polynomial $\sum_{i=0}^{d-1} z^i$.

Any eigenvalue of $\tilde{A}$ can be written as $e^{\mu\tau}$, where μ is an eigenvalue of A, and $|e^{\mu\tau}| = e^{\mathrm{Re}(\mu)\tau}$. Under Assumption 2, we have $\mathrm{Re}(\mu) \geq 0$ and thus $|e^{\mu\tau}| \geq 1$. Now we show $e^{\mu\tau}$ is not a root of the polynomial $\sum_{i=0}^{d-1} z^i$. If $\mu = 0$, then $e^{\mu\tau} = 1$, which is obviously not a root of the polynomial $\sum_{i=0}^{d-1} z^i$. Suppose $\mu \neq 0$. If $\sum_{i=0}^{d-1}(e^{\mu\tau})^i = 0$, then $\sum_{i=0}^{d-1}(e^{\mu\tau})^i(e^{\mu\tau}-1) = e^{\mu\tau d} - 1 = 0$, which implies $\mathrm{Re}(\mu) = 0$, $\mathrm{Im}(\mu) = \frac{2k\pi}{\tau d}$, $k = \pm1, \pm2, \ldots$. Thus $\mu = \sqrt{-1}\frac{2k\pi}{\tau d}$. As two different eigenvalues of A, μ and $-\mu$ have the same real part, and satisfy $\mathrm{Im}(\mu d\tau - (-\mu d\tau)) = 4k\pi$, which is contradictable with Assumption 4.

(2) $(\tilde{A}^d, \tilde{B})$ is controllable and $\tilde{B}$ has full column rank.

First, it will be proven that matrices $\int_0^{d\tau} e^{As}ds$ and $\int_0^{\tau} e^{As}ds$ are both nonsingular. Denote the Jordan canonical form of A by $J \doteq T^{-1}AT$. Then

$$\int_0^{d\tau} e^{As}ds = T\int_0^{d\tau} e^{Js}dsT^{-1}. \tag{11}$$

Since e^{Js} is an upper triangular matrix with diagonal elements $e^{\lambda_j(A)s}$, $j = 1, \ldots, n$, $\int_0^{d\tau} e^{Js}ds$ is an upper triangular matrix with diagonal elements $\int_0^{d\tau} e^{\lambda_j(A)s}ds$, $j = 1, \ldots, n$. It can be observed that

$$\int_0^{d\tau} e^{\lambda_j(A)s}ds \neq 0,\ j = 1, \ldots, n. \tag{12}$$

Actually, if $\lambda_j(A) = 0$, $\int_0^{d\tau} e^{\lambda_j(A)s}ds = d\tau \neq 0$. If $\lambda_j(A) \neq 0$, there holds $\int_0^{d\tau} e^{\lambda_j(A)s}ds = (e^{\lambda_j(A)d\tau} - 1)/\lambda_j(A)$. Suppose $\int_0^{d\tau} e^{\lambda_j(A)s}ds = 0$, then $e^{\lambda_j(A)d\tau} = 1$. In a manner similar to that of (1), a contradiction with Assumption 4 is derived. So (12) is true and thus $\int_0^{d\tau} e^{Js}ds$ is nonsingular. Similarly, the invertibility of the matrix $\int_0^{\tau} e^{As}ds$ can be shown. Secondly, the discretization system of the continuous-time system with the sample period equal to $d\tau$ is $(\tilde{A}^d, \int_0^{d\tau} e^{As}dsB)$. According to Michael (1970), Assumption 4 ensures that this discretization system is controllable. Since $\int_0^{d\tau} e^{As}ds$ is invertible and is commutable with A, $(\tilde{A}^d, B)$ is also controllable. Similarly, $(\tilde{A}^d, \tilde{B})$ is controllable. Because B has full column rank, $\tilde{B} = \int_0^{\tau} e^{As}dsB$ also has full column rank.

(3) The inequality $\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} < \frac{1}{\prod_{j=1}^n |\lambda_j(\tilde{A})|^d}$ is (9).

Step 2: We will show that the system (1) and the controller (10) correspond to the system (4) and the controller (5), respectively, by discretization. Note that $u_i(t)$ is constant at the interval $[k\tau, k\tau+\tau)$. By defining $\tilde{u}_i(k) \doteq u_i(k\tau)$, and $\tilde{x}_i(k) \doteq x_i(k\tau)$, system (1) becomes

$$\tilde{x}_i(k+1) = \tilde{A}\tilde{x}_i(k) + \tilde{B}\tilde{u}_i(k), \tag{13}$$

which is just the system (4). Since $u_i(t)$ is constant at the interval $[kd\tau, kd\tau + d\tau)$, it is known that $\tilde{u}_i(kd+d) = \tilde{u}_i(kd+d+1) = \cdots = \tilde{u}_i(kd+2d-1)$. Together with (13), it yields that

$$\tilde{x}_i(kd) = \tilde{A}^{v-d}\tilde{x}_i(kd+d-v) - \sum_{m=1}^{d-v}\tilde{A}^{-m}\tilde{B}\tilde{u}_i(kd), \tag{14}$$

where v is any integer in $[1, d]$. Define assistant variables

$$z_i(k) \doteq \tilde{x}_i(kd) + \sum_{m=1}^{d}\tilde{A}^{-m}\tilde{B}\tilde{u}_i(kd). \tag{15}$$

Substituting (14) into (15), it follows that

$$z_i(k) = \tilde{A}^{v-d}\tilde{x}_i(kd+d-v) + \sum_{m=d-v+1}^{d}\tilde{A}^{-m}\tilde{B}\tilde{u}_i(kd). \tag{16}$$

(10) implies that

$$\begin{aligned}\tilde{u}_i(kd+d) = K\sum_{j=1}^{N} a_{ij}[&\tilde{A}^{s_{ij}-d}\tilde{x}_j(kd+d-s_{ij}) \\ &+ \sum_{m=d-s_{ij}+1}^{d}\tilde{A}^{-m}\tilde{B}\tilde{u}_j(kd+d-s_{ij}) \\ &- \tilde{A}^{s_{ii}-d}\tilde{x}_i(kd+d-s_{ii}) \\ &- \sum_{m=d-s_{ii}+1}^{d}\tilde{A}^{-m}\tilde{B}\tilde{u}_i(kd+d-s_{ii})].\end{aligned}$$

Note that (16) is valid for any $v \in [1, d]$. So the above equation can be rewritten as

$$\tilde{u}_i(kd+d) = K\sum_{j=1}^{N} a_{ij}[z_j(k) - z_i(k)]. \tag{17}$$

Together with (15), it yields

$$\begin{aligned}\tilde{u}_i(kd+d) = K\sum_{j=1}^{N} a_{ij}[&\tilde{x}_j(kd) - \tilde{x}_i(kd) \\ &+ \sum_{m=1}^{d}\tilde{A}^{-m}\tilde{B}(\tilde{u}_j(kd) - \tilde{u}_i(kd))].\end{aligned} \tag{18}$$

Also, the feedback matrix K above is the same as the one in the controller (5). Therefore, (10) is just (5). Thus, by applying Lemma 1, we conclude that

$$\lim_{k\to\infty}[\tilde{x}_i(k) - \tilde{x}_j(k)] = 0,\quad \lim_{k\to\infty}\tilde{u}_i(k) = 0. \tag{19}$$

Step 3: It remains to show that $\lim_{t\to\infty}[x_i(t) - x_j(t)] = 0,\ \forall i, j \in \bar{N}$. For any $t \geq 0$, there exists a unique integer $w(t)$ such that $t \in [w(t)\tau, w(t)\tau + \tau)$. Denote $g(t) \doteq t - w(t)\tau$. From (1), it follows that

$$x_i(t) = e^{Ag(t)}\tilde{x}_i(w(t)) + \int_0^{g(t)} e^{A\sigma}d\sigma B\tilde{u}_i(w(t)),$$

which means

$$\begin{aligned}x_i(t) - x_j(t) = &e^{Ag(t)}[\tilde{x}_i(w(t)) - \tilde{x}_j(w(t))] \\ &+ \int_0^{g(t)} e^{A\sigma}d\sigma B[\tilde{u}_i(w(t)) - \tilde{u}_j(w(t))].\end{aligned} \tag{20}$$

Owing to $g(t) \in [0, \tau)$, $e^{Ag(t)}$ and $\int_0^{g(t)} e^{A\sigma}d\sigma$ are bounded. Together with (19), it can be derived that $\lim_{t\to\infty}[x_i(t) - x_j(t)] = 0$. This ends the proof. □

Remark 3. Comments on the condition (9) are in order. For marginally stable agents, A has purely imaginary eigenvalues, which means that the right hand side of (9) is $+\infty$. That is, consensus can always be achieved for any time delay. If A has strictly unstable eigenvalues, then the right hand side is a finite positive number. In this case, (9) gives a good characterization about the delay bound. More specifically, the delay bound is closely related to the connectivity (λ_2) and synchronizability (λ_N) of the network as well as the growth rate of the agents' states ($\mathrm{tr}(A)$).

5. Performance analysis of consensus protocols

In this section, we will consider a special case that all the communication delays between different pairs of agents are the same, i.e., $s_{ij} = \bar{s} = 1$. According to Theorem 1, the protocol (10) where K satisfies $\rho(\tilde{A} - \lambda_i\tilde{A}^{-1}\tilde{B}K) < 1, i = 2, \ldots, N$ renders the multi-agent system (1) consensus. The convergence speed of the error states $x_i(t) - x_j(t)$, $i, j \in \bar{N}$, determines the consensus speed. Due to the existence of the delayed inputs in (10), it is difficult to analyse the asymptotic behaviour of the error states directly. However, from the proof of Lemma 1 and Theorem 1, it is known that the closed-loop system of (1) under (10) corresponds to a delay-free discrete-time system. Now we will discuss the error states by means of this delay-free discrete-time system. Note that

$$\begin{aligned}x_i(t) - x_j(t) = &e^{Ag(t)}[\tilde{x}_i(k) - \tilde{x}_j(k)] + \int_0^{g(t)} e^{A\sigma}d\sigma B \\ &\times [\tilde{u}_i(k) - \tilde{u}_j(k)],\ t \in [k\tau, k\tau+\tau),\end{aligned}$$

where $g(t) = t - k\tau,\ t \in [k\tau, k\tau + \tau)$, is bounded. From (13), (18), and (6)–(8), it can be derived that

$$\begin{aligned} &\tilde{x}_i(k) - \tilde{x}_j(k) \\ &= z_i(k) - z_j(k) - \tilde{A}^{-1}\tilde{B}K\{\sum_{s=1}^{N} a_{is}[z_s(k-1) \\ &-z_i(k-1)] - \sum_{s=1}^{N} a_{js}[z_s(k-1) - z_j(k-1)]\}, \end{aligned} \tag{21}$$

and $\tilde{u}_i(k) = K\sum_{j=1}^{N} a_{ij}[z_j(k-1) - z_i(k-1)]$, where $z_i(k),\ i \in \bar{N}$, reaches consensus and obeys the dynamics

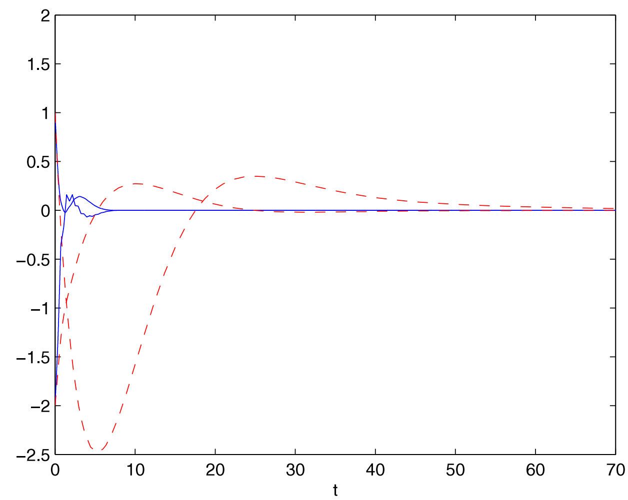
$$z_i(k+1) = \tilde{A}z_i(k) + \tilde{A}^{-1}\tilde{B}K \sum_{j=1}^{N} a_{ij}[z_j(k) - z_i(k)]. \tag{22}$$

According to You and Xie (2011b), the consensus speed of the above dynamics is characterized by the asymptotic convergence factor $r_1(K) \doteq \max_{i=2,\ldots,N} \rho(\tilde{A} - \lambda_i \tilde{A}^{-1}\tilde{B}K)$. Thus the convergence speed of the error states $x_i(t) - x_j(t),\ i, j \in \bar{N}$ is determined by $r_1(K)$. In summary, among all the matrix gains which render system (1) consensus, the optimal one that minimizes the above $r_1(K)$ yields the fastest consensus speed. Together with Proposition 1 in the Appendix, we can obtain the following result.

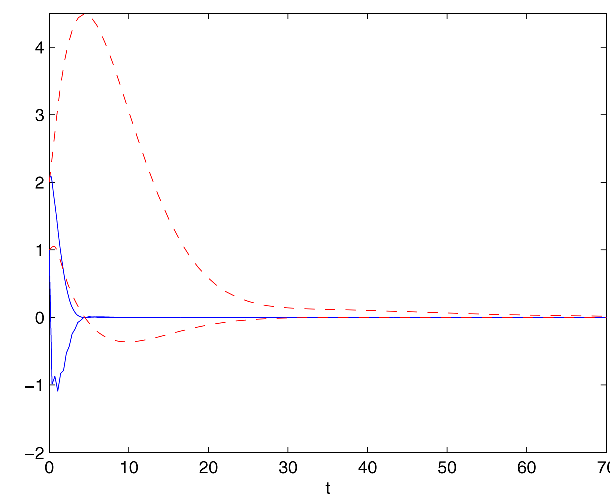
Theorem 2. *Suppose $u_i(t) \in \mathbb{R}$ and the conditions of Theorem 1 hold. Let the characteristic equation of $\tilde{A}$ be $f_{\tilde{A}}(z) = \sum_{i=0}^{n} c_i z^i$ with $c_n = 1$ and Π_1 be the unique nonsingular matrix such that $(\Pi_1^{-1}\tilde{A}\Pi_1, \Pi_1^{-1}\tilde{B})$ is the controllable canonical form of $(\tilde{A}, \tilde{B})$. Denote $R = \left(\frac{e^{tr(A)\tau}(\lambda_N - \lambda_2)}{\lambda_N + \lambda_2}\right)^{\frac{1}{n}}$. Then the control gain $K^\star = \hat{K}\Pi_1^{-1}\tilde{A}$ is optimal in the sense that it yields a faster consensus speed for system (1) under protocol (10) than other gains, where $\hat{K} = \begin{pmatrix} K_1 & \cdots & K_n \end{pmatrix}$ with*

$$K_{i+1} = \begin{cases} \dfrac{c_{n-i}R^{n-2i}(\lambda_N - \lambda_2) - c_i(\lambda_N + \lambda_2)}{2\lambda_2\lambda_N}, & n \text{ is even}, \\ \dfrac{-c_{n-i}R^{n-2i}(\lambda_N - \lambda_2) - c_i(\lambda_N + \lambda_2)}{2\lambda_2\lambda_N}, & n \text{ is odd}, \end{cases}$$

for $i = 0, \ldots, n-1$.



(a) components of $x_2(t) - x_1(t)$.



(b) components of $x_3(t) - x_1(t)$.

Fig. 1. Comparison between our method and that of Wang et al. (2015).

6. Numerical examples

6.1. Single-delay case

Consider the multi-agent system (1) with $A = \begin{pmatrix} 0 & 1 \\ -0.015 & 0.25 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The network topology is given by $2 —— 1 —— 3$. Suppose the communication delay is τ, i.e., $s_{ij} = \bar{s} = 1$. According to (9), the delay bound for consensus under protocol (10) is $\tau < 2.7726$. On the other hand, Wang et al. (2015) provide the following delay bound for consensus under protocol (3):

$$\tau < \frac{1}{2\mathrm{tr}(A)} \max_{q>0} q[1 - \frac{(\lambda_3 + \lambda_2)^2 nq^2 e^{2q}}{4\lambda_3\lambda_2}] = 0.3627.$$

So Wang et al. (2015) cannot deal with delay $\tau \in [0.3627, 2.7726)$. Consider $\tau = 0.36$. First, employing the control protocol (10) with $K^\star = \begin{pmatrix} 1.098 & 2.2269 \end{pmatrix}$ given in Theorem 2, the trajectories of state errors between agent 1 and other agents are shown in the solid line of Fig. 1. Secondly, using protocol (3) with $K = \begin{pmatrix} 0.0207 & 0.5189 \end{pmatrix}$ given in Wang et al. (2015), the corresponding trajectories of state errors are shown in the dashed line of Fig. 1, from which it is seen that our method achieves a faster consensus speed than that of Wang et al. (2015).

6.2. Multiple-delay case

Consider the multi-agent system (1) with $A = 1$ and $B = 1$. The network topology is $2 —— 1 —— 3$. The communication delays between agents 1 and 2 and between agents 1 and 3 are 2τ and τ with $\tau = 0.3$, respectively. The trajectories of the state errors between agent 1 and other agents under the control protocol (10) with feedback gain $K = 2.0192$ given in Theorem 1 are derived in Fig. 2, which verifies that this protocol renders the system (1) consensus.

7. Conclusions

In this paper, we have proposed a novel and simple technique for consensus control of a network of continuous-time agents with communication delays between neighbouring agents. High order dynamic models are allowed for each agent. Our consensus algorithm requires communication between neighbouring agents only at sampling instants, not at all the time. A delay bound below which consensus can be achieved is proposed. For marginally

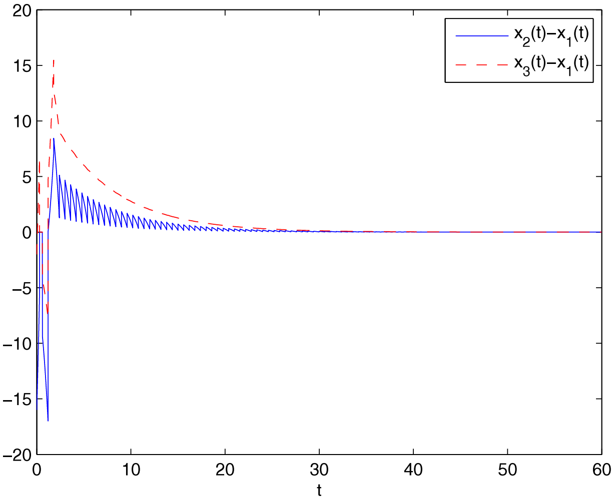


Fig. 2. State errors between agent 1 and other agents.

stable agents, this bound is infinite and for unstable agents, it is a finite number depending on the ratio λ_2/λ_N and the trace of agent system matrix. Consensus control gains are designed in terms of a modified Riccati inequality. In the single-input case, an optimal control gain leading to faster consensus speed than other gains is presented analytically. Future study can be directed to generalize the results to time-varying and directed graphs and time-varying delays.

Appendix. On the fastest consensus speed

To make the following multi-agent system

$$x_i(k+1) = \mathscr{A}x_i(k) + \mathscr{B}u_i(k),\ i \in \bar{N},\ k \geq 0, \tag{A.1}$$

achieve consensus under the protocol

$$u_i(k) = K\sum_{j=1}^{N} a_{ij}[x_j(k) - x_i(k)], \tag{A.2}$$

in the fastest convergence speed, one needs to find the solution to the optimization problem

$$r^\star = \min_K r(K),\ \ r(K) = \max_{i=2,\ldots,N} \rho(\mathscr{A} - \lambda_i \mathscr{B}K). \tag{A.3}$$

This problem is studied in Xiao and Boyd (2004) and You and Xie (2011b) which focus on scalar and two-order multi-agent systems, respectively. To the best of our knowledge, it has not been solved for high-order systems. Here, we will give an analytic solution to this problem for single-input agents. Due to the limitation of paper length, the result is presented below without proof.

Proposition 1. *Suppose all the eigenvalues of $\mathscr{A}$ lie on or outside the unit circle, $\mathscr{B} \in \mathbb{R}^n$, $(\mathscr{A}, \mathscr{B})$ is controllable, and the inequality $\frac{\lambda_N-\lambda_2}{\lambda_N+\lambda_2} < \frac{1}{\prod_{j=1}^n |\lambda_j(\mathscr{A})|}$ holds. Then one of the optimal solutions to the optimization problem (A.3) is given by the following $K^\star$:*

$$K^\star \doteq \hat{K}\Pi^{-1},\ \hat{K} \doteq \begin{pmatrix} K_1 & \cdots & K_n \end{pmatrix},$$

$$K_{i+1} \doteq \frac{c_{n-i}R^{n-2i}(\lambda_N - \lambda_2) - c_i(\lambda_N + \lambda_2)}{2\lambda_2\lambda_N},\ \text{if } c_0 > 0,$$

$$K_{i+1} \doteq \frac{-c_{n-i}R^{n-2i}(\lambda_N - \lambda_2) - c_i(\lambda_N + \lambda_2)}{2\lambda_2\lambda_N},\ \text{if } c_0 < 0.$$

Therein, the matrix Π is the unique nonsingular matrix such that $(\Pi^{-1}\mathscr{A}\Pi, \Pi^{-1}\mathscr{B})$ is the controllable canonical form of $(\mathscr{A}, \mathscr{B})$, i.e.,

$$\Pi^{-1}\mathscr{A}\Pi = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-1} \end{pmatrix},\ \Pi^{-1}\mathscr{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

c_n is equal to 1, and R is given by

$$R = \left(\frac{\prod_{j=1}^n |\lambda_j(\mathscr{A})|(\lambda_N - \lambda_2)}{\lambda_N + \lambda_2} \right)^{\frac{1}{n}}.$$

In addition, the associated optimal asymptotic convergence factor is $r^\star = R$.

References

Artstein, Z. (1982). Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, *AC-27*(4), 869–879.

Cepeda-Gomez, R. (2015). Finding the exact delay bound for consensus of linear multi-agent systems. *International Journal of Systems Science*, *47*(11), 1–9.

Fax, J. A., & Murray, R. M. (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, *49*(9), 1465–1476.

Kar, S., & Moura, J. M. F. (2010). Distributed consensus algorithms in sensor networks: quantized data and random link failures. *IEEE Transactions on Signal Processing*, *58*(3), 1383–1400.

Kar, S., Moura, J. M. F., & Ramanan, K. (2012). Distributed parameter estimation in sensor networks: nonlinear observation models and imperfect communication. *IEEE Transactions on Information Theory*, *58*(6), 3575–3605.

Li, L., & Fu, M. (2016). Consensus control for high order continuous-time agents with communication delays, In *2016 American control conference* (pp. 4458–4463).

Ma, C. Q., & Zhang, J. F. (2010). Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, *55*(5), 1263–1268.

Michael, G. J. (1970). Controllability of linear time-invariant sampled-data systems with nonconstant shaping functions. *IEEE Transactions on Automatic Control*, *15*(5), 594–595.

Mou, Y., Xing, H., Lin, Z., & Fu, M. (2015). Decentralized optimal demand-side-management for PHEV charging in a smart grid. *IEEE Transactions on Smart Grid*, *6*(2), 726–736.

Munz, U., Papachristodoulou, A., & Allgower, F. (2010). Delay robustness in consensus problems. *Automatica*, *46*(8), 1252–1265.

Olfati-Saber, R., & Murray, R. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Paganini, F., Doyle, J., & Low, S. (2001). Scalable laws for stable network congestion control. In *40th IEEE conference on decision and control* (pp. 185–190).

Tanner, H. G., Jadbabaie, A., & Pappas, G. J. (2007). Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control*, *52*(5), 863–868.

Wang, Z., Zhang, H., & Fu, M. (2015). Consensus problems in networks of agents with communication delay. In *2015 IEEE conference on control applications (pp. 1069–1073)*.

Xiao, L., & Boyd, S. (2004). Fast linear iterations for distributed averaging. *Systems & Control Letters*, *53*(4), 65–78.

Xu, J., Zhang, H., & Xie, L. (2013). Input delay margin for consensusability of multi-agent systems. *Automatica*, *49*(6), 1816–1820.

You, K., & Xie, L. (2011a). Consensusability of discrete-time multi-agent systems over directed graphs. In *30th Chinese control conference (pp. 6413–6418)*.

You, K., & Xie, L. (2011b). Network topology and communication data rate for consensusability of discrete-time multi-agent systems. *IEEE Transactions on Automatic Control*, *56*(10), 2262–2275.

Zhou, B., & Lin, Z. (2014). Consensus of high order multi-agent systems with large input and communication delays. *Automatica*, *50*(2), 452–464.

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