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Optimal Regulation of Linear Discrete-Time Systems with Multiplicative Noises

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Abstract: This paper studies optimal regulation problem for networked linear discrete-time systems with fading channel. The uncertainties in fading channels are modeled as multiplicative noises. The regulation performance is measured by a quadratic function. The optimal state feedback is designed by the mean-square stabilization solution to a modified Algebraic Riccati equation (MARE). The necessary and sufficient condition to the existence of the mean-square stabilization is presented in terms of the inherent characterizations of the systems. It is a nature extension for the result in standard optimal discrete-time linear quadratic regulation (LQR) problem. We also show that this optimal state feedback design problem is an eigenvalue problem (EVP). And then a design algorithm is developed for this optimal control problem.

Key Words: Optimal Control, Multiplicative noise, Linear Regulation, Algebraic Riccati equation

1 INTRODUCTION

During last two decades, networked control system has attracted many research interests. In these works, the main issues include modeling communication channel uncertainties such as data rate limits, quantization errors, channel fading, data package drop, channel delays etc., analyzing design constraints on feedback systems caused by these uncertainties and developing feedback control design methods for networked systems. It is shown that the multiplicative noise model may be an efficient way in modeling uncertainties which appear in communication channels in feedback systems, such as, packet loss ([23], [24] and [32]), quantization errors ([20], [26]), fading channels [7] and etc. These motive further research in networked control and stochastic control areas. Since linear time invariant systems with multiplicative noises involve nonlinearities, several issues in stabilization and optimal control problems for the systems are still opened meanwhile LQG control theory has been well established for LTI systems with additive noises.

The studies for LTI systems with multiplicative noises can be traced back to the later of 1960's. In 1971, Willems and Blankenship [28] formulated the mean-square stability problem for linear time-invariant (LTI) SISO feedback systems with multiplicative noises, respectively, and presented the sufficient and necessary condition of the stability for the systems. These results are extended to MIMO systems in [16]. In [13], [14] and [17], the criterion of mean-square stability is studied for LTI systems with multiplicative noises. And then, the mean-square stabilization via state feedback is studied for the systems (for example see [1] and [29]). The optimal control is another issue which has been studied widely. The earlier works were reported in [30] and [31] where optimal regulation problem is formulated based on a quadratic cost function. It is shown in these works that the optimal state feedback is determined by a positive semi-definite solution to a modified algebraic Riccati equation (MARE). After then, the necessary and sufficient condition for this optimal control design problem is presented in terms of the solvability of a linear matrix inequality (LMI) in [19]. In [6] and [34], the relation between the existence of the solution to the optimal regulation problem and the mean-square stabilizability, observability is studied. The sufficient conditions are presented for the optimal design in several cases.

In [7], Elia revisited the mean-square stabilization problem for a MIMO system with multiplicative noises which are used to model communication channel uncertainties in networked systems. Sinopoli *et. al.* [23], [24] studied Kalman filtering with packet loss in communication channels, where the channel uncertainty caused by packet loss is modeled as a multiplicative noise. Sufficient conditions are presented for the convergence of the Kalman filter with intermittent observations. Xiao *et. al.* [32] studied the stabilization problem for networked systems with packet loss and presented an explicit connection between mean-square stabilization and signal-to-noise ratios of control communication channels. In networked setting, we [20], [22], [26] formulated the optimal tracking problem with quantization error by applying multiplicative noise models and solved optimal tracking problem via output feedback for a minimum phase LTI system.

In this paper, networked feedback systems with quantization effects and communication channels are considered. The quantization errors and/or uncertainties in communication channels are modeled by multiplicative noises. By this model, we formulates an optimal regulation control problem in mean-square sense for the networked systems. Following the stochastic small gain theorem [16], it is turned out that the optimal state feedback gain in this problem is determined by a positive semi-definite solution, refereed as to mean-square stabilization solution in this work, to a MARE. And the necessary and sufficient condition for the existence of this solution is presented. It extends the standard opti-

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mal linear quadratic regulation (LQR) state feedback design for LTI systems to LTI systems with multiplicative noises. Moreover, it is found that the optimal LQR design problem is an eigenvalue problem, i.e., the global optimal solution exists and can be solved by a set of linear matrix inequalities (LMI) and line search technique. The remainder of this paper is organized as follows. In Section 2, we formulate the optimal design problems for an LTI system with multiplicative noises. In Section 3, optimal regulation design via state feedback is discussed. Section 4 concludes the paper.

The notation used throughout this paper is fairly standard. We denotes the set of real n-dimensional vector space by R^n. Denote the transpose of a matrix by $(\cdot)^T$, and rank of a matrix by Rank$(\cdot)$, respectively. Denote the mathematical expectation operator by E$(\cdot)$ and the spectral radius by $\rho(\cdot)$, respectively. We denote block-diagonal matrix formed from the arguments by diag$\{\cdots\}$. For conjugate symmetric matrix X, Y, the notation $X > Y$(respectively, X$\geq Y$) is used to denote $X - Y$ is positive definite (positive semidefinite).

2 Problem Formulation

This paper study optimal regulation control problem for networked LTI systems with quantization effects (or input multiplicative noises). The networked feedback system under study is shown in Fig. 1. In the system, P is a plant, K is a state feedback controller, i.e, $u = Fx$. The matrix F is a state feedback gain to be designed and $x \in R^n$ is the state of the plant. The channel is modeled as a quantization law and noise free communication channel. The control signal u_q from R^m is quantized in the sending side and the quantized control signal u_q is received at the receiving side perfectly.

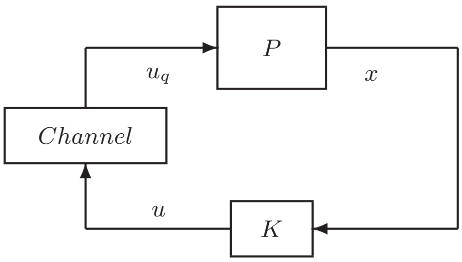

Fig. 1: The networked system with quantization effects and a state noise

The model of the plant with quantizers (or input multiplicative noises) is given as below:

$$x(k+1) = Ax(k) + Bu_q(k). \tag{1}$$

The quantization errors $d(k)$ of the control signal $u(k)$ are modeled with multiplicative noises $\omega_1, \cdots, \omega_m$, i.e.,

$$d(k) = u_q(k) - u(k) = \omega(k)u(k) \tag{2}$$

where $\omega = \text{diag}\{\omega_1, \cdots, \omega_m\}$, diag $\{*, \cdots, *\}$ is a diagonal matrix and $*$ is an entry in the diagonal of the matrix. ω is referred to as the relative quantization error.

Assumption 1 The noises $\omega_i(k), i \in \{1, \cdots, m\}$ are mutually independent white noise processes with

$$E\{\omega_i(k)\} = 0,\ E\{\omega_i(k_1)\omega_i(k_2)\} = \begin{cases} \sigma_i^2, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases},$$

for $i \neq j$,

$$E\{\omega_i(k_1)\omega_j(k_2)\} = 0.$$

Let the initial state of the plant $x(0)$ be a random variable with zero mean. Its covariance is $n \times n$ identity matrix. The performance of the system is measured by the cost function J_{LQR}

$$J_{LQR} = \mathbf{E}\sum_{k=0}^{\infty}\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \tag{3}$$

where $Q \geq 0$ and $R \geq 0$.

The optimal regulation control problem is to find the optimal state feedback law $u = Fx$ such that the cost function J_{LQR} is minimized, i.e., to find $F_{2,opt}$

$$F_{2,opt} = arg\inf_F J_{LQR}. \tag{4}$$

The closed-loop system of the plant with a state feedback controller and quantized control inputs are diagrammed as the system shown in Fig. 2 where ω is the relative quantization error given by the model (2). The nominal closed-loop

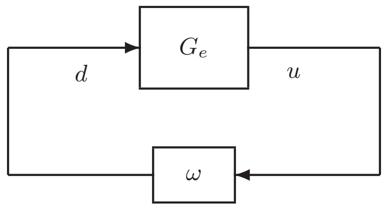

Fig. 2: The closed-loop system with multiplicative noises

system G_e is partitioned as below

$$G_e = \begin{bmatrix} G_{z0} & G_{z1} & \cdots & G_{zm} \\ G_{10} & G_{11} & \cdots & G_{1m} \\ & \cdots & \cdots & \\ G_{m0} & G_{m1} & \cdots & G_{mm} \end{bmatrix} \tag{5}$$

which is compatible to the sizes of signals $v(k), d(k), z(k)$ and $u(k)$ in the closed-loop system.

Now, define the mean-square stability for the closed-loop systems in Fig. 1, and the mean-square stabilizibility for the plants where the quantization errors in the control inputs are modeled as multiplicative input noises by Assumption 1, respectively.

Definition 1 It is referred to as that the closed-loop system shown in Fig. 2 is mean-square stable if, for any bounded initial state of the system with zero inputs, the covariance of the state is convergent to zero.

Definition 2 It is referred to as that a plant above is mean-square stabilizable, if there exists a state control law $u(k) = Fx(k)$ such that the resultant closed-loop system is mean-square sable, i.e., for any bounded initial state $x(0)$ with zero input, $\lim_{k\to\infty} E\{x(k)^T x(k)\} = 0$.

To study the mean-square stability of the closed-loop system, let

$$G = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ & \cdots & \\ G_{m1} & \cdots & G_{mm} \end{bmatrix}. \tag{6}$$

Then, let

$$\hat{G} \triangleq \begin{bmatrix} \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \ddots & \\ \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix} \tag{7}$$

and

$$\Sigma \triangleq \operatorname{diag}\left\{\sigma_1^2, \cdots, \sigma_m^2\right\}. \tag{8}$$

Lemma 1 (see [16]) Suppose that the nominal closed-loop system G_e of the system shown in Fig. 2 is stable. The system is mean-square stable if and only if the spectral radius of the matrix $\hat{G}\Sigma$ is less than one, i.e.,

$$\rho(\hat{G}\Sigma) < 1. \tag{9}$$

3 Optimal Regulation Control

In this section, the optimal regulation control is studied for the plant (1). From the quantization error model (2), the plant is decomposed to the nominal plant

$$x(k+1) = Ax(k) + Bd(k) + Bu(k) \tag{10}$$
$$z(k) = \begin{bmatrix} Q^{\frac{1}{2}}x(k) \\ R^{\frac{1}{2}}u(k) \end{bmatrix}$$

and the multiplicative noise part

$$d(k) = \omega(k)u(k). \tag{11}$$

From the diagram in Fig. 2, we can see that the nominal part G_e in the closed-loop system of the plant (1) with a state feedback controller K is determined by the nominal plant (10) and the state feedback gain F. It is given by

$$G_e = \begin{bmatrix} C_F(zI - A_F)^{-1} & C_1(zI - A_F)^{-1}B_1 & \cdots & \cdots & C_1(zI - A_F)^{-1}B_m \\ F_1(zI - A_F)^{-1} & F_1(zI - A_F)^{-1}B_1 & \cdots & \cdots & F_1(zI - A_F)^{-1}B_m \\ & \cdots & \cdots & \cdots & \\ F_m(zI - A_F)^{-1} & F_m(zI - A_F)^{-1}B_1 & \cdots & \cdots & F_m(zI - A_F)^{-1}B_m \end{bmatrix} \tag{12}$$

where $A_F = A + BF$, $C_F = \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}F \end{bmatrix}$, B_i and F_i are i-th column of B and i-th row of F, respectively.

Suppose the initiate state $x(0)$ of the system is a random vector with zero mean. The covariance of $x(0)$ is the identity matrix. The cost function J_{LQR} of the closed-loop system is obtained as follows:

Lemma 2 The cost function J_{LQR} of the closed-loop system is given by

$$J_{LQR} = \|G_{z0}\|_2^2 + \begin{bmatrix} \|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2 \end{bmatrix} \Sigma(I - \hat{G}\Sigma)^{-1} \times \begin{bmatrix} \|G_{10}\|_2^2 \\ \vdots \\ \|G_{m0}\|_2^2 \end{bmatrix}. \tag{13}$$

Proof is presented in Appendix .

Let

$$\hat{G}_e = \begin{bmatrix} \|G_{z0}\|_2^2 & \|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2 \\ \|G_{10}\|_2^2 & \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \cdots & \cdots & \\ \|G_{m0}\|_2^2 & \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix}. \tag{14}$$

Lemma 3 Consider the closed-loop system shown in Fig. 2 with given $\sigma_1, \cdots, \sigma_m$. For any given $\sigma_0 > 0$, it holds that

$$J_1 < \frac{1}{\sigma_0^2} \quad \text{and} \quad \rho(\hat{G}\Sigma) < 1 \tag{15}$$

if and only if there exists a $\sigma_0 > 0$ so that

$$\rho(\hat{G}_e\Sigma_e) < 1$$

where $\Sigma_e \triangleq \operatorname{diag}\left\{\sigma_0^2, \sigma_1^2, \cdots, \sigma_m^2\right\}$.

Remark 1 Lemma 3 is a general version of Theorem 4.1 in [16] which is available in the case when v and z are scalars.

From Lemma 3, we can see that the spectral radius $\rho(\hat{G}_e\Sigma_e)$ of the matrix $\hat{G}_e\Sigma_e$ plays key role in the optimal state feedback design. Hence, the feature of positive matrix is needed.

Lemma 4 (see [12]) For any square matrix T, if all entries of the matrix are greater than zero, then its spectral radius $\rho(T)$ (or largest eigenvalue) is given as follows

$$\rho(T) = \inf_{\Gamma} \|\Gamma T \Gamma^{-1}\|_\infty \triangleq \inf_{\Gamma} \max_{j} \sum_{i} \gamma_i^2 t_{ij} \frac{1}{\gamma_j^2} \tag{16}$$

where $\Gamma = \operatorname{diag}\left\{\gamma_1^2, \cdots, \gamma_m^2\right\} > 0$.

To study the optimal state feedback design for the regulation problem, the standard optimal H_2 state feedback design is reviewed for the plant (17)

$$x(k+1) = Ax(k) + Bu(k) \tag{17}$$
$$z(k) = \begin{bmatrix} C \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ D \end{bmatrix} u(k).$$

Lemma 5 The discrete algebraic Riccati equation (18) has the unique stabilizing solution $X \geq 0$ (i.e., all eigenvalues of $A + BF$ with $F = -(D^TD + B^TXB)^{-1}BXA$ are in the open unit disc) if and only if (A, B) is stabilizable and (A, C) has no unobservable pole in the unit circle,

$$X = A^TXA - A^TXB(D^TD + B^TXB)^{-1}BXA + CC^T. \tag{18}$$

The proof follows [18] and [27].

Lemma 6 The minimum H_2 norm of the closed-loop system (17) via state feedback and optimal state feedback gain are given by

$$\min_F \left\| \begin{bmatrix} C \\ DF \end{bmatrix} (zI - A - BF)^{-1}B \right\|_2^2 = \operatorname{tr}\left\{B^TXB\right\}$$

and

$$F = -(D^TD + B^TXB)^{-1}BXA,$$

respectively, where X is a positive semi-definite solution to (18).

The proof follows that of Theorem 6.4.1 in [3].

New, these lemmas are generalized to the plant (1)-(2). Let X be a modified algebraic Riccati equation (MARE) (19). If the state feedback law with the feedback gain F given by (20) stabilizes the plant (1)-(2) in mean-square sense, X is referred as to a mean-square stabilizing solution to the MARE.

Theorem 1 For given $\sigma_1, \cdots, \sigma_m$, the MARE (19) has a mean-square stabilizing solution

$$\begin{aligned} X =& A^T XA + Q \\ & - A^T XB \left(\Phi + B^T XB + R\right)^{-1} B^T XA \end{aligned} \quad (19)$$

where $\Phi = \mathrm{diag}\left\{\sigma_1^2 B_1^T XB_1, \cdots, \sigma_m^2 B_m^T XB_m\right\}$ if and only if the plant is mean-square stabilizable, $(A, R^{\frac{1}{2}})$ has no unobservable in the unit circle.

Furthermore, the optimal H_2 state feedback gain for the plant (10-11) is given by

$$F = -\left(\Phi + B^T XB + R\right)^{-1} B^T XA \quad (20)$$

where X is a positive semi-definite solution to (19)

The minimum regulation cost of the plant (10-11) via state feedback is

$$J_{LQR} = \mathrm{tr}\left\{X\right\}.$$

In the light of the proof for Theorem 1, this theorem can be extended straightforwardly to the case where the plant (1) has state dependent multiplicative noises ω_{si}, $i = 1, \cdots, m_s$ as below:

$$x(k+1) = Ax(k) + \sum_{i=1}^{m_s} A_i \omega_{si}(k) + B_2 u_q(k) \quad (21)$$

where the matrixes A_i, $i = 1, \cdots, m_s$ have rank one. The noises ω_{si}, $i = 1, \cdots, m_s$ are *i.i.d.* processes with zero mean and variances σ_{si}, respectively. Moreover, ω_{si}, $i = 1, \cdots, m_s$, ω_i, $i = 1, \cdots, m$ and v are mutually independent.

Corollary 1 The MARE (22) has a mean-square stabilizing solution

$$\begin{aligned} X =& A^T XA + \sum_{i=1}^{m_s} \sigma_{si}^2 A_i^T XA_i + Q \\ & - A^T XB \left(\Phi + B^T XB + R\right)^{-1} B^T XA. \end{aligned} \quad (22)$$

if and only if the plant is mean-square stabilizable via state feedback and $\left(A, \begin{bmatrix} Q^{\frac{1}{2}} \\ A_1 \\ \vdots \\ A_{m_s} \end{bmatrix}\right)$ has no unobservable pole in the unit circle.

The optimal state feedback gain in the optimal regulation problem is given by

$$F = -\left(\Phi + B^T XB + R\right)^{-1} B^T XA$$

and the minimum regulation cost J_{LQR} is given by

$$J_{LQR} = \mathrm{tr}\left\{X\right\}.$$

Following the proof of Theorem 1, we have the optimal design algorithm for the optimal state feedback design problem as below:

Theorem 2 The optimal state feedback gain F for the system (10-11) is given by

$$F = -(\Gamma_{opt} + B^T X_{opt} B + R)^{-1} B^T X_{opt} A$$

where X_{opt} and Γ_{opt} are the solution to the following inequalities when σ_0 is maximized,

$$\begin{bmatrix} A^T XA - X + Q & A^T XB \\ B^T XA & \Gamma + B^T XB \end{bmatrix} \geq 0, \qquad X \geq 0 \quad (23)$$

and

$$\mathrm{tr}\{X\} \leq \frac{1}{\sigma_0^2}, \qquad B_i^T XB_i \leq \frac{1}{\sigma_i^2} e_i^T \Gamma e_i, \qquad i = 1, \cdots, m. \quad (24)$$

4 Conclusions

In this paper, the optimal regulation control via state feedback for discrete-time systems with quantization effects has been studied. Under the multiplicative noise model, the necessary and sufficient condition is presented for the solvability of this optimal control problem. Furthermore, we have shown that this optimal control problem is a generalized eigenvalue problem. It can be solved by LMI and line search techniques.

5 Appendix: Proof of Lemma 2

Let $\{G_e(0), G_e(1), G_e(2), \cdots\}$ be impulse response associated with the transfer function G_e in (5). Following the partition of G_e in (5), we write $G_e(k)$, $k = 0, 1, 2, \cdots$ as below:

$$G_e(k) = \begin{bmatrix} g_{z0}(k) & g_{z1}(k) & \cdots & g_{zm}(k) \\ g_{10}(k) & g_{11}(k) & \cdots & g_{1m}(k) \\ \vdots & \vdots & \ddots & \vdots \\ g_{m0}(k) & g_{m1}(k) & \cdots & g_{mm}(k) \end{bmatrix}.$$

Note the fact that the nominal plant (10) has a strict proper function from input (v, d, u) to output (z, y). The transfer function G_e of the plant with a proper feedback control law is also strict proper. Hence, it holds that $g_{ij}(0) = 0$, $i = 1, \cdots, m$, $j = 0, \cdots, m$.

When the initial state of the system in Fig. 2 is at rest, the output of the system is given by

$$\begin{bmatrix} z(k) \\ u(k) \end{bmatrix} = \sum_{i=0}^{k} G_e(i) \begin{bmatrix} v(k-i) \\ d(k-i) \end{bmatrix} \quad (25)$$

with $d(k-i) = \omega(k-i)u(k-i)$.

Subsequently, the output $z(k)$ is written a summation as below:

$$z(k) = z_v(k) + \sum_{j=1}^{m} z_{d_j}(k) \quad (26)$$

where

$$z_v(k)=\sum_{i=0}^{k}G_{zd}(i)v(k-i) \quad (27)$$

$$z_{d_j}(k)=\sum_{i=0}^{k}G_{zj}(i)d_j(k-i),\ j=1,\cdots,m.$$

Following Assumption 1 and assumption on v, we obtain that

$$\mathbf{E}z^T(k)z(k)=\mathbf{E}z_v^T(k)z_v(k)+\sum_{j=1}^{m}\mathbf{E}z_{d_j}^T(k)z_{d_j}(k). \quad (28)$$

Denote the autocorrelation matrixes and power spectral density of z_v by $R_{zv}(\tau)$ and S_{zv}, i.e.

$$R_{zv}(\tau)=\lim_{k\to\infty}\frac{1}{k+1}\sum_{i=0}^{k}\mathbf{E}z_v(i+\tau)z_v^T(i).$$

From the definition of the averaged power, it holds that

$$\mathbf{E}\|z_v\|_p^2=\mathrm{tr}\left\{R_{zv}(0)\right\}=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{tr}\left\{S_{zv}(e^{j\omega})\right\}d\omega. \quad (29)$$

Noting (27), we rewrite (29) as below:

$$\mathbf{E}\|z_v\|_p^2=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{tr}\left\{G_{z0}(e^{j\omega})S_v(e^{j\omega})G_{z0}^*(e^{j\omega})\right\}d\omega.$$

Since the power spectral density S_v of the signal v is the identity matrix, it holds that

$$\mathbf{E}\|z_v\|_p^2=\|G_{z0}\|_2^2. \quad (30)$$

Denote the power spectral densities of z_{d_j} and u_j by S_{d_j} and S_{u_j}, respectively. Following the process on the discussion above, we obtain that

$$\mathbf{E}\|z_{d_j}\|_p^2=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{tr}\left\{G_{zj}(e^{j\omega})S_{d_j}(e^{j\omega})G_{zj}^*(e^{j\omega})\right\}d\omega. \quad (31)$$

Note the fact that

$$S_{d_j}(e^{j\omega})=\sum_{\tau=-\infty}^{\infty}R_{d_j}(\tau)e^{-j\tau\omega} \quad \text{and} \quad d_j(k)=\omega_j(k)u_j(k). \quad (32)$$

It follows from Assumption 1 that

$$R_{d_j}(0)=\lim_{k\to\infty}\frac{1}{k+1}\sigma_j^2\mathbf{E}\sum_{i=0}^{k}u_j^2(i)=\sigma_j^2\mathbf{E}\|u_j\|_p^2 \quad (33)$$

and

$$R_{d_j}(\tau)=0,\ \tau=1,2,\cdots. \quad (34)$$

Substituting (33) and (34) into (32) yields that

$$S_{d_j}(e^{j\omega})=\sigma_j^2\mathbf{E}\|u_j\|_p^2. \quad (35)$$

Subsequently, (31) is rewritten as

$$\mathbf{E}\|z_{d_j}\|_p^2=\sigma_j^2\|G_{zj}\|_2^2\mathbf{E}\|u_j\|_p^2. \quad (36)$$

With (28), (30) and (36), we have that

$$\mathbf{E}\|z\|_p^2=\|G_{z0}\|_2^2+\sum_{i=1}^{m}\sigma_i^2\|G_{zi}\|_2^2\mathbf{E}\|u_i\|_p^2. \quad (37)$$

In the light of the discussion above, we can obtain straightforwardly that

$$\mathbf{E}\|z_{u_j}\|_p^2=\|G_{j0}\|_2^2+\sum_{i=1}^{m}\sigma_i^2\|G_{ji}\|_2^2\mathbf{E}\|u_i\|_p^2,\quad j=1,\cdots,m \quad (38)$$

or

$$\begin{bmatrix}\mathbf{E}\|z_{u_1}\|_p^2\\ \vdots\\ \mathbf{E}\|z_{u_m}\|_p^2\end{bmatrix}=\begin{bmatrix}\mathbf{E}\|G_{10}\|_p^2\\ \vdots\\ \mathbf{E}\|G_{m0}\|_p^2\end{bmatrix}+\hat{G}\Sigma\begin{bmatrix}\mathbf{E}\|z_{u_1}\|_p^2\\ \vdots\\ \mathbf{E}\|z_{u_m}\|_p^2\end{bmatrix}. \quad (39)$$

From (37) and (39), we obtain (13).

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