Distributed Augmented Lambda-iteration Method For Economic Dispatch in Smart Grid

Hao Xing, Zhiyun Lin, and Minyue Fu

***Abstract*— The economic dispatch problem (EDP) is of vital importance in the operation and planning of power systems. In this paper, we study the EDP with general cost functions, transmission losses and prohibited operating zones. The equality constraint of the EDP is nonlinear due to transmission losses, while the feasible region is discontinuous due to prohibited operating zones, meaning that the EDP is a non-convex optimization problem. In order to solve this problem, this paper proposes the distributed augmented lambda-iteration method. Compared with the conventional lambda-iteration method, the proposed method has the advantages that it can get the optimal dispatch in the presence of prohibited operating zones and effectively avoid the oscillatory behavior that the conventional lambda-iteration method often exhibits. Furthermore, the proposed method is distributed in the sense that the nodes conduct local computations and communicate with their neighbors in a connected undirected graph. Simulation results are given to show the performance of our method.**

I. INTRODUCTION

The economic dispatch problem, which targets the minimum aggregate costs of electricity generation in a cooperative way, has been intensively investigated in the power industry and many centralized algorithms have been developed to solve the EDP. Recently, following the trend of smart grid and distribution algorithms, a number of distributed algorithms for the EDP have also appeared. In [1] and [2], the authors propose the incremental cost consensus (ICC) algorithm to solve the EDP, where the average consensus algorithm is used to guarantee the balance between demand and supply. In [3], the authors propose a consensus based distributed algorithm, which enables the generators to collectively learn the mismatch between demand and total supply for feedback. Consensus-based distributed bisection method is also proposed in [4] and [5] to solve the EDP with general convex functions on strongly connected directed graphs.

Nevertheless, some practical constraints, e.g., transmission losses and prohibited operating zones, are not taken into consideration in [1]–[5]. Neglecting transmission losses causes an imbalance between demand and supply, possibly threatening the stability of power systems. Also, neglecting prohibited operating zones may lead to power assignments that endanger stable operation of generators. However, transmission losses lead to the nonlinearity of the equality constraints, while prohibited operating zones lead to the discontinuity of the feasible region, both rendering the EDP non-convex [6]. To handle the non-convexity due to transmission losses, the conventional lambda-iteration method has been widely used in the power industry, which solves the non-convex EDP by iteratively solving an approximated convex problem. However, it has two major drawbacks:

- As widely reported in the literature, e.g., [7], [8], the conventional lambda-iteration method may exhibit the oscillatory behavior in large-scale mixed generation systems.
- The conventional lambda-iteration method cannot directly deal with the discontinuity incurred by the prohibited operating zones [9], [10].

In consideration of the growing trend of distributed algorithms and the necessity of including more practical constraints in the EDP, this paper aims at solving the EDP with transmission losses and prohibited operating zones in a distributed fashion and propose the *distributed augmented lambda-iteration* method. The challenges mainly stem from the non-convexity of the EDP and the constraint of distributed algorithmic architecture. We not only adopt the idea of the conventional lambda-iteration method to overcome the non-convexity challenge, but also extend it by defining *pseudo marginal cost* to deal with the discontinuity of the feasible region. Compared with the conventional lambda-iteration method, the proposed method is *augmented* in the following senses:

- We introduce a damping mechanism to avoid the oscillatory behavior that the conventional lambda-iteration method usually exhibits;
- The proposed method is able to solve the EDP with prohibited operating zones;
- The proposed method is applicable to not only the EDP with quadratic cost functions, but also the EDP with general convex cost functions.

Furthermore, the proposed algorithm is distributed in the sense that it does not rely on any pre-assigned central/leader node. Based on the average consensus algorithm [11], in our method all the nodes conduct local computation and communicate with their neighbors in a connected undirected network to solve the EDP.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Basics of Graph Theory

An undirected graph $G = (V, E)$ consists of a non-empty finite set of nodes $V = \{1, 2, ..., n\}$ and a finite set of unordered edges $E \subseteq V \times V$. Let us denote the neighbor set

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of node $i \in V$ by $N_i = \{j \in V - \{i\} : (j,i) \in E\}$, which implies that node i can communicate with its neighbors bi-directionally. The degree of node i, denoted by $d_i = |N_i|$, is defined as the cardinality of N_i. Since G is undirected, for any i and j, $(i,j) \in E$ implies $(j,i) \in E$. An undirected graph is connected if there is a path from any node to any other node. Self-loops are included, i.e., $\forall i \in V,\ (i,i) \in E$. A non-negative matrix $Q \in \mathbb{R}^{n\times n}$ is associated with graph G, where $[Q]_{ij} > 0$ if and only if $(j,i) \in E$.

B. Average Consensus Algorithm

Let us consider the undirected graph $G = (V,E)$, where $V = \{1,\ldots,n\}$. Each node $i \in V$ holds a state denoted by $x \in \mathbb{R}$. Denote by $x = [x_1,\ldots,x_n]^T \in \mathbb{R}^n$ the aggregate state. Define the *Metropolis* weight matrix $Q \in \mathbb{R}^{n\times n}$ associated with graph G as

$$q_{ij} = \begin{cases} \dfrac{1}{\max(d_i,d_j)+1}, & \text{if } j \in N_i, \\ 1 - \sum\limits_{j\in N_i} q_{ij}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where q_{ij} is the entry of Q on the ith row and the jth column.

With the iteration index denoted by $\kappa = 0,1,\ldots,$ the average consensus algorithm is given by

$$x_i(\kappa+1) = q_{ii}x_i(\kappa) + \sum_{j\in N_i} q_{ij}x_j(\kappa),\ \forall i = 1,\ldots,n. \tag{2}$$

where $x(0)$ is the initial aggregate state at $\kappa = 0$.

We say that algorithm (2) solves the *average consensus problem asymptotically*, i.e., for any initial states $x_i(0)$'s, it follows

$$\lim_{\kappa\to\infty} x_i(\kappa) = \big(\sum_{j=1}^{n} x_j(0)\big)/n,\ \forall i = 1,\ldots,n.$$

The average consensus algorithm is fully distributed since iteration (2) can be implemented in a distributed fashion. See more details in [11].

C. Problem Formulation

Suppose that there are in total n generators in the power grid, labelled from 1 to n. Let us denote the total load demand and the output of the ith generator by P^{load} and P_i, respectively. Taking into consideration the transmission losses and prohibited operation zones of generators, we formulate the EDP as follows:
Objective:

$$\min \quad \sum_{i=1}^{n} F_i(P_i), \tag{3}$$

where $F_i(P_i)$ is the cost function associated with the ith generator.
Power balance constraint:

$$\sum_{i=1}^{n} P_i - P^{loss} - P^{load} = 0, \tag{4}$$

where P^{loss} is the total transmission losses over the grid.
Capacity constraints of generators:

$$\underline{P}_i \leqslant P_i \leqslant \bar{P}_i,\ \forall i = 1,\ldots,n, \tag{5}$$

where $\underline{P}_i$ and $\bar{P}_i$ are the lower bound and upper bound of the output of the ith generator, respectively.
Prohibited operating zones of generators: For each generator $i \in V$, suppose that there are in total M_i prohibited operating zones, and define $\underline{P}_i^j$ and $\bar{P}_i^j$ as the lower and upper bounds of the jth prohibited operating zone, respectively. Therefore the permitted operating zones of the ith generator consist of $M_i + 1$ disjoint regions:

$$\begin{cases} \underline{P}_i \leqslant P_i \leqslant \underline{P}_i^1, \text{ or} \\ \bar{P}_i^j \leqslant P_i \leqslant \underline{P}_i^{j+1}, \text{ for } j = 1,\ldots,M_i - 1, \text{ or} \\ \bar{P}_i^{M_i} \leqslant P_i \leqslant \bar{P}_i. \end{cases} \tag{6}$$

In this paper, we deal with the EDP with general cost functions satisfying the assumption below:

Assumption 1: For every $1 \leqslant i \leqslant n$, $F_i(P_i) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly convex and twice continuously differentiable with

$$\frac{df_i(P_i)}{dP_i} \geqslant 0, \quad \forall P_i \in \mathbb{R}_+,$$

where $f_i = \frac{dF_i(P_i)}{dP_i}$ is the first derivative of function $F_i(P_i)$, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, and the equality holds at isolated points only.

We assume that the total transmission losses are a function of the generator outputs P_i's and we use the **B** *matrix loss formula* (**B** coefficients) to represent P^{loss}, given by

$$P^{loss} = \sum_{i=1}^{n}\sum_{j=1}^{n} P_i B_{ij} P_j + \sum_{i=1}^{n} B_{0i}P_i + B_{00}, \tag{7}$$

where $B_{ij} = B_{ji}$, B_{0i}, and B_{00} are computed according to the line parameters and the average daily operating status of the power systems. The equality constraint (4) contains a quadratic term due to the loss formula, which also leads to the non-convexity of the EDP (3)-(6). Besides, prohibited operating zones also lead to the discontinuity (non-convexity) of the feasible region.

Two connected undirected communication networks with self-loops are established in this paper: one consists of pure generation bus and the other one consists of m buses associated with pure load, pure generation, or both, respectively denoted by $G = (V,E)$ and $G' = (V',E')$, where $V' = \{1,2,...,n,n+1,...,m\}$ and $V = \{1,2,...,n\}$.

III. Conventional Lambda-iteration Method

The Lagrange function of the EDP (3)-(5) is given by

$$L = \sum_{i=1}^{n} F_i(P_i) - \lambda\big(\sum_{i=1}^{n} P_i - P^{loss} - P^{load}\big),$$

where λ is the Lagrange multiplier.

With prohibited operating zones neglected, the optimal solution P_i^* and the optimal Lagrange multiplier λ^* satisfy

$$\frac{\partial L}{\partial P_i^*} = f_i(P_i^*) - \lambda^*\left(1 - \frac{\partial P^{loss}}{\partial P_i^*}\right) = 0, \forall i, \tag{8}$$

where $f_i(r) = \alpha_i r + \beta_i$. Combining the cost functions (3) and the inequality constraints (5), we have the optimality condition that for all $i \in V$,

$$\begin{cases} f_i(P_i^*)pf_i > \lambda^*, & \text{for } P_i^* = \underline{P}_i, \\ f_i(P_i^*)pf_i = \lambda^*, & \text{for } \underline{P}_i < P_i^* < \bar{P}_i, \\ f_i(P_i^*)pf_i < \lambda^*, & \text{for } P_i^* = \bar{P}_i, \end{cases} \tag{9}$$

where pf_i is the *penalty factor*:

$$pf_i = 1/\left(1 - \frac{\partial P^{loss}}{\partial P_i}\right). \tag{10}$$

We can obtain the optimal solution P_i^*'s and the optimal Lagrange multiplier λ^* by combining (4) and (9). But the equality constraint (4) has a quadratic term and the inequation (9) is non-linearly piecewise, therefore it is extremely hard to solve (4) and (9) directly.

We now introduce the conventional lambda-iteration method [6], which can solve (4) and (9) iteratively. Denote the iteration step denoted by $k = 0, 1, \ldots$

Step 1: At $k = 0$, the generators pick initial values $P_i[0]$'s such that $\sum_{i=1}^n P_i[0] - P^{load} = 0$.

Step 2: Compute the penalty factors $pf_i[k]$'s and the total transmission losses $P^{loss}[k]$ according to (10) and (7).

Step 3: Solve the following equations to get $P_i[k+1]$'s and $\lambda[k+1]$.

$$\sum_{i=1}^n P_i[k+1] - P^{loss}[k] - P^{load} = 0, \tag{11}$$

$$\begin{cases} f_i(P_i[k+1])pf_i[k] > \lambda[k+1], & P_i[k+1] = \underline{P}_i, \\ f_i(P_i[k+1])pf_i[k] = \lambda[k+1], & P_i[k+1] \in (\underline{P}_i, \bar{P}_i), \\ f_i(P_i[k+1])pf_i[k] < \lambda[k+1], & P_i[k+1] = \bar{P}_i, \end{cases} \tag{12}$$

Step 4: Go back to Step 2 and loop until convergence.

IV. Distributed Augmented Lambda-iteration Method

A. Distributed Collection of Demand Information

The problem formulation in Section II-C is based on an implicit assumption that the aggregate demand information is known to each generator. Nevertheless, in practical power grids the demand is spatially distributed at almost all the buses, i.e., $P^{load} = \sum_{j=1}^m P_j^{load}$, where P_j^{load} is the power demand at bus j. As pointed out in [5], it is unnecessary for the generators to know the aggregated demand P^{load}. Instead, we apply the average consensus algorithm (2) in order that each node in V gets the average demand $\eta^* = P^{load}/n$, which as we will show, is sufficient to solve the EDP.

In graph $G^{'} = (V^{'}, E^{'})$, using (1), define an associated doubly stochastic matrix $Q^{'} \in \mathbb{R}^{m \times m}$ using (1). where the superscript $'$ indicates that the parameters are defined with regard to graph $G^{'}$.

For every node $i \in V^{'}$, we establish two variables $p_i(\kappa)$ and $s_i(\kappa)$, respectively initialized by $p_i(0) = P_i^{load}$, and

$$s_i(0) = \begin{cases} 1, & i = 1, 2, ..., n, \\ 0, & i = n+1, n+2, ...m. \end{cases}$$

Note that initialization of $s_i(\kappa)$ is in a heterogenous fashion. We then run the following average consensus algorithms simultaneously until convergence:

$$p_i(\kappa+1) = q_{ii}^{'} p_i(\kappa) + \sum_{j \in N_i^{'}} q_{ij}^{'} p_j(\kappa), \tag{13}$$

$$s_i(\kappa+1) = q_{ii}^{'} s_i(\kappa) + \sum_{j \in N_i^{'}} q_{ij}^{'} s_j(\kappa). \tag{14}$$

Defining $p^* = \lim_{\kappa \to \infty} p_i(\kappa)$ and $s^* = \lim_{\kappa \to \infty} s_i(\kappa)$, we have:

$$p^* = P^{load}/m, \ s^* = n/m.$$

For every node $i \in V$, we have:

$$\eta^* = \frac{p^*}{s^*} = P^{load}/n. \tag{15}$$

B. Distributed Determination of $P_i[0]$'s

This subsection aims at the determination of $P_i[0]$'s in a distributed fashion, satisfying the inequality constraints (5) and

$$\sum_{i=1}^n P_i[0] = P^{load} + B_{00} = n(\eta^* + B_{00}/n).$$

Note that we extract the term B_{00} from the loss formula and put it in the above equation, for B_{00} is a constant and B_{00}/n is assumed to be known by each generator.

To estimate the total generation capacity in the network, each node $i \in V$ establishes two auxiliary variables $\underline{x}_i(\kappa)$ and $\bar{x}_i(\kappa)$, initialized by

$$\underline{x}_i(0) = \underline{P}_i, \ \bar{x}_i(0) = \bar{P}_i.$$

Then run the following average consensus iterations till convergence,

$$\underline{x}_i(\kappa+1) = q_{ii}\underline{x}_i(\kappa) + \sum_{j \in N_i} q_{ij}\underline{x}_j(\kappa), \ \forall i \in V, \tag{16}$$

$$\bar{x}_i(\kappa+1) = q_{ii}\bar{x}_i(\kappa) + \sum_{j \in N_i} q_{ij}\bar{x}_j(\kappa), \ \forall i \in V, \tag{17}$$

where q_{ij} is given by (1). When iterations (16) and (17) converge, each node $i \in V$ will get the common values $\underline{x}^*$ and $\bar{x}^*$ similarly given by

$$\underline{x}^* = \left(\sum_{i=1}^n \underline{P}_i\right)/n, \ \bar{x}^* = \left(\sum_{i=1}^n \bar{P}_i\right)/n. \tag{18}$$

After obtaining $\underline{x}^*$ and $\bar{x}^*$, the nodes can get the $P_i[0]$'s following

$$P_i[0] = \underline{P}_i + \frac{x^* - \underline{x}^*}{\bar{x}^* - \underline{x}^*}(\bar{P}_i - \underline{P}_i), \ \forall i \in V, \tag{19}$$

where $x^* = \eta^* + B_{00}/n$.

C. Distributed Computation of $pf_i[k]$'s and $P_a^{loss}[k]$

We assume that each node i knows the **B** coefficients associated with itself, i.e., $B_{ij},\ \forall j \in V$. The key to calculating $pf_i[k]$ is to calculate $\sum_{j=1}^{n} B_{ij}P_j[k]$ in a distributed fashion. For this purpose, each node $i \in V$ establishes an auxiliary variable $y_i^j(\kappa)$, where the superscript j represents y_i^j is meant for the calculation of $pf_j[k]$.

We initialize y_i^j's according to

$$y_i^j(0) = B_{ij}P_i[k]. \tag{20}$$

Then run the following iteration till convergence,

$$y_i^j(\kappa+1) = q_{ii}y_i^j(\kappa) + \sum_{l\in N_i} q_{il}y_l^j(\kappa),\ \forall i \in V. \tag{21}$$

Denote the convergence value of (21) by y^{j*}, it follows that

$$y^{j*} = \left(\sum_{i=1}^{n} B_{ij}P_i[k]\right)/n.$$

Therefore the penalty factor pf_j is given by

$$pf_j[k] = 1/\left(1 - 2ny^{j*} - B_{0j}\right). \tag{22}$$

Loop until all the nodes $j \in V$ obtains their $pf_j[k]$'s.

We then compute the total transmission losses $P^{loss}[k]$. Since the constant term B_{00} has already been included in x^*, we only need to compute

$$\begin{aligned} P_a^{loss}[k] =& P^{loss}[k] - B_{00} \\ =& \sum_{i=1}^{n}\sum_{j=1}^{n} P_i[k]B_{ij}P_j[k] + \sum_{i=1}^{n} B_{0i}P_i[k] \\ =& \sum_{i=1}^{n}\left(ny^{i*}[k]P_i[k] + B_{0i}P_i[k]\right). \end{aligned} \tag{23}$$

For this purpose, each node establishes an auxiliary variable $y_i(\kappa)$, initialized by

$$y_i(0) = ny^{i*}[k]P_i[k] + B_{0i}P_i[k]. \tag{24}$$

And then run the following average consensus algorithm till convergence,

$$y_i(\kappa+1) = q_{ii}y_i(\kappa) + \sum_{j\in N_i} q_{ij}y_j(\kappa),\ \forall i \in V. \tag{25}$$

Denote the convergence value of (25) by y^*, it follows that

$$y^* = P_a^{loss}[k]/n. \tag{26}$$

D. Distributed Bisection Algorithm for $P_i[k+1]$

With the $pf_i[k]$'s and $P_a^{loss}[k]$ obtained in a distributed fashion, we now proceed to the updates of $P_i[k+1]$'s and $\lambda[k+1]$, which corresponds to step 3 of the conventional lambda-iteration method.

The updating of $P_i[k+1]$'s and $\lambda[k+1]$ is equivalent to solving the following optimization problem:

$$\min \quad \sum_{i=1}^{n} pf_i[k]F_i(P_i) \tag{27}$$

$$\text{s.t.} \quad P_i \in \Omega_i,\ \forall i \in V, \tag{28}$$

$$\sum_{i=1}^{n} P_i - P^{loss}[k] - P^{load} = 0, \tag{29}$$

where for all i, Ω_i is defined as the set of real numbers that satisfies the constraint (6). One can verify that the equations (11) and (12) which are solved in step 3 of the conventional lambda-iteration method, are the necessary optimality condition of problem (27)-(29) with constraints (28) replaced by (5). Note that though the $pf_i[k]$ and $P^{loss}[k]$ are constant, the optimization problem (27)-(29) is still non-convex, as the sets Ω_i's are discontinuous and therefore non-convex. Besides, the optimality condition (9) does not apply here due to the prohibited operating zones.

Let us consider the problem from a Lagrange dual perspective. The Lagrange function of the problem (27)-(29) is given by

$$L = \sum_{i=1}^{n} pf_i[k]F_i(P_i) - \lambda\left(\sum_{i=1}^{n} P_i - P^{loss}[k] - P^{load}\right). \tag{30}$$

For a given Lagrange multiplier λ, consider the following optimization problem:

$$\begin{aligned} &\min \quad L, \\ &\text{s.t.} \quad P_i \in \Omega_i,\ \forall i \in V. \end{aligned} \tag{31}$$

Since the Lagrange multiplier λ is given, the objective of (31) is equivalent to

$$\min \quad L^o = \sum_{i=1}^{n} pf_i[k]F_i(P_i) - \lambda\sum_{i=1}^{n} P_i. \tag{32}$$

Note that in the optimization problem (31), the P_i's are mutually independent, for the equality constraint is null (reflected indirectly in λ). With $pf_i[k] > 0$, the problem (31) can be equivalently divided into n subproblems:

$$\begin{aligned} &\min \quad L_i^o = F_i(P_i) - \frac{\lambda P_i}{pf_i[k]}, \\ &\text{s.t.} \quad P_i \in \Omega_i, \end{aligned} \tag{33}$$

which is simply the minimization of a scalar function on a discontinuous region.

The first derivative of L_i^o is given by

$$\frac{dL_i^o}{dP_i} = f_i(P_i) - \frac{\lambda}{pf_i[k]}.$$

If there are no inequality constraints, the optimal solution to (33), denoted by P_i^o, is such that $\partial L^o/\partial P_i = 0$, i.e.,

$$P_i^o = g_i\left(\frac{\lambda}{pf_i[k]}\right) = h_i(\lambda), \tag{34}$$

where $g_i(\cdot)$ is the inverse function of $f_i(\cdot)$. If P_i^o happens to belong to the set Ω_i, then P_i^o minimizes L_i^o with respect to not only the set Ω_i, but also the whole domain of real numbers. If P_i^o does not belong to Ω_i, there are 3 cases:

1) $P_i^o > \bar{P}_i$: From the assumption 1, it follows that $f_i(P_i)$ is continuously increasing in $\mathbb{R}_+$, therefore dL_i^o/dP_i is also continuously increasing in $\mathbb{R}_+$. Since $P_i^o > \bar{P}_i$ and $dL_i^o/dP_i(P_i^o) = 0$, it follows that $dL_i^o/dP_i(P_i) < 0$, for $P_i < P_i^o$. Consequently L_i^o is monotonically decreasing in Ω_i. Therefore in this case $P_i^\star = \bar{P}_i$ is the minimizer of L_i^o in Ω_i.

2) $P_i^o < \underline{P}_i$: This case can be analyzed by the same technique used above. If $P_i^o < \underline{P}_i$, we have $P_i^\star = \underline{P}_i$.

3) $\underline{P}_i^j < P_i^o < \bar{P}_i^j$: In last two cases, P_i^o is infeasible because it is beyond the generator's capacity, while in this case it falls into one of the prohibited operating zones. Since $dL_i^o/dP_i(P_i^o) = 0$ and dL_i^o/dP_i is monotonically increasing,

- for $P_i < P_i^o,\ P_i \in \Omega_i$, $dL_i^o/dP_i(P_i) < 0$, meaning that L_i^o is monotonically decreasing;
- for $P_i > P_i^o,\ P_i \in \Omega_i$, $dL_i^o/dP_i(P_i) > 0$, meaning that L_i^o is monotonically increasing.

Therefore the optimal solution is either $\underline{P}_i^j$ or $\bar{P}_i^j$. We now investigate

$$L_i^o(\bar{P}_i^j) - L_i^o(\underline{P}_i^j) = F_i(\bar{P}_i^j) - F_i(\underline{P}_i^j) - \frac{\lambda(\bar{P}_i^j - \underline{P}_i^j)}{pf_i[k]}.$$

Let us define $\lambda_i^j = \frac{F_i(\bar{P}_i^j) - F_i(\underline{P}_i^j)}{\bar{P}_i^j - \underline{P}_i^j}$.

- If $L_i^o(\bar{P}_i^j) - L_i^o(\underline{P}_i^j) > 0$, then $\lambda < pf_i[k]\lambda_i^j$, $P_i^\star = \underline{P}_i^j$;
- If $L_i^o(\bar{P}_i^j) - L_i^o(\underline{P}_i^j) < 0$, then $\lambda > pf_i[k]\lambda_i^j$, $P_i^\star = \bar{P}_i^j$;
- If $L_i^o(\bar{P}_i^j) - L_i^o(\underline{P}_i^j) = 0$, then $\lambda = pf_i[k]\lambda_i^j$, and $P_i^\star$ can take either $\underline{P}_i^j$ or $\bar{P}_i^j$.

The prohibited operating zones lead to the discontinuity of the feasible region, but the natural domain of the cost functions is the entire domain of real numbers. Therefore, in order to fix the discontinuity of the marginal cost function, we define the *pseudo marginal cost* λ_i^j associated with the jth prohibited operating zone, which is a constant in $(\underline{P}_i^j, \bar{P}_i^j)$. Due to the strong convexity of $F_i(P_i)$, we have

$$f_i(\underline{P}_i^j) < \lambda_i^j < f_i(\bar{P}_i^j).$$

Let us define the following mapping:

$$\mathcal{P}_i(\lambda) = \begin{cases} \underline{P}_i, & h_i(\lambda) < \underline{P}_i, \\ h_i(\lambda), & h_i(\lambda) \in \Omega_i, \\ \underline{P}_i^j, & \underline{P}_i^j < h_i(\lambda) < \bar{P}_i^j, \lambda < pf_i[k]\lambda_i^j, \\ \bar{P}_i^j, & \underline{P}_i^j < h_i(\lambda) < \bar{P}_i^j, \lambda \geqslant pf_i[k]\lambda_i^j, \\ \bar{P}_i, & h_i(\lambda) > \bar{P}_i. \end{cases} \quad (35)$$

One can verify that the mapping (35) is monotonically increasing with respect to λ. So $P_i[k+1] = \mathcal{P}_i(\lambda[k+1])$ is also monotonically increasing with respect to $\lambda[k+1]$, which enables us to use the bisection method. The detailed procedures are as follows.

Each node establishes two commonly shared variables λ^+ and λ^- such that the optimal Lagrange multiplier must lie in the interval $[\lambda^-, \lambda^+]$. The initial λ^+ and λ^- can be selected to be extremely large and small, respectively, for the convergence of bisection is very fast.

Let $t = 0, 1, \ldots$ denote the bisection steps. At step t, each node computes $\lambda(t+1) = (\lambda^-(t) + \lambda^+(t))/2$.

Each node then obtains $P_i(\lambda(t+1))$ according to (35) with $\lambda[k+1]$ replaced by $\lambda(t+1)$, and then establishes a variable $z_i(\kappa)$, which is initialized by $z_i(0) = P_i(\lambda(t+1))$. Run the following iteration till convergence,

$$z_i(\kappa+1) = q_{ii} z_i(\kappa) + \sum_{j \in N_i} q_{ij} z_j(\kappa), \ \forall i \in V. \quad (36)$$

After convergence, denote the convergence value by z^*. Then each node updates $\lambda^-(t+1)$ and $\lambda^+(t+1)$ according to

- For $z^* < y^*[k] + x^*$,
$$\lambda^+(t+1) = \lambda^+(t), \ \lambda^-(t+1) = \lambda(t+1).$$
- For $z^* = y^*[k] + x^*$,
$$\lambda[k+1] = \lambda(t+1), \text{ and the bisection stops.}$$
- For $z^* > y^*[k] + x^*$,
$$\lambda^+(t+1) = \lambda(t+1), \ \lambda^-(t+1) = \lambda^-(t).$$

Recompute $P_i(\lambda(t+1))$'s and circulate the bisection until convergence. Then each node obtains $P_i[k+1]$ and $\lambda[k+1]$.

E. Damping Mechanism to Avoid Oscillations

As aforementioned, the conventional lambda-iteration method may exhibit oscillatory behavior. That is, after a certain amount of iterations, the results of each lambda-iteration oscillate periodically between 2 (seldom more than 2) values, instead of converging to a unique solution. To the best of the authors' knowledge, none of the existing works has effectively avoided this oscillatory behavior.

Herein we assume that after enough iteration steps, the results oscillate between two values, i.e., for some integer $K > 0$ and $l = 1, 2, \ldots$, $P[k] = P[k+2l]$, and $P[k] \neq P[k+1]$, $\forall k > K$, where $P[k] = [P_1[k], \ldots, P_n[k]]^T \in \mathbb{R}^n$. We note that both in subsection IV-C and in the step 2 of the conventional lambda-iteration method, the computation of $pf_i[k]$'s and $P_a^{loss}[k]$ only depends on $P_i[k]$'s, without using the results of previous iterations. Define the following *damping operator*

$$D[k] = \frac{(P[k] + P[k-1])}{2},$$

which takes the average of the computational results at iterations k and $k-1$. We then use $D[k]$ instead of $P[k]$ to compute $pf_i[k]$'s and $P_a^{loss}[k]$. We have the following proposition:

Proposition 1: The damping operator $D[k]$ prevents the lambda-iteration from oscillating between 2 values.

Proof: Since the lambda-iteration method is an implicit form of the fixed-point iteration method, it can be expressed by $P[k+1] = H(P[k])$, where $H(\cdot)$ is the implicit function corresponding to the lambda-iteration without $D[k]$. Therefore the lambda-iteration with $D[k]$ is given by $P[k+1] = H((P[k] + P[k-1])/2)$. We then prove Proposition 1 by contradiction. Assume that after enough iteration steps,

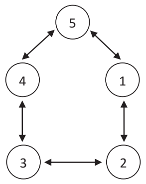

Fig. 1. Communication network of the 5 generators.

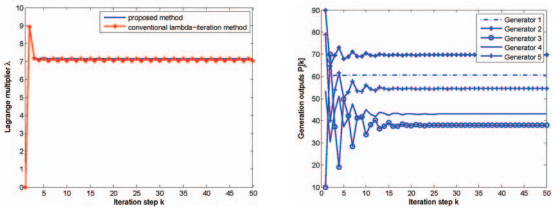

Fig. 2. Results of Case 1

the lambda-iteration with $D[k]$ oscillates between u and v. Therefore we have for some k,

$$P[k] = u,\ P[k+1] = v,\ P[k+2] = u,\ P[k+3] = v,$$

where $u \neq v$. Note that

$$P[k+2] = H\left(\frac{v+u}{2}\right) = P[k+3],$$

which contradicts the assumption that $u \neq v$. ∎

In practice, there is a very small chance that the lambda-iterations oscillate between $l > 2$ values. To deal with this, the damping operator can be generalized in the following way:

$$D[k] = \frac{P[k] + \ldots + P[k-l+1]}{l}.$$

V. Numerical Simulation

A. Case 1: EDP without Prohibited Operating Zones

In this case, we apply our distributed algorithm to the EDP on the IEEE 14-bus system. We consider quadratic cost functions transmission losses, while neglect the prohibited zones temporarily, which means that the conventional lambda-iteration method [6] can also solve this EDP.

The generation parameters are adopted from [5] and $\gamma_i = 0$ is set 0 MU for all i. The communication network is shown in Fig. 1. In this case the total power demand is $P^{load} = 250$ MW. We set $\lambda^+(0) = 10$ MU/MW and $\lambda^-(0) = 0$ MU/MW, which are sufficient to ensure $\lambda^* \in [\lambda^-(0), \lambda^+(0)]$. In the calculation of $P_i[k]$'s and $\lambda[k]$, we artificially set the bisection number to 15, such that for each k, $|\lambda(15) - \lambda[k]| \leqslant \frac{1}{2}|\lambda^+(14) - \lambda^-(14)| = \frac{1}{2^{15}}|\lambda^+(0) - \lambda^-(0)| \approx 0.0003$.

The evolutions of $\lambda[k]$ and $P_i[k]$'s are shown in Fig. 2(a) and Fig. 2(b), respectively. From Fig. 2(a), we can see that the conventional lambda-iteration method exhibits the oscillatory behavior after about 5 iterations, while the proposed method with the damping operator converges to the unique optimal solution.

B. Case 2: EDP with Prohibited Operating Zones

In this case we consider the EDP with prohibited operating zones. The prohibited zones are $20 \sim 40\%$ and $60 \sim 80\%$ of each generator's capacity range and the total demand $P^{load} = 150$ MW.

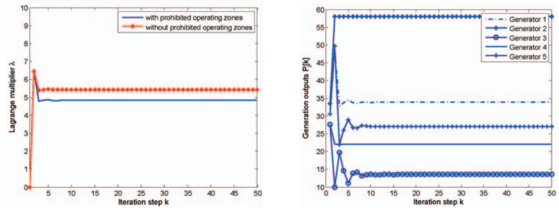

Fig. 3. Results of Case 2

The evolutions of $\lambda[k]$ and $P_i[k]$'s are shown in Fig. 3(a) and 3(b), respectively. One counter-intuitive finding from Fig. 3(a) is that the optimal Lagrange multiplier $\lambda^* = 4.833$ MU/MW with the prohibited operating zones is lower than $\lambda^{*'} = 5.426$ MU/MW without the prohibited operating zones. Intuitively, due to the existence of prohibited operating zones, the optimal power assignments are forced away from those without prohibited operating zones, probably driving the marginal cost up. But on the other hand, the optimal aggregated cost with the prohibited zones is $F^* = 648.33$ MU, which is larger than $F^{*'} = 631.81$ MU without the prohibited operating zones. For generators 2 and 4, their optimal power assignments are 58 MW and 22 MW, which take the boundary value of their prohibited operating zones, while in the absence of prohibited operating zones their optimal assignment are 40.14 MW and 20.09 MW, respectively.

VI. Concluding Remarks

This paper aims at solving the EDP with general cost functions, transmission losses, and prohibited operating zones, for which we propose the distributed augmented lambda-iteration method. Future work would take into consideration more constraints, e.g., transmission line capacity and spinning reserve.

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