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Brief paper

Robust filtering for uncertain linear discrete-time descriptor systems☆

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Abstract

This paper is concerned with the problem of robust filtering for uncertain linear discrete-time descriptor systems. The matrices of the system state-space model are uncertain, belonging to a given polytope. A linear matrix inequality based method is proposed for designing a linear stationary filter that guarantees the asymptotic stability of the estimation error and gives an optimized upper bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function, which is shown to outperform parameter-independent ones.

Keywords: Robust Kalman filters; Descriptor systems; Uncertain linear systems; Parameter-dependent Lyapunov functions; Discrete-time systems

1. Introduction

Descriptor systems (also known as singular, implicit or differential-/difference-algebraic systems) are an important class of dynamic systems from both a theoretical and practical points of view due to their capacity to describe algebraic constraints between physical variables (see, e.g., Xu & Lam, 2006 and the reference therein).

A great deal of interest has been devoted in the last decade or so to Kalman filtering methods for linear discrete-time descriptor systems; see, for instance, Bianco, Ishihara, and Terra (2005), Dai (1989), Deng and Liu (1999), Ishihara, Terra, and Campos (2005), Nikoukhah, Willsky, and Levy (1992), Nikoukhah, Campbell, and Delebecque (1999) and Zhang, Xie, and Soh (1999). The aforementioned methods rely on the knowledge of a perfect system model and they may fail to provide a guaranteed error variance when only an approximate model is available. In the context of robust Kalman filtering, only very recently this problem was addressed in Ishihara, Terra, and Campos (2004) for descriptor systems with norm-bounded parameter uncertainties, where a Riccati equation approach was proposed. Although norm-bounded parameter uncertainties are important to consider, most uncertain system models are much better described by polytopic structures; see, e.g., Boyd, El Ghaoui, Feron, and Balakrishnan (1994). Indeed, polytopic structures arise naturally when there are multiple real-valued uncertain parameters. Using norm-bounded structures typically over-estimates the uncertainties in the system. To the best of the authors' knowledge, the problem of robust Kalman filtering for linear discrete-time descriptor systems with polytopic uncertainty has not yet been investigated.

This paper addresses the problem of robust filter design for linear discrete-time descriptor systems with polytopic-type uncertainties, namely the matrices in the system state-space model are uncertain and assumed to belong to a given polytopic set. We develop a linear matrix inequality (LMI) method for designing a linear stationary filter that provides the asymptotic stability of the estimation error and an optimized upper bound on the asymptotic error variance, irrespective of the parameter uncertainty. The proposed robust filter design is based on a parameter-dependent Lyapunov function to achieve improved performance. An example is presented in the paper to illustrate this feature.

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Notation. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are the set of n-dimensional real vectors and $n\times m$ real matrices, I_n is the $n\times n$ identity matrix, 0_n is the $n\times n$ matrix of zeros, $\mathrm{Tr}[\cdot]$ denotes matrix trace, $\mathrm{diag}\{\cdots\}$ denotes block-diagonal matrix, and $\otimes$ is the Kronecker product. For a real matrix S, S^{T} denotes its transpose, $\mathrm{Her}\{S\}$ stands for $S+S^{\mathrm{T}}$ and $S>0$ means that S is symmetric and positive definite. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the blocks outside the main diagonal block and $\mathbf{E}\{\cdot\}$ stands for mathematical expectation.

2. Problem formulation

Consider a linear descriptor system in the following singular value decomposition (SVD) normal form:

$$\begin{cases} x(k+1)=A_1x(k)+A_2\phi(k)+Bw(k),\\ 0=H_1x(k)+H_2\phi(k)+H_3w(k),\\ y(k)=C_1x(k)+C_2\phi(k)+Dw(k),\\ s(k)=L_1x(k)+L_2\phi(k), \end{cases} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the dynamic state, $\phi(k)\in\mathbb{R}^{n_\phi}$ is the algebraic state, $w(k)\in\mathbb{R}^{n_w}$ is a zero-mean white noise signal (including process and measurement noises) with an identity covariance matrix and uncorrelated with $x(0)$ and $\phi(0)$, $y(k)\in\mathbb{R}^{n_y}$ is the measurement, $s(k)\in\mathbb{R}^{n_s}$ is the signal to be estimated, $A_i, C_i, H_i, L_i, i=1,2, B, H_3$ and D are real matrices with H_2 being a square matrix.

It is assumed that the matrices of system (1) are unknown but they are bounded by the following polytope:

$$\Omega=\left\{\Pi:\Pi=\sum_{i=1}^{\ell}\alpha_i\Pi_i,\ \alpha_i\geqslant 0:\sum_{i=1}^{\ell}\alpha_i=1\right\}, \tag{2}$$

where

$$\Pi=\begin{bmatrix} A_1 & A_2 & B\\ H_1 & H_2 & H_3\\ C_1 & C_2 & D\\ L_1 & L_2 & 0\end{bmatrix},\quad \Pi_i=\begin{bmatrix} A_{1i} & A_{2i} & B_i\\ H_{1i} & H_{2i} & H_{3i}\\ C_{1i} & C_{2i} & D_i\\ L_{1i} & L_{2i} & 0\end{bmatrix} \tag{3}$$

and $A_{ki}, C_{ki}, H_{ki}, L_{ki}, k=1,2, B_i, H_{3i}$ and D_i are given real constant matrices.

This paper is aimed at designing a stationary asymptotically stable linear filter $\mathscr{F}$ which provides an estimate $\hat{s}$ of the signal s with a guaranteed performance bound for the asymptotic variance of the estimation error, irrespective of the uncertainty. We seek a filter of order n with a state-space realization

$$\begin{cases} \mathscr{F}:\hat{x}(k+1)=A_f\hat{x}(k)+B_fy(k),\quad \hat{x}(0)=0,\\ \hat{s}(k)=C_f\hat{x}(k), \end{cases} \tag{4}$$

where $\hat{x}\in\mathbb{R}^n$ and the matrices A_f, B_f and C_f are to be designed. Given a performance bound $\gamma>0$, the design task is to find, if possible, a filter $\mathscr{F}$ in (4) such that

$$\max_{\Omega}\{\mathrm{var}\{e\}\}\leqslant\gamma, \tag{5}$$

where $\mathrm{var}\{e\}:=\lim_{k\to\infty}\mathbf{E}\{[s(k)-\hat{s}(k)]^{\mathrm{T}}[s(k)-\hat{s}(k)]\}$.

Observe that considering (1) and (4), the dynamics of the estimation error $e:=s-\hat{s}$ can be described by the following state-space model:

$$\begin{cases} \xi(k+1)=\tilde{A}_1\xi(k)+\tilde{A}_2\phi(k)+\tilde{B}w(k),\\ 0=\tilde{H}_1\xi(k)+H_2\phi(k)+H_3w(k),\\ e(k)=\tilde{L}_1\xi(k)+L_2\phi(k), \end{cases} \tag{6}$$

where

$$\xi=\begin{bmatrix} x\\ \hat{x}\end{bmatrix},\quad \tilde{A}_1=\begin{bmatrix} A_1 & 0\\ B_fC_1 & A_f\end{bmatrix},\quad \tilde{A}_2=\begin{bmatrix} A_2\\ B_fC_2\end{bmatrix}, \tag{7}$$

$$\tilde{B}=\begin{bmatrix} B\\ B_fD\end{bmatrix},\quad \tilde{H}_1=[\,H_1\quad 0\,],\quad \tilde{L}_1=[\,L_1\quad -C_f\,]. \tag{8}$$

Next, we recall a version of Finsler's lemma needed in the next section.

Lemma 1 (Boyd et al., 1994). *Given matrices* $\Psi_i=\Psi_i^{\mathrm{T}}\in\mathbb{R}^{n\times n}$ *and* $\mathscr{N}_i\in\mathbb{R}^{m\times n}$, $i=1,\ldots,v$, *then*

$$x_i^{\mathrm{T}}\Psi_ix_i<0,\quad \forall x_i\in\mathbb{R}^n:\mathscr{N}_ix_i=0, x_i\neq 0; i=1,\ldots,v$$

iff there exist matrices K_i, $i=1,\ldots,v$, *such that*

$$\Psi_i+K_i\mathscr{N}_i+\mathscr{N}_i^{\mathrm{T}}K_i^{\mathrm{T}}<0,\quad i=1,\ldots,v. \tag{9}$$

In the next remark we comment on the generality of the problem formulation given above.

Remark 1. We claim that the above problem formulation can be applied to a seemingly more general class of uncertain causal descriptor systems represented by

$$\begin{cases} \mathbf{J}(\theta)\zeta(k+1)=\mathbf{A}(\theta)\zeta(k)+\mathbf{B}(\theta)w(k),\\ y(k)=\mathbf{C}(\theta)\zeta(k)+\mathbf{D}(\theta)w(k),\\ s(k)=\mathbf{L}(\theta)\zeta(k), \end{cases} \tag{10}$$

where $\zeta\in\mathbb{R}^{n_\zeta}$ is the state, $s(k)$, $w(k)$ and $y(k)$ are as before, $\theta\in\mathbb{R}^p$ is a vector of uncertain constant parameters belonging to a given polytope, $\mathbf{J}(\theta)$, $\mathbf{A}(\theta)$, $\mathbf{B}(\theta)$, $\mathbf{C}(\theta)$, $\mathbf{D}(\theta)$ and $\mathbf{L}(\theta)$ are real matrices affine in θ, and

$$\min_{\theta}\mathrm{rank}\{\mathbf{J}(\theta)\}=n<n_\zeta.$$

Since the matrix $\mathbf{J}(\theta)$ is uncertain, an SVD transformation cannot be directly used to obtain an equivalent system in the SVD normal form of (1) (Bender & Laub, 1987). However, introducing the following decompositions[1] :

$$\mathbf{A}(\theta)=\mathbf{A_0}+\mathscr{A}(\theta\otimes I_n),\quad \mathscr{A}=[\mathbf{A_1}\ldots\mathbf{A_p}],$$

$$\mathbf{J}(\theta)=\mathbf{J_0}+\mathscr{J}(\theta\otimes I_n),\quad \mathscr{J}=[\mathbf{J_1}\ldots\mathbf{J_p}]$$

and defining the augmented state vector:

$$x_a=[\zeta^{\mathrm{T}}\quad \eta^{\mathrm{T}}]^{\mathrm{T}},\quad \eta=\theta\otimes\zeta$$

[1] As the matrices $\mathbf{A}(\theta)$ and $\mathbf{J}(\theta)$ are affine functions of θ, it is always possible to find such decompositions.

system (10) can be rewritten as

$$\begin{cases} Jx_a(k+1) = A_a(\theta)x_a(k) + B_a(\theta)w(k), \\ y(k) = C_a(\theta)x_a(k) + D_a(\theta)w(k), \\ s(k) = L_a(\theta)x_a(k), \end{cases} \tag{11}$$

where

$$J = \begin{bmatrix} \mathbf{J_0} & \mathscr{J} \\ 0 & 0 \end{bmatrix}, \quad A_a(\theta) = \begin{bmatrix} \mathbf{A_0} & \mathscr{A} \\ \theta \otimes I_n & -I \end{bmatrix},$$

$$B_a(\theta) = \begin{bmatrix} \mathbf{B}(\theta) \\ 0 \end{bmatrix},$$

$$C_a(\theta) = [\,\mathbf{C}(\theta) \quad 0\,], \quad L_a(\theta) = [\,\mathbf{L}(\theta) \quad 0\,], \quad D_a(\theta) = \mathbf{D}(\theta).$$

Now, since in system (11) the matrix J does not depend on θ, this system can be transformed, via an SVD of the matrix J, into an equivalent system of the form of (1). It should be noted that although the state vector x_a of (11) is of dimension $n_\zeta + pn_\zeta$, the matrix J as above has rank equal to n. This implies that the dynamic state in the resulting difference-algebraic model (1) is of dimension n, which is also the dimension of the dynamic state in the original system (10). This is an important fact because the order of the filter to be developed in this paper is identical to the dimension of the dynamic state in (1).

Remark 2. Note that although system (1) could be transformed into a linear system without algebraic constraints (when the matrix H_2 is nonsingular over the polytope Ω), such a transformation destroys the polytopic uncertainty structure, i.e., the transformed system will not be a polytopic uncertain linear system.

3. Filter performance analysis

This section is concerned with the problem of filter performance analysis. More specifically, given an uncertain system (1), a filter (4) and a scalar $\gamma > 0$, we want to know how to check if the asymptotic variance of the estimation error is bounded by γ for all admissible uncertainties. The solution to this problem will be useful in developing conditions for the synthesis of a robust filter.

We first give necessary and sufficient conditions for the filter performance analysis for the case where the system matrix Π is perfectly known.

Theorem 1. *Consider system* (1) *with known matrices. Given a filter* (4) *and a scalar* $\gamma > 0$, *the estimation error system* (6) *is asymptotically stable, causal and* $\mathrm{var}\{e\} < \gamma$ *if and only if either of the equivalent conditions below holds*:

(a) *There exist matrices* $P > 0, \Xi > 0, M$ *and* N *satisfying the following LMIs*:

$$\begin{bmatrix} \mathrm{Her}\{M\mathscr{H}_1\} - \tilde{P} & \star & \star \\ P\tilde{A} & -P & \star \\ \tilde{L} & 0 & -I \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} \tilde{\Xi} + \mathrm{Her}\{N\mathscr{H}_2\} & \star & \star \\ P\tilde{B}_a & P & \star \\ \tilde{L}_2 & 0 & I \end{bmatrix} > 0, \tag{13}$$

$$\gamma - \mathrm{Tr}[\Xi] > 0, \tag{14}$$

where

$$\tilde{A} = [\,\tilde{A}_1 \quad \tilde{A}_2\,], \quad \tilde{B}_a = [\,\tilde{B} \quad \tilde{A}_2\,], \quad \tilde{L} = [\,\tilde{L}_1 \quad L_2\,], \tag{15}$$

$$\tilde{L}_2 = [\,0 \quad L_2\,], \quad \mathscr{H}_1 = [\,\tilde{H}_1 \quad H_2\,], \quad \mathscr{H}_2 = [\,H_3 \quad H_2\,], \tag{16}$$

$$\tilde{P} = \mathrm{diag}\{P, 0_{n_\phi}\}, \quad \tilde{\Xi} = \mathrm{diag}\{\Xi, 0_{n_w}\}. \tag{17}$$

(b) *There exist matrices* $P > 0, \Xi > 0, G, M$ *and* N *satisfying the LMIs of* (14) *and*

$$\begin{bmatrix} \mathrm{Her}\{M\mathscr{H}_1\} - \tilde{P} & \star & \star \\ G\tilde{A} & P - G - G^{\mathrm{T}} & \star \\ \tilde{L} & 0 & -I \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} \tilde{\Xi} + \mathrm{Her}\{N\mathscr{H}_2\} & \star & \star \\ G\tilde{B}_a & G + G^{\mathrm{T}} - P & \star \\ \tilde{L}_2 & 0 & I \end{bmatrix} > 0. \tag{19}$$

Proof. (a) We first show the causality of system (6). Note that if (12) holds, then $\mathrm{Her}\{M\mathscr{H}_1\} - \tilde{P} < 0$, which implies that the matrix H_2 is nonsingular and thus the causality of system (6) follows (Xu & Lam, 2006). Since (6) is causal, it can be rewritten in the form

$$\begin{aligned} \xi(k+1) &= \bar{A}\xi(k) + \bar{B}w(k), \\ e(k) &= \bar{L}\xi(k) + \bar{D}w(k), \end{aligned} \tag{20}$$

where

$$\begin{cases} \bar{A} = \tilde{A}_1 - \tilde{A}_2 H_2^{-1}\tilde{H}_1, & \bar{B} = \tilde{B} - \tilde{A}_2 H_2^{-1} H_3, \\ \bar{L} = \tilde{L}_1 - L_2 H_2^{-1}\tilde{H}_1, & \bar{D} = -L_2 H_2^{-1} H_3. \end{cases} \tag{21}$$

It is well known that system (20) is asymptotically stable and $\mathrm{var}\{e\} < \gamma$ if and only if there exist matrices $P > 0$ and $\Xi > 0$ satisfying the inequalities (Boyd et al., 1994):

$$\bar{A}^{\mathrm{T}} P\bar{A} - P + \bar{L}^{\mathrm{T}}\bar{L} < 0, \tag{22}$$

$$\Xi - \bar{B}^{\mathrm{T}} P\bar{B} - \bar{D}^{\mathrm{T}}\bar{D} > 0, \tag{23}$$

$$\gamma - \mathrm{Tr}[\Xi] > 0. \tag{24}$$

In view of (21), it can be readily verified that (22) and (23) are equivalent to, respectively,

$$\eta^{\mathrm{T}}\Lambda_1\eta < 0, \quad \eta = \Phi_1 x, \quad \forall x \in \mathbb{R}^n, x \neq 0, \tag{25}$$

$$\upsilon^{\mathrm{T}}\Lambda_2\upsilon > 0, \quad \upsilon = \Phi_2 w, \quad \forall w \in \mathbb{R}^{n_w}, w \neq 0, \tag{26}$$

where

$$\Lambda_1 = \tilde{A}^{\mathrm{T}} P \tilde{A} - \tilde{P} + \tilde{L}^{\mathrm{T}} \tilde{L}, \quad \Lambda_2 = \tilde{\Xi} - \tilde{B}_a^{\mathrm{T}} P \tilde{B}_a - \tilde{L}_2^{\mathrm{T}} \tilde{L}_2, \quad (27)$$

$$\Phi_1^{\mathrm{T}} = [\, I \quad -(H_2^{-1} \tilde{H}_1)^{\mathrm{T}} \,], \quad \Phi_2^{\mathrm{T}} = [\, I \quad -(H_2^{-1} H_3)^{\mathrm{T}} \,]. \quad (28)$$

Considering that

$$\{\eta : \eta = \Phi_1 x, \forall x \in \mathbb{R}^n, x \neq 0\} = \{\eta : \mathscr{H}_1 \eta = 0, \eta \neq 0\},$$

$$\{\upsilon : \upsilon = \Phi_2 w, \forall w \in \mathbb{R}^{n_w}, w \neq 0\} = \{\upsilon : \mathscr{H}_2 \upsilon = 0, \upsilon \neq 0\}$$

it follows from (25) and (26) that (22) and (23) are also equivalent to, respectively,

$$\eta^{\mathrm{T}} \Lambda_1 \eta < 0, \quad \forall \eta \in \mathbb{R}^{n_\eta} : \mathscr{H}_1 \eta = 0, \eta \neq 0, \quad (29)$$

$$\upsilon^{\mathrm{T}} \Lambda_2 \upsilon > 0, \quad \forall \upsilon \in \mathbb{R}^{n_\upsilon} : \mathscr{H}_2 \upsilon = 0, \upsilon \neq 0. \quad (30)$$

By Lemma 1, (29) and (30) hold if and only if there exist matrices M and N of appropriate dimensions such that

$$\Lambda_1 + \mathrm{Her}\{M \mathscr{H}_1\} < 0, \quad (31)$$

$$\Lambda_2 + \mathrm{Her}\{N \mathscr{H}_2\} > 0. \quad (32)$$

Applying Schur's complement, it follows that (31) and (32) are identical to (12) and (13), respectively.

(b) The equivalence of conditions (a) and (b) will be established.

(a) $\Rightarrow$ (b): If there exist matrices P, Ξ, M and N satisfying (12)–(14), then it follows that (18), (19) and (14) hold with $G = P$ and the same matrices Ξ, M, and N.

(b) $\Rightarrow$ (a): Pre- and post-multiplying (18) by $[I \ \ \tilde{A}^{\mathrm{T}} \ \ \tilde{L}^{\mathrm{T}}]$ and its transpose, and (19) by $[I \ \ \tilde{B}_a^{\mathrm{T}} \ \ \tilde{L}_2^{\mathrm{T}}]$ and its transpose, respectively, leads to

$$\mathrm{Her}\{M \mathscr{H}_1\} - \tilde{P} + \tilde{A}^{\mathrm{T}} P \tilde{A} + \tilde{L}^{\mathrm{T}} \tilde{L} < 0, \quad (33)$$

$$\tilde{\Xi} + \mathrm{Her}\{N \mathscr{H}_2\} - \tilde{B}_a^{\mathrm{T}} P \tilde{B}_a - \tilde{L}_2^{\mathrm{T}} \tilde{L}_2 > 0. \quad (34)$$

Since $P > 0$, applying Schur's complement to (33) and (34) it follows that these inequalities are equivalent to (12) and (13), respectively. □

It should be remarked that $V(\xi) = \xi^{\mathrm{T}}(k) P \xi(k)$, with P satisfying part (a) or (b) of Theorem 1, is a Lyapunov function for the unforced system of (6).

Since the inequalities (14), (18) and (19) are affine in the system matrices, the result (b) in Theorem 1 can be readily extended to the case where the matrices of system (1) belong to a polytope Ω. This robust filter performance analysis result is presented in the next theorem. In this result, we allow the Lyapunov matrix P and Ξ to be polytopic, but the matrices G, M and N are constrained to be fixed. Due to these constraints, this result provides only a sufficient condition for robust performance.

Theorem 2. *Consider system* (1) *with uncertain matrices and let Ω be a given polytope of admissible system matrices Π. Given a filter* (4) *and a scalar $\gamma > 0$, the error system* (6) *is asymptotically stable, causal and* $\mathrm{var}\{e\} < \gamma$ *over Ω if there exist matrices $P_i > 0, \Xi_i > 0$, $i = 1, \ldots, \ell$, G, M and N satisfying the following LMIs*:

$$\Psi_{1i}(P_i, G, M) < 0, \quad i = 1, \ldots, \ell, \quad (35)$$

$$\Psi_{2i}(P_i, \Xi_i, G, N) > 0, \quad i = 1, \ldots, \ell, \quad (36)$$

$$\gamma - \mathrm{Tr}[\Xi_i] > 0, \quad i = 1, \ldots, \ell, \quad (37)$$

where $\Psi_{1i}(\cdot)$ and $\Psi_{2i}(\cdot)$ denote the left-hand side of (18) *and* (19), *respectively, with the system matrices at the ith vertex of Ω, and P and Ξ replaced by P_i and Ξ_i. Moreover, $V(\xi) = \xi^{\mathrm{T}}(k) P(\alpha) \xi(k)$ is a Lyapunov function for the unforced system of* (6), *where $P(\alpha)$ is a parameter-dependent Lyapunov matrix given by $P(\alpha) = \sum_{i=1}^{\ell} \alpha_i P_i$ and where α_i are the scalars as in* (2).

Remark 3. Note that, although the causality of a polytopic uncertain descriptor system is not ensured by the causality of the systems for the vertices of the uncertainty polytope, it turns out that (35) ensures the causality of systems (1) and (6) for any system matrix Π belonging to the polytope Ω. Indeed, as $\Psi_{1i}(\cdot)$ is affine in Π, by convexity, (35) implies that $\mathrm{Her}\{M \mathscr{H}_1\} < \tilde{P}$ over Ω and thus the matrix H_2 is nonsingular over Ω, which implies the causality of systems (1) and (6) for any Π in Ω.

4. Synthesis of robust filters

In this section, we develop an LMI based method to design filters with an optimized asymptotic error variance for system (1). Note that a direct application of Theorem 1(b) or Theorem 2 would lead to bilinear matrix inequalities (BMIs), which are, in general, difficult to be solved numerically. However, it turns out that, applying appropriate congruence transformations and parameterization of the matrix G and the filter matrices, these BMIs can be transformed into LMIs. The next theorem deals with the case where the system matrix Π is perfectly known and is derived from part (b) of Theorem 1.

Theorem 3. *Consider system* (1) *with known matrices. Given a scalar $\gamma > 0$, there exists a filter* (4) *such that the estimation error system* (6) *is asymptotically stable, causal and* $\mathrm{var}\{e\} < \gamma$ *if and only if there exist matrices $X > 0, \Xi > 0$, $F_1, F_2, N, Q, R, S, Z, Y$ and W satisfying the LMIs of* (14) *and*

$$\begin{bmatrix} \mathrm{Her}\{F_1 \tilde{H}_1\} - X & \star & \star & \star \\ F_2 \tilde{H}_1 + H_2^{\mathrm{T}} F_1^{\mathrm{T}} & \mathrm{Her}\{F_2 H_2\} & \star & \star \\ \mathscr{A}_1 & \mathscr{A}_2 & X - \Upsilon - \Upsilon^{\mathrm{T}} & \star \\ \mathscr{L}_1 & L_2 & 0 & -I \end{bmatrix} < 0, \quad (38)$$

$$\begin{bmatrix} \tilde{\Xi} + \mathrm{Her}\{N \mathscr{H}_2\} & \star & \star \\ \mathscr{B}_a & \Upsilon + \Upsilon^{\mathrm{T}} - X & \star \\ \tilde{L}_2 & 0 & I \end{bmatrix} > 0, \quad (39)$$

where $\tilde{L}_2$, $\mathscr{H}_2$ *and* $\tilde{\Xi}$ *are as in* (16) *and* (17), *and*

$$\mathscr{A}_1 = \begin{bmatrix} RA_1 & 0 \\ SA_1 + YC_1 + Q & Q \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} R & W^{\mathrm{T}} \\ S & W \end{bmatrix},$$

$$\mathscr{B}_a^{\mathrm{T}} = \begin{bmatrix} \mathscr{B}^{\mathrm{T}} \\ \mathscr{A}_2^{\mathrm{T}} \end{bmatrix}, \quad \mathscr{A}_2 = \begin{bmatrix} RA_2 \\ SA_2 + YC_2 \end{bmatrix},$$

$$\mathscr{B} = \begin{bmatrix} RB \\ SB + YD \end{bmatrix}, \quad \mathscr{L}_1^{\mathrm{T}} = \begin{bmatrix} L_1^{\mathrm{T}} - Z^{\mathrm{T}} \\ -Z^{\mathrm{T}} \end{bmatrix}.$$

If the LMIs above are satisfied, the desired filter (4) *has the following transfer function matrix*:

$$H_{\mathscr{F}}(z) = Z(zI - W^{-1}Q)^{-1}W^{-1}Y. \tag{40}$$

Proof. We will establish the equivalence between the conditions in Theorem 1(b) and those in Theorem 3.

Sufficiency: We will show that the LMIs (14), (38) and (39) imply that the conditions (14), (18) and (19) of Theorem 1(b) hold with the filter (40).

Since $X > 0$, (38) implies that $\Upsilon + \Upsilon^{\mathrm{T}} > 0$, and thus R and W are nonsingular matrices. Parameterize G as follows:

$$G = \begin{bmatrix} I & -I \\ 0 & V^{-\mathrm{T}} \end{bmatrix} \begin{bmatrix} R & 0 \\ S & U \end{bmatrix}, \tag{41}$$

where U and V are $n \times n$ nonsingular matrices with V being a free parameter and $U = WV^{-1}$. It is clear that G is nonsingular.

Next, we define

$$\mathscr{T} = G^{-\mathrm{T}} \begin{bmatrix} R^{\mathrm{T}} & S^{\mathrm{T}} \\ 0 & U^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} I & 0 \\ V & V \end{bmatrix} \tag{42}$$

and let the following realization for the filter (40):

$$A_f = VW^{-1}QV^{-1}, \quad B_f = VW^{-1}Y, \quad C_f = ZV^{-1}.$$

By performing straightforward matrix manipulations, it can be readily verified that

$$\mathscr{T}^{\mathrm{T}} G\tilde{A}_1 \mathscr{T} = \mathscr{A}_1, \quad \mathscr{T}^{\mathrm{T}} G\tilde{A}_2 = \mathscr{A}_2, \quad \tilde{H}_1 \mathscr{T} = \tilde{H}_1, \tag{43}$$

$$\tilde{L}_1 \mathscr{T} = \mathscr{L}_1, \quad \mathscr{T}^{\mathrm{T}} G\tilde{B} = \mathscr{B}, \quad \mathscr{T}^{\mathrm{T}}(G + G^{\mathrm{T}})\mathscr{T} = \Upsilon + \Upsilon^{\mathrm{T}}. \tag{44}$$

Take

$$P = \mathscr{T}^{-\mathrm{T}} X \mathscr{T}^{-1}, \quad M = \mathrm{diag}\{\mathscr{T}^{-\mathrm{T}}, I_{n_\phi}\}[F_1^{\mathrm{T}} \;\; F_2^{\mathrm{T}}]^{\mathrm{T}}, \tag{45}$$

$$\mathbb{T}_1 = \mathrm{diag}\{\mathscr{T}^{-1}, I_{n_\phi}, \mathscr{T}^{-1}, I_{n_s}\},$$

$$\mathbb{T}_2 = \mathrm{diag}\{I_{n_w}, I_{n_\phi}, \mathscr{T}^{-1}, I_{n_s}\}.$$

Pre- and post-multiplying (38) by $\mathbb{T}_1^{\mathrm{T}}$ and $\mathbb{T}_1$, respectively, and using (43)–(45), we obtain (18). Similarly, pre- and post-multiplying (39) by $\mathbb{T}_2^{\mathrm{T}}$ and $\mathbb{T}_2$, respectively, and considering (43)–(45) yields (19).

Necessity: Suppose there exist a filter with a state-space realization (A_f, B_f, C_f) and matrices $P > 0$, G, Ξ, M and N satisfying inequalities (14), (18) and (19). We will show that (14), (38) and (39) also hold. First, (19) implies that the matrix G is nonsingular. Let G^{-1} be partitioned as follows:

$$G^{-1} := \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix},$$

where all the blocks are $n \times n$ matrices. Since the inequalities (18) and (19) are strict, we may assume that G_i, $i = 1, \ldots, 4$, are all nonsingular. Indeed, if this is not the case, we may perturb the given G slightly to achieve so without violating (18) and (19). The above assumption and the nonsingularity of G imply that $G_4 G_2^{-1} G_1 - G_3$ is also a nonsingular matrix. Define

$$R = G_1^{-1}, \quad U = (G_4 G_2^{-1} G_1 - G_3)^{-1}, \quad V = G_2 G_1^{-\mathrm{T}},$$

$$W = UV, \quad S = -UG_3G_1^{-1}, \quad Q = UA_fV,$$

$$Y = UB_f, \quad Z = C_fV.$$

The above definitions imply that G can be written as in (41). Taking $\mathscr{T}$ as in (42), it turns out that (38) and (39) can be derived similarly to the sufficiency part by reversing the steps there. □

The next result presents the robust filter design method. In the light of Theorem 2 and considering that (38) and (39) are affine in the matrices of system (1), the result follows directly from Theorem 3 by convexity arguments.

Theorem 4. *Let system* (1) *with uncertain matrices and a given polytope* Ω *of admissible system matrices* Π. *Given a scalar* $\gamma > 0$, *there exists a filter* (4) *such that the estimation error system* (6) *is asymptotically stable, causal and* $\mathrm{var}\{e\} < \gamma$ *over* Ω *if there exist matrices* $X_i > 0$, $\Xi_i > 0$, $i = 1, \ldots, \ell$, F_1, F_2, N, Q, R, S, Z, Y *and* W *satisfying the following LMIs*:

$$\Gamma_{1i} < 0, \quad \Gamma_{2i} > 0, \quad \gamma - \mathrm{Tr}[\Xi_i] > 0, \quad i = 1, \ldots, \ell, \tag{46}$$

where Γ_{1i} *and* Γ_{2i} *denote the left-hand side of* (38) *and* (39), *respectively, with the system matrices at the ith vertex of* Ω, *and* X *and* Ξ *replaced by* X_i *and* Ξ_i. *Moreover, the transfer function of a suitable filter is as in* (40).

Note that the feasibility of the LMIs of (46) requires the nonsingularity of the matrix H_2 over Ω.

5. An illustrative example

Let the uncertain descriptor system, which is adapted from Ishihara et al. (2004), described by (10) with

$$\mathbf{J}(\theta) = \mathbf{J_0} + \mathbf{J_1}\theta, \quad \mathbf{A}(\theta) = \mathbf{A_0} + \mathbf{A_1}\theta, \quad \mathbf{C}(\theta) = \mathbf{C_0} + \mathbf{C_1}\theta,$$

$$\mathbf{L}(\theta) = I_3, \quad \mathbf{B}(\theta) = \mathrm{diag}\{\sqrt{1.2}, \sqrt{1.6}, \sqrt{2}\}, \quad \mathbf{D}(\theta) = \sqrt{1.6},$$

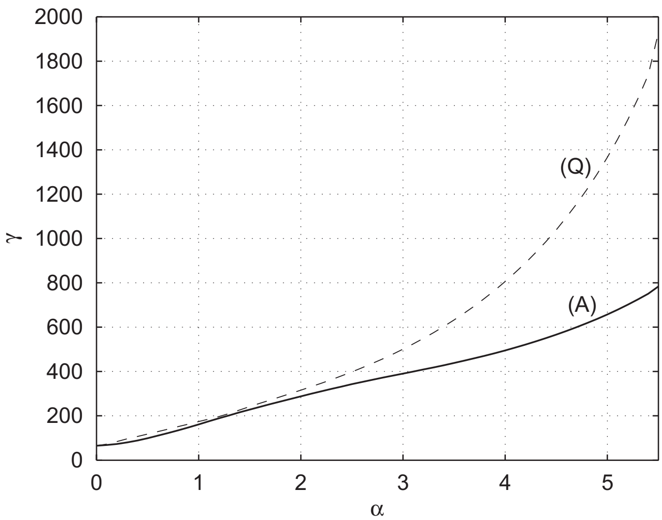

Fig. 1. Upper bound γ for designs (A) and (Q) for $|\theta| \leqslant \alpha$.

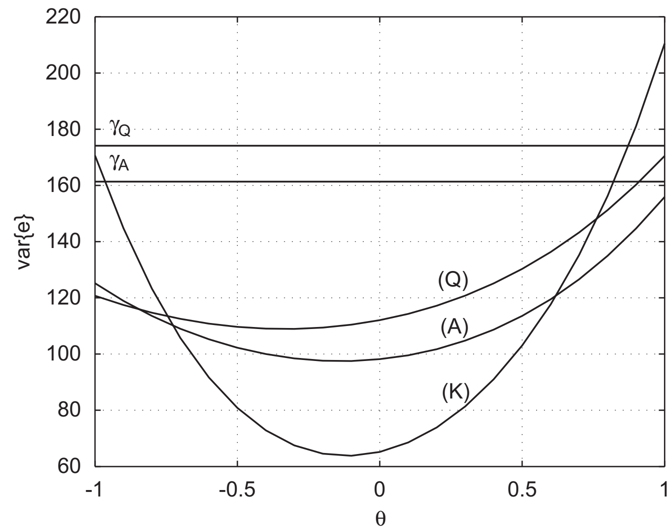

Fig. 2. Actual error variance for the filters of methods (A), (Q) and (K) and upper bounds γ_A and γ_Q for $|\theta| \leqslant 1$.

where θ is an uncertain parameter satisfying $|\theta| \leqslant \alpha$ and

$$\mathbf{J_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{J_1} = 0.01 \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A_0} = 0.1 \begin{bmatrix} 9 & 0 & 10 \\ 0 & 8 & 0 \\ 2 & 2 & 2 \end{bmatrix}, \quad \mathbf{A_1} = 0.01 \begin{bmatrix} 5 & 10 & 10 \\ 5 & 10 & 10 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C_0} = \begin{bmatrix} 1.4 \\ 0.8 \\ 1 \end{bmatrix}^T, \quad \mathbf{C_1} = 0.8 \begin{bmatrix} 0.659 \\ 5.931 \\ 0.659 \end{bmatrix}^T.$$

Robust filters are designed for the system with different values of α using Theorem 4. To this end, this system needs to be rewritten in the difference-algebraic form of (1). This can be achieved by first using the procedure outlined in Remark 1 to get an equivalent model of the form (11) and then apply an SVD transformation. Note that the dynamic state of the resulting model (1) is of dimension 2, which will be the order of the robust filter.

Theorem 4 is applied using two types of Lyapunov functions: (i) parameter-dependent Lyapunov function affine in θ; (ii) parameter-independent Lyapunov function, which are referred to as methods (A) and (Q), respectively. The method (Q) is obtained from Theorem 4 by setting the matrices X_i to a common matrix X. Fig. 1 displays the minimum upper bound γ on the asymptotic error variance versus α for both methods. Note that, for this example, both methods can solve the problem as long as the system is asymptotically stable, namely for $\alpha \leqslant 5.5$. The method (A) gives a significant performance improvement compared with (Q) for $\alpha > 2$.

To further illustrate the behaviors of the designed filters, the actual asymptotic variance of the estimation error is calculated for each method as a function of θ and for $\alpha = 1$. In addition, we also compare the later results with the asymptotic error variance produced by the Kalman filter designed for the nominal system with $\theta = 0$ and when applied to the system for different values of θ—referred to as (K). Fig. 2 shows these results along with the error variance upper bounds γ_A and γ_Q given by the methods (A) and (Q), respectively. The robustness of the two robust filters and the advantage of using parameter-dependent Lyapunov function are clearly demonstrated in the figure. We also observe that the upper bound given by (A), $\gamma_A = 161.3359$, is close to the worst-case asymptotic variance of 155.9167, which is achieved for $\theta = 1$.

6. Conclusions

This paper has addressed the design of minimum variance filters for uncertain linear discrete-time descriptor systems represented by a difference-algebraic state-space model. The matrices of the system state-space model are uncertain and assumed to belong to a given polytope. Our first main result, namely Theorem 1, considers a filter performance analysis problem for a known system model, namely without uncertainties, and gives necessary and sufficient conditions for a linear stationary filter to guarantee that the asymptotic variance of the estimation error is less than a given bound. The result is formulated in such a way that it can be extended to uncertain systems. The extension is done in the second main result, Theorem 2, where a polytopic uncertainty structure is considered for the descriptor system and a sufficient condition is given for a linear filter to guarantee a given performance bound in the worse case. These two analysis results are then used to produce two filter synthesis results, Theorems 3 and 4 for known systems and uncertain systems, respectively. The proposed robust filter design method is based on a parameter-dependent Lyapunov function.

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