Maximum Principle For Mckean-Vlasov Type Semi-Linear Stochastic Evolution Equations

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***Abstract*—We investigate the optimal control problem with a final state constraint for the controlled Mckean-Vlasov type semi-linear stochastic evolution equations. By mean of Ekeland's variation principle, we give the maximum principle. It turns out that the form of the maximum principle is quite different from the one corresponding to the case of linear diffusion.**

I. Introduction

A. Background

Consider the following Mckean-Vlasov type stochastic process described by a semi-linear stochastic evolution equation in a real separable Hilbert space H:

$$\begin{cases} dx_t = Ax_t dt + b(x_t, \mu_t)dt + \sigma(x_t, \mu_t)dw_t, \\ x_0 = \xi, \quad t \in [0, T] \end{cases} \tag{1}$$

where μ_t is the probability distribution induced by x_t, w is a given H-valued cylindrical Wiener process, $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $\{e^{At}\}$. b, σ are appropriate functionals defined on $H \times \mathcal{P}_{\lambda^2}(H)$, where $\mathcal{P}_{\lambda^2}(H)$ denotes a proper subset of probability measure on H, and ξ is a given H-valued random variable.

Recently, this system attracted a lot of research attention (see [1], [2], [6]), because there are situations where the nonlinear drift term b amd diffusion term σ depend not only on the state of the process but also on the probability distribution of the process. For example, a (biological, chemical or physical) interacting particle system in which each particle moves in the space H according to the dynamics described by (1) with μ_t being replaced by the empirical distribution $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_t^i}$ of the N particles $x_t^1, .., x_t^N$. In other words, we have a system of N coupled semi-linear stochastic evolution equations:

$$dx_t^i = Ax_t^i dt + \frac{1}{N}\sum_{j=1}^N b(x_t^i, x_t^j)dt + \frac{1}{N}\sum_{j=1}^N \sigma(x_t^i, x_t^j)dw_t^i;$$

$$x_0^i = \xi, \quad i = 1, .., N.$$

According to McKean-Vlasov theory (see, for example [3], [7]), under proper conditions, the empirical measure-valued process μ^N converges in probability as N goes to infinity to a deterministic measure-valued function μ, which corresponds to the probability distribution of the state process determined by (1).

B. Contribution

This paper investigates the optimal control problem of the general Mckean-Vlasov type semi-linear stochastic evolution equations. There are two main differences from the existed result given by [1]: on the one hand, via Riesz's representation we give a 1-order expansion of the state variable, and then get a different variation equation; on the other hand, via the conception of the copy probability space, the martingale representation theorem holds, we then obtain a different adjoint equation compared with the one appeared in [1]. In addition, this paper can be looked as an extension of the result of [4] to the case of Mckean-Vlasov type semi-linear stochastic evolution equations. It is well known that stochastic differential equations of Ito type generate linear diffusion. We remake here because of the appearance of the probability distribution of the state process itself in both the drift and diffusion term, this makes corresponding diffusion nonlinear. The form of the maximum principle turns out to be quite different from the one corresponding to the case of linear diffusion.

II. Preliminaries and Statement of the problem

A. Framework

We introduce some notations, definitions and spaces which will be used throughout this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $\{\mathcal{F}_t\}$. A cylindrical Wiener process on E, a real separable Hilbert space, is a collection indexed by t of Gaussian linear random functionals $w_t(\cdot)$, such that, for any $e \in E$, $w_t(e)$ is a real $\{\mathcal{F}_t\}$ Wiener process, with the correlation function

$$Ew_t(e_1)w_s(e_2) = \langle e_1, e_2 \rangle_E \min(t, s), \ e_1, e_2 \in E, \ \forall\, t, s \geq 0,$$

where $\langle \cdot, \cdot \rangle_E$ is the inner product on E, we assume

$$\mathcal{F}_t = \sigma(w_s, s \leq t).$$

Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}) = (\Omega \times \widetilde{\Omega}, \mathcal{F} \bigotimes \widetilde{\mathcal{F}}, P \bigotimes \widetilde{P})$ be the product of $(\Omega, \mathcal{F}, P)$ with itself, where $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}) \triangleq (\Omega, \mathcal{F}, P)$. We endow this product space with the filtration $\overline{\mathcal{F}} = \{\overline{\mathcal{F}}_t = \mathcal{F}_t \bigotimes \widetilde{\mathcal{F}}, t \in I\}$, where $I \triangleq [0,T]$. Let H be a real separable Hilbert space, for any Borel measurable space $(E, \mathcal{E})$ and any random variable $\xi : (\Omega, \mathcal{F}, P; H) \to (E, \mathcal{E})$, we put $\widetilde{\xi}(\widetilde{\omega}) \triangleq \xi(\widetilde{\omega}), \widetilde{\omega} \in \widetilde{\Omega} = \Omega$ and $\xi(\omega, \widetilde{\omega}) \triangleq \xi(\omega)$, $\widetilde{\xi}(\omega, \widetilde{\omega}) \triangleq \widetilde{\xi}(\widetilde{\omega})$, $(\omega, \widetilde{\omega}) \in \Omega \times \widetilde{\Omega}$. It is easy to observe that $\widetilde{\xi}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ a copy of ξ on $(\Omega, \mathcal{F}, P)$, and $\widetilde{\xi}, \xi$ are i.i.d under $\overline{P}$.

We shall denote the norm and the inner product in real separable Hilbert spaces H by $|\cdot|_H$ and $\langle \cdot, \cdot \rangle_H$ respectively, and denote by $\mathcal{L}(X, Y)$ the Banach space of continuous linear operators $T : X \to Y$ and $\mathcal{L}(X)$ when $X = Y$. We shall consider also the functional space $L^2_{\mathcal{F}}(I \times \Omega; \mathcal{L}^2(X, Y))$, which is the space of stochastic processes $\xi_{(\cdot)}$ with values in $\mathcal{L}^2(X, Y)$ (the space of Hilbert-Schmidt operators with Hilbert-Schmidt norm $\|\cdot\|_{\mathcal{L}^2(X,Y)}$), adapted to the filtration $\mathcal{F}_t$, such that $E\int_I \|\xi_t\|^2_{\mathcal{L}^2(X,Y)} dt < \infty$. Note that $L^2_{\mathcal{F}}(I \times \Omega; \mathcal{L}^2(X, Y))$ is Hilbert space with the norm

$$\Big[E \int_I \| \ \cdot \|^2_{\mathcal{L}^2(X,Y)} dt\Big]^{1/2}.$$

For $k \in \mathbb{N}$, we denote by $C^k(H)$ the space of all functions which are continuous on H togheter with all their Fréchet derivatives up to order k. Let $\lambda(x) = 1 + |x|_H, x \in H$, the following notation we shall use frequently

$$C_{\lambda^2;Lip}(H) = \Big\{\varphi \in C^0(H) : \|\varphi\|_{\lambda^2;Lip} \triangleq \sup_{x \in H} \frac{|\varphi(x)|}{\lambda^2(x)} + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < \infty\Big\},$$

For each $k = 1, 2, ...$, we define the following notation

$$C_{\lambda^k}(H) = \Big\{\varphi \in C^0(H) : \|\varphi\|_{\lambda^k} \triangleq \sup_{x \in H} \frac{|\varphi(x)|}{\lambda^k(x)} < \infty\Big\}.$$

$$C^1_{\lambda^k}(H) = \Big\{\varphi \in C^1(H) : \|\varphi\|_{\lambda^k;1} \triangleq \sup_{x \in H} \frac{|\varphi(x)|}{\lambda^k(x)} + \sup_{x \in H} \frac{|D\varphi(x)|}{\lambda^k(x)} < \infty\Big\},$$

where D is the 1-order Fréchet derivative.

Let H be a separable real Hilbert space, $\mathcal{B}(H)$ be the Borel σ-algebra generated by all open subsets of H. For $1 \le p < \infty$, let $M_{\lambda^p}(H)$ denote the Banach space of signed Borel measures on $\mathcal{B}(H)$ such that the norm

$$\|\mu\|_{\lambda^p} \triangleq \Big(\int_H \lambda^p(x)|\mu|(dx)\Big)^{1/p} < \infty,$$

where $|\mu|$ is the total variation of μ.

Let $\mathcal{P}(H)$ be the set of all probability measure on H, set $\mathcal{P}_{\lambda^2}(H) \triangleq M_{\lambda^2}(H) \bigcap \mathcal{P}(H)$, and furnish it the following metric topology

$$\rho(\mu, \nu) \triangleq \sup_{\|\varphi\|_{\lambda^2;Lip} \le 1} \Big\{\langle \mu - \nu, \varphi \rangle, \ \varphi \in C_{\lambda^2;Lip}(H)\Big\},$$

where we use the notation $\langle \mu, \varphi \rangle \triangleq \mu(\varphi) = \int_H \varphi(x)\mu(dx)$ whenever this integral makes sense. So $(\mathcal{P}_{\lambda^2}(H), \rho)$ constructs a complete metric space. In addition, we denote also $C(I, \mathcal{P}_{\lambda^2}(H))$ the complete metric space of continuous functions from I to $\mathcal{P}_{\lambda^2}(H)$ with the metric:

$$dis(\mu, \nu) \triangleq \sup_{t \in I} \rho(\mu_t, \nu_t),$$

where $\mu, \nu \in C(I, \mathcal{P}_{\lambda^2}(H))$.

Remark 1. *Let $C^*_{\lambda^2;Lip}(H)$ be the topological dual space of $C_{\lambda^2;Lip}(H)$, with the norm*

$$\|l\|_{C^*_{\lambda^2;Lip}(H)} \triangleq \sup_{\|\varphi\| \le 1} \frac{|l(\varphi)|_{\mathbb{R}}}{\|\varphi\|_{\lambda^2;Lip}}.$$

So, $(C^*_{\lambda^2;Lip}(H), \|\cdot\|_{C^*_{\lambda^2;Lip}(H)})$ is a Banach space. Note that there is an injection

$$\begin{array}{rccl} i : & M_{\lambda^2}(H) & \hookrightarrow & C^*_{\lambda^2;Lip}(H) \\ & \mu & \mapsto & l^\mu \end{array}$$

for any $\varphi \in C_{\lambda^2;Lip}(H)$, there is a l^μ such that $\langle \mu, \varphi \rangle \triangleq \int_H \varphi(x)\mu(dx) = l^\mu(\varphi)$. It is easy to see that the left-hand aside above is well-defined from the definition of $C_{\lambda^2;Lip}(H)$ and $M_{\lambda^2}(H)$. We denote $(M_{\lambda^2}(H), \rho) \triangleq (M_{\lambda^2}(H), \|\cdot\|_{C^*_{\lambda^2;Lip}(H)})$, the metric subspace with the constraint topology on the subset $M_{\lambda^2}(H)$ of metric space $(C^*_{\lambda^2;Lip}(H), \|\cdot\|_{C^*_{\lambda^2;Lip}(H)})$. Obviously, $(\mathcal{P}_{\lambda^2}(H), \rho)$ defined above is a closed bounded subset of the closed unit ball of $(M_{\lambda^2}(H), \rho)$.

In addition, we remark that although $\|\cdot\|_{\mathcal{P}_{\lambda^2}(H)}$ makes no sense because $(\mathcal{P}_{\lambda^2}(H), \rho)$ is only a metric space, we prefer to use this notation and it implies the norm on $C^*_{\lambda^2;Lip}(H)$, i.e., $\|\cdot\|_{C^*_{\lambda^2;Lip}(H)}$, so is the case of $\|\cdot\|_{\mathcal{M}_{\lambda^2}(H)}$.

B. The optimal Control Problem

1) Controlled Semi-linear Mckean-Vlasov Stochastic Evolution Equations: Let us assume that $\mathcal{O}$ is a real separable Hilbert space with norm $|\cdot|_{\mathcal{O}}$, induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{O}}$ on it. Let U be a non-empty subset (maybe non-convex) of Hilbert space $\mathcal{O}$. We choose the set of all $\mathcal{F}_t$-adapted U-valued random process which satisfies the following condition

$$\sup_{t \in I} E|u_t|^2 < \infty,$$

as the admissible control set and denote it by $\mathcal{U}_{ad}$.

We discuss the following controlled semi-linear MV stochastic evolution equation

$$dx_t = Ax_t dt + b(t, x_t, P(x_t), u_t)dt + \sigma(t, x_t, P(x_t))dw_t, \quad (2)$$

where where $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $\{e^{At}\}$, $w_\cdot$ is a cylindrical Wiener process on real separable Hilbert space E, and $P(x_t)$ denotes the probability law induced by x_t itself. $u \in \mathcal{U}_{ad}$ is the control variable.

Let $b(\cdot, x, \mu, \cdot), \sigma(\cdot, x, \mu)$ be stochastic processes depending on $(x, \mu) \in H \times \mathcal{P}_{\lambda^2}(H)$, and satisfies the following assumptions:

Assumption 1. *1) There exist $C, L > 0$, such that for any $x, x_1, x_2 \in H, \mu, \mu_1, \mu_2 \in \mathcal{P}_{\lambda^2}(H), u \in U$,*

$$|b(t,x,\mu,u)|^2 + |\sigma(t,x,\mu)|^2_{\mathcal{L}^2(E,H)} \leq C(1 + |x|^2 + |\mu|^2_{\mathcal{P}_{\lambda^2}(H)}),$$
$$|b(t,x_1,\mu_1,u) - b(t,x_2,\mu_2,u)|^2 + |\sigma(t,x_1,\mu_1) - \sigma(t,x_2,\mu_2)|^2 \leq L(|x_1 - x_2|^2_H + \rho^2(\mu_1,\mu_2)),$$

hold uniformly with respect to $(t,u) \in I \times U$.
2) The infinitesimal generator A of a C_0-semigroup $\{e^{At}, t \geq 0\}$, on the Hilbert space H satisfying

$$\sup_{t\in I} \|e^{At}\|_{\mathcal{L}(H)} \leq M.$$

We have the following existence and uniqueness theorem for above controlled MV stochastic evolution equation (see [1] for the proof).

Theorem 1. *Suppose Assumption 1 holds. Then for every $\mathcal{F}_0$ measurable H-valued random variable $x_0 \in L^2_{\mathcal{F}_0}(\Omega; H)$, there exists a unique mild solution $x. \in \mathbb{S}_{\mathcal{F}}(I; H)$, the Banach space of all $\{\mathcal{F}_t\}$-adapted H-valued processes $\eta_t(\omega)$ having the norm $(\sup_{t\in I} E|\eta_t|^2)^{1/2} < \infty$, along with $P(x.) \in C(I, (\mathcal{P}_{\lambda^2}(H), \rho))$, satisfying the following Mckean-Vlasov semi-linear stochastic evolution equation*

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)} b(s, x_s, P(x_s), u_t) ds + \int_0^t e^{A(t-s)} \sigma(s, x_s, P(x_s)) dw_s. \qquad \square$$

2) Cost Functional: The cost functional is given by

$$J(u.) = E \int_I l(t, x_t, P(x_t), u_t) dt + Eg(x_T, P(x_T)), \tag{3}$$

where l is measurable in the first argument and continuous w.r.t the rest of arguments, g is continuous in their all arguments.

Problem: The optimal control problem is to minimize $J(u.)$, over $u \in \mathcal{U}_{ad}$, subject to (2) and the following final state constraint

$$EG(x_T, P(x_T)) = 0, \tag{4}$$

where $G : H \times \mathcal{P}_{\lambda^2}(H) \to \mathbb{R}^m$ is continuous in (x, μ).

C. The continuous Dependence Theorem

Let $(\mathbb{A}, d)$ be a complete metric space. For a given $a \in \mathbb{A}$, consider the mild solution equation

$$x^a_t = \int_0^t e^{A(t-s)} \underline{b}(s, x^a_s, P(x^a_s), a) d_s + \int_0^t e^{A(t-s)} \underline{\sigma}(s, x^a_s, P(x^a_s), a) dw_s, \tag{5}$$

where x^a is the state trajectory corresponding with the "control variable" a.

Assumption 2.

$$\underline{b}(t, x, \mu, a) : I \times H \times \mathcal{P}_{\lambda^2}(H) \times \mathbb{A} \to H,$$
$$\underline{\sigma}(t, x, \mu, a) : I \times H \times \mathcal{P}_{\lambda^2}(H) \times \mathbb{A} \to \mathcal{L}^2(E, H),$$

are joint Borel measurable and satisfy the Assumption 2 (replace u with a and U with $\mathbb{A}$ there). Further,

$$E \int_I |\underline{b}(t, x^{a_0}_t, P(x^{a_0}_t), a) - \underline{b}(t, x^{a_0}_t, P(x^{a_0}_t), a_0)|^2 dt \to 0,$$
$$E \int_I |\underline{\sigma}(t, x^{a_0}_t, P(x^{a_0}_t), a) - \underline{\sigma}(t, x^{a_0}_t, P(x^{a_0}_t), a_0)|^2 dt \to 0, \tag{6}$$

as $a \to a_0$ under the distance d.

Theorem 2. *Under Assumption 1-2, we have*

$$\lim_{d(a,a_0)\to 0} \sup_{t\in I} E|x^a_t - x^{a_0}_t|^2 = 0,$$
$$\lim_{d(a,a_0)\to 0} \sup_{t\in I} \rho^2(P(x^a_t), P(x^{a_0}_t)) = 0.$$

Proof. We only prove the case of

$$x^a_t = \int_0^t e^{A(t-s)} \underline{\sigma}(s, x^a_s, P(x^a_s), a) dw_s.$$

Let (x^a, x^{a_0}) be the state trajectory with respect to (a, a_0) respectively. Note that

$$E|x^a_t - x^{a_0}_t|^2 \leq 2MLE \int_0^t [|x^a_s - x^{a_0}_s|^2 + \rho^2(P(x^a_s), P(x^{a_0}_s))] ds + 2ME \int_0^t |\underline{\sigma}(s, x^{a_0}_s, P(x^{a_0}_s), a) - \underline{\sigma}(s, x^{a_0}_s, P(x^{a_0}_s), a_0)|^2 ds.$$

Fixed $t \in I$, from the definition of the distance ρ, we have

$$\begin{aligned} &\rho(P(x^a_t), P(x^{a_0}_t)) \\ &= \sup_{\|\varphi\|_{\lambda^2;Lip}\leq 1} \left\{ \int_H \varphi(x)\mu^a_t(dx) - \int_H \varphi(x)\mu^{a_0}_t(dx) \right\} \\ &= \sup_{\|\varphi\|_{\lambda^2;Lip}\leq 1} \left\{ E[\varphi(x^a_t) - \varphi(x^{a_0}_t)] \right\} \\ &\leq CE|x^a_t - x^{a_0}_t|_H, \end{aligned} \tag{7}$$

where μ^a, μ^{a_0} denotes the probability measure induced by $x^a_t, x^{a_0}_t$ respectively. Thus, Assumption 2 and Gronwall's inequality give the desired result. $\square$

III. Necessary Conditions of Optimality

Let (x^o, u^o) be an optimal solution of the problem (2-4). For any given $u \in \mathcal{U}_{ad}$, $t_0 \in [0, T), 0 \leq \theta \leq T - t_0$, define the spike variation

$$u^\theta_t = \begin{cases} u, & t \in [t_0, t_0 + \theta] \\ u^o_t, & otherwise. \end{cases}$$

Let x^θ be the state trajectory with respect to u^θ, we shall derive the variational equation blow.

A. Variational Equation

To start, we introduce the following assumption.

Assumption 3. *The drift $b = b(t, x, \mu, u)$ and the diffusion operator $\sigma = \sigma(t, x, \mu)$ are Borel measurable in all the arguments and once continuously $Fréchet$ differentiable in* *their second and third argument, and the Fréchet derivatives are uniformly bounded on $I\times H\times\mathcal{P}_{\gamma^2}(H)\times U$ and measurable in the uniform operator topology.*

From Assumption 3, we see $b_\mu(t,x,\mu,u),\sigma_\mu(t,x,\mu)\in C_{\lambda^2}(H)$, when we evaluate some value for μ. In order to derive relevant approximation below, we have to add an additional assumption.

Assumption 4. *Both of $b_\mu(t,x,\mu,u),\sigma_\mu(t,x,\mu)\in C_{\lambda^2}(H)$ also lie in $C^1_{\lambda^2}(H)$, which implies $Db_\mu(t,x,(\mu',x';\cdot),u),D\sigma_\mu(t,x,(\mu',x';\cdot))$ are bound and continuous in the argument $x'\in H$, where D denote the Fréchet derivation w.r.t. x'. (See the remark below)*

Remark 2. *We explain the notation appearing in the Assumption 4. In the Banach space $(M_{\lambda^2}(H),\|\cdot\|_{\lambda^2})$, we firstly calculate the Fréchet derivation of $\sigma(t,x,\mu)$ in μ. Let $\mu^\theta=P(x^\theta)$, $\mu^o=P(x^o)$. From the definition of Fréchet derivation (for simplicity, we use the Landau notation), we have*

$$\begin{aligned}\sigma(t,x,\mu^\theta)-\sigma(t,x,\mu^o)=\sigma_\mu(t,x,(\mu,\zeta))(\mu^\theta-\mu^o)\\ +o(\|\mu^\theta-\mu^o\|),\end{aligned}$$

where

$$\begin{aligned}&\sigma_\mu(t,x,(\mu,\zeta))(\mu^\theta-\mu^o)\\ =&\int_H\sigma_\mu(t,x,(\mu,\zeta))\mu^\theta(d\zeta)-\int_H\sigma_\mu(t,x,(\mu,\zeta))\mu^o(d\zeta)\\ =&\widetilde{E}[\sigma_\mu(t,x,(\mu,\widetilde{x}^\theta))-\sigma_\mu(t,x,(\mu,\widetilde{x}^o))]\\ =&\widetilde{E}[D\sigma_\mu(t,x,(\mu,\widetilde{x}^o;\widetilde{x}^\theta-\widetilde{x}^o))+o(|\widetilde{x}^\theta-\widetilde{x}^o|_H).\end{aligned}$$

$\forall\ h\in H$, letting $\delta x^o=x^\theta-x^o$, we know

$$\Big\langle h,\widetilde{E}[D\sigma_\mu(t,x,(\mu,\widetilde{x}^o;\delta\widetilde{x}^o))\Big\rangle_H$$

is bilinear in argument h and $\delta\widetilde{x}^o$. Thus, by Riesz' representation, there is an operator $\Sigma_t\in\mathcal{L}(H)$ such that

$$\Big\langle h,\widetilde{E}[D\sigma_\mu(t,x,(\mu,\widetilde{x}^o;\delta\widetilde{x}^o))\Big\rangle_H=\langle h,\Sigma_t\delta\widetilde{x}^o\rangle_H.$$

From here on, with a slight abuse of notation, we denote $\Sigma_t=\widetilde{E}D\sigma_\mu(t,x,\widetilde{\mu})(\widetilde{x}^o)$, similarly, we use the correspond notation about the case of b.

In the following, we introduce a linear equation called variational equation.

$$\begin{aligned}dz_t=&Az_tdt+\Big\{b_x(t,x^o_t,\mu^o_t,u^o_t))z_t\\ &+\widetilde{E}[(Db_\mu(t,x^o_t,\widetilde{\mu}^o_t,u^o_t)(\widetilde{x}^o_t))\widetilde{z}_t]\\ &+\theta^{-1}(b(t,x^o_t,\mu^o_t,u^\theta_t)-b(t,x^o_t,\mu^o_t,u^o_t))\Big\}dt\\ &+\Big\{\sigma_x(t,x^o_t,\mu^o_t)z_t+\widetilde{E}[(D\sigma_\mu(t,x^o_t,\widetilde{\mu}^o_t)(\widetilde{x}^o_t))\widetilde{z}_t]\Big\}dw_t,\end{aligned}\tag{8}$$

where $\mu^o_t=P(x^o_t)$ and the equation starts from the initial zero.

From the previous section, we have the existence and uniqueness of the solution of (8). Indeed, (8) is of the form

$$\begin{aligned}dz_t=&Az_tdt+\{\bar{b}^1_tz_t+\widetilde{E}[\bar{b}^2_t\widetilde{z}_t]+\phi_t\}dt\\ &+\{\bar{\sigma}^1_tz_t+\widetilde{E}[\bar{\sigma}^2_t\widetilde{z}_t]+\psi_t\}dw_t,\end{aligned}\tag{9}$$

where $\bar{b}^1,\bar{\sigma}^1,\phi$ and ψ are $\mathcal{F}_t$-progressively measurable processes and $\bar{b}^2,\bar{\sigma}^2$ are $(\mathcal{F}_t\bigotimes\mathcal{F})$-progressively measurable processes. So, even though we are dealing with possible random coefficients, the supremum norm defined on H can give it a similar estimation with Theorem 1.

Lemma 1.

$$\lim_{\theta\to0}\sup_{t\in I}E\Big|\frac{x^\theta_t-x^o_t}{\theta}-z_t\Big|^2=0.$$

Proof. This proof is redundant and for space limiting, we omit here. □

B. The Variational Inequality

In this section, we transform the optimal control problem of the final state constraint (2-4) using Ekeland's variational principle to a free final state optimal control problem and then obtain the variational inequality.

Lemma 2. *(Ekeland's variational principle) Let (V,d_V) be a complete metric space, and let $F:V\to R$ be lower semicontinuous and bounded from below. If for $\epsilon>0$, there exists $u\in V$ such that*

$$F(u)\le\inf_{v\in V}F(v)+\epsilon.$$

Then there exists $u^\epsilon\in V$, satisfying

1) $F(u^\epsilon)\le F(u)$,
2) $d_V(u^\epsilon,u)\le\epsilon^{1/2}$,
3) $F(v)+\epsilon^{1/2}d_V(u^\epsilon,v)>F(u^\epsilon),\quad\forall u^\epsilon\neq v.$ □

Let $(x^o_\cdot,u^o_\cdot)$ be an optimal solution of (2) and (3) under the final state constraint (4). We employ the following admissible subset and distance appearing in [4].

Fix $u\in U$, set

$$\begin{aligned}&V=\Big\{u_\cdot\in\mathcal{U}_{ad}\Big|\sup_{t\in I}E|u_t|^2\le\sup_{t\in I}E|u^o_t|^2+|u|^2_{\mathcal{O}}\Big\}.\\ &d_V(u^1_\cdot,u^2_\cdot)=\lambda\Big\{t\in I\Big|E|u^1_t-u^2_t|^2>0\Big\},\quad\forall u^1_\cdot,u^2_\cdot\in V,\end{aligned}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.

Proposition 1. *(V,d_V) is a complete metric space. J^ϵ is continuous and bounded on (V,d_V), where*

$$J^\epsilon(u)=\Big((J(u)-J(u^o)+\epsilon)^2+|EG(x_T,P(x_T))|^2_{\mathbb{R}^m}\Big)^{1/2},$$

for any $u\in V$ and $(x,P(x))$ is the mild solution of (2).

Proof. The first statement sees [4], we only prove the second one. Let (x^n,x) be the state trajectory w.r.t. $(u^n,u)\in(V,d_V)$. Based on the fact

$$|\mu^n_s|^2_{P_{\gamma^2}(H)}\le C\int_H(1+|x|_H)^2\mu^n_s(dx)\le CE(1+|x^n_s|_H)^2.$$

Hence, Gronwall inequality yields $\sup_{t\in I} E|x_t^n|^2 \le C$.

Then, the Lebesgue dominated convergence theorem ensure (6) holds and the continuous dependence theorem yields

$$\Big[\sup_{t\in I} E\Big|x_t^n - x_t\Big|^2\Big]^{1/2} \to 0;$$
$$\sup_{t\in I} \rho(P(x_t^n), P(x_t)) \to 0, \quad as\ n\to\infty.$$

Thus, the definition of $J^\epsilon(u)$ yields $J^\epsilon(u)$ is Cauchy continuous on V. Finally, the boundedness is obvious. □

Now, the optimal control problem of the final state constraint (2-4) is changed into the free final state optimal control problem: $\inf_{u\in V} J^\epsilon(u.)$, subject to the constraint (2).
Note that

$$0 \le \inf_{u\in V} J^\epsilon(u.) \le \inf_{u\in V} J^\epsilon(u^o.) = \epsilon,$$

Ekeland's variational principle holds and we have

$$\begin{aligned} &1)\ J^\epsilon(u^\epsilon) \le J^\epsilon(u^o) = \epsilon,\\ &2)\ d_V(u^\epsilon, u^o) \le \epsilon^{1/2},\\ &3)\ J^\epsilon(v) + \epsilon^{1/2} d_V(u^\epsilon, v) \ge J^\epsilon(u^\epsilon), \quad \forall v\in V. \end{aligned} \tag{10}$$

Via the spike variation, for any $\theta > 0$, we can construct a control $u^{\epsilon\theta} \in \mathcal{U}_{ad}$ such that $u^{\epsilon\theta} \in V$ as well, and $d_V(u^\epsilon, u^{\epsilon\theta}) \le \theta$. Let $x^\epsilon, x^{\epsilon\theta}$ be the state trajectory w.r.t. $u^\epsilon, u^{\epsilon\theta}$ respectively. In this case, we look u^ϵ as the "optimal control" and $u^{\epsilon\theta}$ as a spike perturbation of u^ϵ.

We rewrite the variational equation (8) as

$$\begin{aligned} dz_t^{\epsilon\theta} &= Az_t^{\epsilon\theta}dt + \Big\{b_x(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon))z_t^{\epsilon\theta}\\ &+ \widetilde{E}[(Db_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon, u_t^\epsilon)(\widetilde{x}_t^\epsilon))\widetilde{z}_t^{\epsilon\theta}]\\ &+ \theta^{-1}(b(t, x_t^\epsilon, \mu_t^\epsilon, u_t^{\epsilon\theta}) - b(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon))\Big\}dt\\ &+ \Big\{\sigma_x(t, x_t^\epsilon, \mu_t^\epsilon)z_t^{\epsilon\theta} + \widetilde{E}[(D\sigma_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon)(\widetilde{x}_t^\epsilon))\widetilde{z}_t^{\epsilon\theta}]\Big\}dw_t, \end{aligned} \tag{11}$$

where $\mu_t^\epsilon = P(x_t^\epsilon)$, and have the same result as Lemma 1, that is

$$\lim_{\theta\to 0}\sup_{t\in I} E\Big|\frac{x_t^{\epsilon\theta} - x_t^\epsilon}{\theta} - z_t^{\epsilon\theta}\Big|^2 = 0.$$

Assumption 5. *We assume that l, g, G are Gâteaux differentiable w.r.t. x, μ, and their derivatives l_x, l_μ and g_x, g_μ, G_x, G_μ are continuous w.r.t. (x, μ, u) and (x, μ) respectively and satisfy*

$$\begin{aligned} |l_x(t, x, \mu, u)|_H &\le C(1 + |x|_H + |\mu|_{\mathcal{P}_{\lambda^2}(H)}),\\ \|l_\mu(t, x, \mu, u)\|_{\lambda^2;Lip} &\le C(1 + |x|_H + |\mu|_{\mathcal{P}_{\lambda^2}(H)}),\\ &\forall\ (t, x, \mu, u) \in I\times H\times \mathcal{P}_{\lambda^2}(H)\times U;\\ |g_x(x, \mu)|_H &\le C(1 + |x|_H + |\mu|_{\mathcal{P}_{\lambda^2}(H)}),\\ \|g_\mu(x, \mu)\|_{\lambda^2;Lip} &\le C(1 + |x|_H + |\mu|_{\mathcal{P}_{\lambda^2}(H)}),\\ &\forall\ (x, \mu) \in H\times \mathcal{P}_{\lambda^2}(H);\\ |G_x|_H + \|G_\mu\|_{\lambda^2;Lip} &\le C, \forall\ (x, \mu)\in H\times\mathcal{P}_{\lambda^2}(H). \end{aligned}$$

In addition, we assume additionally $Dl_\mu(t, x, \mu', u)(x')$, $Dg_\mu(x, \mu')(x')$, $DG_\mu(x, \mu')(x')$ are bounded and continuous in x'.

Lemma 3. *(The variational inequality) Under the Assumption 1-5, we have*

$$\begin{aligned} &\alpha^\epsilon E\int_I \Big\{\langle l_x(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon), z_t^{\epsilon\theta}\rangle_H + \widetilde{E}\langle(Dl_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon, u_t^\epsilon)(\widetilde{x}_t^\epsilon))\\ &, \widetilde{z}_t^{\epsilon\theta}\rangle_H + \theta^{-1}[l(t, x_t^\epsilon, \mu_t^\epsilon, u_t^{\epsilon\theta}) - l(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon)]\Big\}dt\\ &+ E\Big\{\langle g_x(x_T^\epsilon, \mu_T^\epsilon), z_T^{\epsilon\theta}\rangle_H + \widetilde{E}\langle(Dg_\mu(x_T^\epsilon, \widetilde{\mu}_T^\epsilon)(\widetilde{x}_T^\epsilon)), \widetilde{z}_T^{\epsilon\theta}\rangle_H\Big\}\\ &+ \langle\beta^\epsilon, E\Big\{G_x(x_T^\epsilon, \mu_T^\epsilon)z_T^{\epsilon\theta} + \widetilde{E}[(DG_\mu(x_T^\epsilon, \widetilde{\mu}_T^\epsilon)(\widetilde{x}_T^\epsilon))\widetilde{z}_T^{\epsilon\theta}]\Big\}\rangle_{\mathbb{R}^m}\\ &\ge O(\theta) - \epsilon^{1/2}, \end{aligned}$$

where $\lim_{\theta\to 0} O(\theta) = 0$, *and* $\lim_{\epsilon\to 0}\alpha^\epsilon = \alpha$, $\lim_{\epsilon\to 0}\beta^\epsilon = \beta$, *satisfy that* $|\alpha|_{\mathbb{R}} + |\beta|_{\mathbb{R}^m} = 1$.

Proof. The proof of this lemma depends on the Taylor's expansion and Lemma 2. For space limiting, we omit it. □

C. Adjoint Equation

Define

$$\begin{aligned} F^\epsilon : L^2_{\mathcal{F}}(I\times\Omega; H)\times L^2_{\mathcal{F}}(I\times\Omega; \mathcal{L}^2(E, H)) &\to \mathbb{R}\\ (\phi\ ,\ \psi) &\mapsto L(z^{\epsilon\theta}) \end{aligned}$$

where (ϕ, ψ) comes from (9) (just change the state variable z into $z^{\epsilon\theta}$ there), its concrete content is given by (11) and

$$\begin{aligned} L(z^{\epsilon\theta}) \triangleq\ &\alpha^\epsilon E\int_I \Big\{\langle l_x(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon), z_t^{\epsilon\theta}\rangle_H\\ &+ \widetilde{E}\langle(Dl_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon, u_t^\epsilon)(\widetilde{x}_t^\epsilon)), \widetilde{z}_t^{\epsilon\theta}\rangle_H\Big\}dt\\ &+ E\Big\{\langle g_x(x_T^\epsilon, \mu_T^\epsilon), z_T^{\epsilon\theta}\rangle_H\\ &+ \widetilde{E}\langle(Dg_\mu(x_T^\epsilon, \widetilde{\mu}_T^\epsilon)(\widetilde{x}_T^\epsilon)), \widetilde{z}_T^{\epsilon\theta}\rangle_H\Big\}\\ &+ E\Big\{\langle(\beta^\epsilon)^T G_x(x_T^\epsilon, \mu_T^\epsilon), z_T^{\epsilon\theta}\rangle_H\\ &+ \widetilde{E}\langle(\beta^\epsilon)^T(DG_\mu(x_T^\epsilon, \widetilde{\mu}_T^\epsilon)(\widetilde{x}_T^\epsilon)), \widetilde{z}_T^{\epsilon\theta}\rangle_H\Big\}. \end{aligned} \tag{12}$$

It is not hard to see the mapping F^ϵ is a continuous linear functional on the Hilbert space $L^2_{\mathcal{F}}(I\times\Omega; H)\times L^2_{\mathcal{F}}(I\times\Omega; \mathcal{L}^2(E, H))$. Thus, by means of Riesz's representation, we know there is a unique

$$(P^\epsilon, Q^\epsilon) \in L^2_{\mathcal{F}}(I\times\Omega; H)\times L^2_{\mathcal{F}}(I\times\Omega; \mathcal{L}^2(E, H)),$$

such that

$$F^\epsilon(\phi, \psi) = E\int_I \langle P_t^\epsilon, \phi_t\rangle_H dt + E\int_I \langle Q_t^\epsilon, \psi_t\rangle_{\mathcal{L}^2(E,H)}dt.$$

Lemma 4. *(The adjoint equation) Under the Assumption 1-5, the adjoint pair (P^ϵ, Q^ϵ) defined above satisfies the following Mean field backward stochastic evolution equation:*

$$\begin{aligned} -dP_t^\epsilon &= A^* P_t^\epsilon dt + \alpha^\epsilon l_x(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon)dt\\ &+ \alpha^\epsilon\widetilde{E}[(Dl_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon, u_t^\epsilon)(\widetilde{x}_t^\epsilon))]dt\\ &+ (b_x^*(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon)P_t^\epsilon dt \end{aligned} \tag{13}$$

$$+ \widetilde{E}[(Db_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon, u_t^\epsilon)(\widetilde{x}_t^\epsilon))^* \widetilde{P}_t^\epsilon])dt$$
$$+ (\sigma_x^*(t, x_t^\epsilon, \mu_t^\epsilon)Q_t^\epsilon$$
$$+ \widetilde{E}[(D\sigma_\mu(t, x_t^\epsilon, \widetilde{\mu}_t^\epsilon)(\widetilde{x}_t^\epsilon))^* \widetilde{Q}_t^\epsilon])dt - Q_t^\epsilon dw_t,$$

where B^* denotes the adjoint of the operator B and

$$P_T^\epsilon = g_x(x_T^\epsilon, \mu_T^\epsilon) + \widetilde{E}(Dg_\mu(\widetilde{x}_T^\epsilon, \mu_T^\epsilon)(x_T^\epsilon)) + (\beta^\epsilon)^T \times G_x(x_T^\epsilon, \mu_T^\epsilon) + (\beta^\epsilon)^T \widetilde{E}(DG_\mu(\widetilde{x}_T^\epsilon, \mu_T^\epsilon)(x_T^\epsilon)). \quad (14)$$

Notice that in this lemma all operators l_x, Dl_μ, b_x^* etc, are evaluated values in the state-control triple $(x^\epsilon, \mu^\epsilon, u^\epsilon)$.

Proof. We only provide an outline. Apply Itô formula (just justify by using Yosida approximation) to $\langle P_t^\epsilon, z_t^{\epsilon\theta}\rangle$ in $[0,T]$, and then take expectation. Finally, compare with the variational inequality in Lemma 4.

$$\alpha^\epsilon E\int_I \theta^{-1}(l(t, x_t^\epsilon, \mu_t^\epsilon, u_t^{\epsilon\theta}) - l(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon))dt$$
$$+E\int_I \langle \theta^{-1}(b(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\theta) - b(t, x_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon)), P_t^\epsilon\rangle dt$$
$$\geq O(\theta) - \epsilon^{1/2}. \quad (15)$$

□

Remark 3. *On the existence and uniqueness of the solution of the mean field backward stochastic evolution equation* (13)*, let us consider the following equation*

$$-dP_t = AP_t dt + \widetilde{E}h(t, P_t, Q_t, \widetilde{P}_t, \widetilde{Q}_t)dt - Q_t dw_t, \quad (16)$$

where

$$\widetilde{E}h(t, P_t, Q_t, \widetilde{P}_t, \widetilde{Q}_t) = \int_{\widetilde{\Omega}} h(t, \omega, \widetilde{\omega}, P_t(\omega), Q_t(\omega), P_t(\widetilde{\omega}), Q_t(\widetilde{\omega}))P(d\widetilde{\omega}).$$

We assume that there exists a constant $C \geq 0$ such that, $\overline{P}$-a.s., for all $t \in I$, $p_1, p_2, \widetilde{p}_1, \widetilde{p}_2 \in H$, $q_1, q_2, \widetilde{q}_1, \widetilde{q}_2 \in \mathcal{L}(E, H)$,

1. $|h(t, p_1, q_1, \widetilde{p}_1, \widetilde{q}_1) - h(t, p_2, q_2, \widetilde{p}_2, \widetilde{q}_2)|_H \leq C(|p_1 - p_2|_H + \|q_1 - q_2\|_{\mathcal{L}^2(E,H)} + |\widetilde{p}_1 - \widetilde{p}_2|_H + \|\widetilde{q}_1 - \widetilde{q}_2\|_{\mathcal{L}^2(E,H)})$,
2. $h(\cdot, 0, 0, 0, 0) \in L^2_{\mathcal{F}}(I, H)$.

Under the assumption above, we can check that $h(t, P_t, Q_t, \widetilde{P}_t, \widetilde{Q}_t) \in L^2_{\mathcal{F}}(I \times \Omega, H)$, and $\widetilde{E}h(t, P_t, Q_t, \widetilde{P}_t, \widetilde{Q}_t) \in L^2_{\mathcal{F}}(I \times \Omega, H)$. Thus, from the Theorem 3.1 in [5], we know there exists a unique mild solution pair $(P, Q) \in L^2_{\mathcal{F}}(I \times \Omega, H) \times L^2_{\mathcal{F}}(I \times \Omega, \mathcal{L}^2(E, H))$ satisfies (16).

D. The maximum Principle

In this section, we give the maximum principle. Let

$$-dP_t = A^* P_t dt + \alpha l_x(t, x_t, \mu_t, u_t)dt$$
$$+ \alpha\widetilde{E}[(Dl_\mu(t, x_t, \widetilde{\mu}_t, u_t)(\widetilde{x}_t))]dt$$
$$+ (b_x^*(t, x_t, \mu_t, u_t)P_t dt \quad (17)$$
$$+ \widetilde{E}[(Db_\mu(t, x_t, \widetilde{\mu}_t, u_t)(\widetilde{x}_t))^* \widetilde{P}_t])dt$$
$$+(\sigma_x^*(t, x_t, \mu_t)Q_t + \widetilde{E}[(D\sigma_\mu(t, x_t, \widetilde{\mu}_t)(\widetilde{x}_t))^* \widetilde{Q}_t])dt - Q_t dw_t,$$

where B^* denotes the adjoint of the operator B, and

$$P_T = g_x(x_T^o, \mu_T^o) + \widetilde{E}(Dg_\mu(\widetilde{x}_T^o, \mu_T^o)(x_T^o)) + \beta^T G_x(x_T^o, \mu_T^o) + \beta^T \widetilde{E}(DG_\mu(\widetilde{x}_T^o, \mu_T^o)(x_T^o)),$$

Finally, we derive the maximum principle which concludes this paper.

Theorem 3. *(The maximum principle) Let (x^o, μ^o, u^o) be an optimal solution of the problem (2)-(4). Suppose the Assumption 1-5 hold. Then there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^m$, with $|\alpha|^2_{\mathbb{R}} + |\beta|^2_{\mathbb{R}^m} = 1$, such that*

1. $(P, Q) \in L^2_{\mathcal{F}}(I \times \Omega; H) \times L^2_{\mathcal{F}}(I \times \Omega; \mathcal{L}^2(E, H))$ *is a unique solution of* (17).
2. $H(t, x_t^o, \mu_t^o, u) \geq H(t, x_t^o, \mu_t^o, u_t^o),\ a.e., a.s.\ \forall\ u \in U,$

where $\mu_t^o = P(x_t^o), t \in I$ and

$$H(t, x_t, \mu_t, u_t) = \alpha l(t, x_t, \mu_t, u_t) + \langle b(t, x_t, \mu_t, u_t), P_t\rangle_H.$$

Proof. We provide only an outline. Let (P, Q) be the solution of the equation (17), we first show

$$\lim_{\epsilon\to 0}\sup_{t\in I} E|P_t^\epsilon - P_t|^2_H = 0,$$
$$\lim_{\epsilon\to 0}\|Q_t^\epsilon - Q_t\|^2_{L_{\mathcal{F}}(I\times\Omega; L^2(E,H))} = 0.$$

Then, let $\theta \to 0$ in (15) (remember the definition of the spike variation), we have

$$\alpha^\epsilon\Big(l(t_0, x_{t_0}^\epsilon, \mu_{t_0}^\epsilon, u_{t_0}) - l(t_0, x_{t_0}^\epsilon, \mu_{t_0}^\epsilon, u_{t_0}^\epsilon)\Big) + \Big\langle b(t_0, x_{t_0}^\epsilon, \mu_{t_0}^\epsilon, u_{t_0}^\theta) - b(t_0, x_{t_0}^\epsilon, \mu_{t_0}^\epsilon, u_{t_0}^\epsilon), P_{t_0}^\epsilon\Big\rangle \geq -\epsilon^{1/2}.$$

Finally, let $\epsilon \to 0$ and we obtain the terminal result. □

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REFERENCES

[1] NU Ahmed. Optimal control of general mckean-vlasov stochastic evolution equations on hilbert spaces and necessary conditions of optimality. *Discussiones Mathematicae, Differential Inclusions, Control and Optimization*, 35(2):165–195, 2015.

[2] NU Ahmed and X Ding. A semilinear mckean-vlasov stochastic evolution equation in hilbert space. *Stochastic Processes and Their Applications*, 60(1):65–85, 1995.

[3] Jürgen Gärtner. On the mckean-vlasov limit for interacting diffusions. *Mathematische Nachrichten*, 137(1):197–248, 1988.

[4] Ying Hu and Shige Peng. Maximum principle for semilinear stochastic evolution control systems. *Stochastics and Stochastic Reports*, 33(3-4):159–180, 1990.

[5] Ying Hu and Shige Peng. Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Analysis and Applications*, 9(4):445–459, 1991.

[6] NI Mahmudov and Mark A McKibben. Abstract second-order damped mckean-vlasov stochastic evolution equations. *Stochastic analysis and applications*, 24(2):303–328, 2006.

[7] Henry P McKean. A class of markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Sciences*, 56(6):1907–1911, 1966.