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Optimal Filtering for Networked Systems with Markovian Communication Delays

HAN Chunyan[1], ZHANG Huanshui[2], FU Minyue[3]

1. School of Electrical Engineering, University of Jinan, Jinan 250022, Shandong, P. R. China
E-mail: cyhan823@hotmail.com

2. School of Control Science and Engineering, Shandong University, Jinan 250061, Shandong, P. R. China
E-mail: hszhang@sdu.edu.cn

3. School of Electrical Engineering and Computer Science, University of Newcastle, Callaghan, NSW 2308, Australia
E-mail: minyue.fu@newcastle.edu.au

Abstract: This paper is concerned with the optimal filter problems for networked systems with random transmission delays, while the delay process is modeled as a multi-state Markov chain which incorporates the data losses naturally. By defining a delay-free observation sequence, the optimal filter problems are transformed into the ones of the standard Markov jumping parameter measurement system. We first present an optimal Kalman filter, which is with time-varying, path-dependent filter gains, and the number of the paths grows exponentially in time delay. Thus an alternative optimal Markov jump linear filter is presented, in which the filter gains just depend on the present value of the Markov chain, and as a result, the obtained filter is again a Markov jump linear system. It can be shown that the proposed Markov jump linear filter converges to the constant-gain filter under appropriate assumptions.

Key Words: optimal filtering, Markov jump linear filtering, discrete-time systems, random communication delays, packet dropouts

1 Introduction

Control over networks, or the so called networked control systems (NCSs), has become a widely used technology in control area and has attracted significant attention in the past few years (see [1] and the references therein). In networked systems, data travel through different networks and communication channels from the sensors to the controller and from controller to actuators. Despite of many advantages that communication networks have, such as low cost, reduced weight and power requirements, simple installation and maintenance, when the data is transmitted over the communication networks with finite bandwidth, random delay and packet dropout are two inevitable problems in networked systems. That is, some packets may suffer a time-delay during transmission through the network or they maybe completely lost due to some reasons such as data collisions, transmission errors and network congestion. These phenomena in data transmission over the network, will deteriorate the performance of the controllers and filters if they are not considered in the design procedure. Hence nowadays the filter and controller design for networked control systems with the consideration of various uncertainties in data transmission in the network has attracted significant interest; see [2],[3].

Up to now, many researchers have studied the filter design for the packet delay or packet dropout cases. For the problem of packet dropout, the initial work can be traced back to Nahi [4] and Hadidi [5], where the phenomena of packet losses is described as the observation uncertainties by a scalar binary random variable. More recently, this problem has been studied using intermittent observation models [6], [7]. In [6] and [7], the stability of the Kalman filter in relation to the data arrival rate is investigated. It is shown that there exists a critical data arrival rate for an unstable system so that the mean filtering error covariance will be bounded for any initial condition. In [8], an optimal H_2 filtering in networked control systems with multiple packet dropout is considered, where the random dropout is represented by two Bernoulli distributed white sequences which taking the values of 0 or 1, and the filter is derived by a convex optimization problem through a set of linear matrix inequalities (LMIs). In [9], Sun and Xie have presented a multiple packet dropout modeling method, and an optimal linear estimator was computed recursively in terms of the solution of a Riccati difference equation. In [10] and [11], the robust filtering and nonlinear H_∞ filtering are designed for the multiple missing measurement systems via the LMI techniques, respectively. The measurement output contains randomly missing data that is modeled by a Bernoulli distributed white sequence with a known conditional probability.

For the problem of random communication delays, several results have been presented under different modeling method and performance index. In the case of observations transmitted to the estimator with irregular times, a recursive linear minimum variance state estimator was proposed via the state augmentation method [12]. Latter, the same estimation problem was considered in networked control systems [13], where the minimum variance state estimator and optimal sensor control strategy were obtained. For the situation that the one-step sensor delay was described as a binary white noise sequence, a reduced-order linear unbiased estimator was designed via state augmentation in [14]. As for the random delay characterized by a set of distributed Bernoulli variables, the Kalman filtering [15], the unscented filtering algorithm [16], the linear and quadratic least-square estimation method [17], the robust filtering [18], and the H_∞ filtering [19], [20] have been developed. The rational of modeling the random delay as Bernoulli variable sequences

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has been justified in those papers. On the other hand, modeling the random delay as a finite state Markov chain is also a reasonable way. The relevant estimation results for this type of modeling can be found in [21], and the references therein. In a very recent study, the optimal filtering [22], and the H_2 filtering [23] problems associated respectively with possible delay of one or more sampling period, uncertain observations and multiple packet dropouts are studied under a unified framework, respectively. It can be noted that the results mentioned above mainly focus on the systems just with one or two step delays. To our best knowledge, there exist few estimation results on multiple random delayed systems [21], [22], [15], especially for the systems with multiple Markovian transmission delays and packet dropouts simultaneously. Further, few results considered the convergence and stability analysis of the designed filters [15]. This motivates us to study this interesting and challenging problem, which has great potential in practical applications.

This paper investigates the optimal filtering problems for networked systems with random transmission delays. The delay process is characterized by a multi-state Markov chain which incorporating the case of packet dropouts naturally. A new delay-free observation sequence is defined by rearranging the received observations up to the present time, and the information contained in the new defined observations is equivalent to that of the original ones. Then the filtering problems are converted into the ones of the standard Markov jumping parameter measurement systems, and the jumping parameter has the same statistic properties as the random delays. An optimal linear mean square filter is first presented based on the innovation analysis method, in which the filter gains is time-varying and sample path dependent, and the sample paths grows exponentially in time delay. Then high computation is required in the filter design, and the convergence analysis to this filter is difficult. Alternatively, an optimal Markov jump linear filter is presented, in which the filter gains are just dependent on the present value of the Markov chains, but not the entire mode history. And at each time, just $\bar{r}$ filter gains are derived, which result in the obtained filter is a Markov jump linear system. It can be shown that the Markov jump linear filter is convergent and can be approximated by a stationary filter with constant gains under appropriate assumptions.

The remainder of this paper is organized as follows. In section 2, we present the problem formulations and some preliminaries. In section 3, the optimal linear mean square filter is designed based on the innovation analysis method, which is with time-varying and sample path dependent filter gains. In section 4, an alternative Markov jump linear filter is presented by using the mean square method, in which the filter gains just determined by the present value of the Markov chain, and thus less pre-computation is required. Finally, the conclusions are drawn in section 5 with some final comments.

Notations: Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the norm bounded linear space of all $m \times n$ matrices with $\mathbb{B}(\mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. For $L \in \mathbb{B}(\mathbb{R}^n)$, L' stands for the transpose of L. As usual, $L \geq 0 (L > 0)$ will mean that the symmetric matrix $L \in \mathbb{B}(\mathbb{R}^n)$ is positive semi-definite (positive definite), respectively. We set $\mathbb{B}(\mathbb{R}^n)^+ = \{L \in \mathbb{B}(\mathbb{R}^n); L = L' \geq 0\}$. Moreover, tr(.) indicates the trace operator, E(.) denotes the mathematical expectation operator, and Prob(.) means the occurrence probability of an event.

2 Problem Formulations and Preliminaries

Consider the following discrete-time systems

$$x(k+1) = Ax(k) + Cw(k), x(0) = x_0, \quad (1)$$
$$z(k) = Hx(k) + Gv(k), \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the state sequence, $z(k) \in \mathbb{R}^m$ is the output sequence, $w(k) \in \mathbb{R}^p$ is the system noise, and $v(k) \in \mathbb{R}^q$ is the output noise. The initial state $x_0, w(k)$ and $v(k)$ are null mean second-order independent wide sense stationary sequences with covariance matrices V, I_p and I_q, respectively. $x_0, w(k)$ and $v(k)$ are mutually independent, and $GG' > 0$.

The measurement $z(k)$ is time-stamped, and transmitted through a digital communication network (DCN), whose goal is to deliver packets from a source to a destination. The DCNs are in general very complex, and thus time delay or even packet dropout is unavoidable between the senders and the receivers. Moreover, the network-induced delays and packet dropouts are often random. Let $r(k)$ denote the transmission delay of the measurement $z(k)$, where $r(k)$ is of a Markov process, and takes values in a finite state space $\{0, 1, \cdots, \bar{r}, \infty\}$. When $r(k) = i (i = 0, \cdots, \bar{r})$, it means that $z(k)$ will be received within $\bar{r}$ time steps. If the measurement transmitted to the receiver with a delay larger than $\bar{r}$, it will be considered as the one lost completely. And for this case, the random delay is set to be ∞. Denote the transition probability matrix of $r(k)$ as $\Lambda = [(\lambda_{ij})]$, where $\lambda_{ij} \triangleq \text{Prob}(r(k+1) = j | r(k) = i)(i, j = 0, \cdots, \bar{r}, \infty)$, and set $\pi(k) = [\pi_0(k) \ \cdots \pi_{\bar{r}}(k), \pi_\infty(k)]'$ with $\pi_i(k) \triangleq \text{Prob}(r(k) = i)(i = 0, \cdots, \bar{r}, \infty)$, then $\pi(k)$ and Λ satisfy the Kolmogorov difference equation $\pi(k+1) = \Lambda' \pi(k)$. Further, we introduce the indicator function of $r(k)$ as follows

$$\phi_{k,i} = \begin{cases} 1, & \text{if } r(k) = i; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

for $i = 0, 1, \cdots, \bar{r}$. As is well known, in the real-time control system, the output $z(k)$ can only be observed at most one time, and thus $\phi_{k,i} (i = 0, 1, \cdots, \bar{r})$ must satisfy the following property:

$$\phi_{k,i} \times \phi_{k,j} = 0, i \neq j. \quad (4)$$

The relation (4) includes two cases:

$$\phi_{k,i} + \cdots + \phi_{k,\bar{r}} = 1, \quad (5)$$

or

$$\phi_{k,i} + \cdots + \phi_{k,\bar{r}} = 0. \quad (6)$$

Based on the above statement, we know that the possible received observations up to time k are

$$\gamma_s^k z(s) = \gamma_s^k Hx(s) + \gamma_s^k Gv(s), 0 \leq s \leq k, \quad (7)$$

where

$$\gamma_s^k = \begin{cases} 1, & \text{if } z(s) \text{ arrives before or at time } k; \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

From this definition it follows that

$$\gamma_s^k = \sum_{i=0}^{\bar{r}} \phi_{s,i}, 0 \le s \le k - \bar{r}, \quad (9)$$

$$\gamma_s^k = \sum_{i=0}^{k-s} \phi_{s,i}, k - \bar{r} < s \le k, \quad (10)$$

For the convenience of discussions, we denote $y_s^k = \gamma_s^k z(s)$, then the optimal filtering problems studied in this paper can be stated as follows:

Problem 1 (Optimal linear mean square filter) Given the observations $\{y_s^k|_{0\le s\le k}\}$ and $\{\gamma_s^k|_{0\le s\le k}\}$, find an optimal linear mean square filter $\hat{x}_o(k|k)$ of the state $x(k)$.

Problem 2 (Optimal Markov jump filter) Given the observations $\{y_s^k|_{0\le s\le k}\}$ and the present value of γ_k^k, find an optimal recursive Markov jump linear filter $\hat{x}_e(k|k)$ of the state $x(k)$.

3 Optimal Linear Mean Square Filter

In this section, we will design the optimal linear minimum mean square error (LMMSE) filter $\hat{x}_o(k|k)$ of the state $x(k)$ based on the projection formula, where

$$\hat{x}_o(k|k) \triangleq \mathrm{E}\{x(k)|y_0^k, \cdots, y_k^k; \gamma_0^k, \cdots, \gamma_k^k\}. \quad (11)$$

We now present the following definition.

Definition 1 *Consider the given time instant k. For $0 \le s \le k$, the LMMSE estimator of $x(s)$ is defined as*

$$\hat{x}_o(s|s-1) \triangleq E\{x(s)|y_0^k, \cdots, y_{s-1}^k; \gamma_0^k, \cdots, \gamma_{s-1}^k\}, \quad (12)$$

while its error covariance matrix $P(s|s-1)$ is defined as

$$P(s|s-1) \triangleq E\{(x(s) - \hat{x}_o(s|s-1))(x(s) - \hat{x}_o(s|s-1))'|y_0^k, \cdots, y_{s-1}^k; \gamma_0^k, \cdots, \gamma_{s-1}^k\}. \quad (13)$$

From the the projection formula, we will obtain the LMMSE estimation result as follows.

Theorem 1 *Consider the system (1), (7), and recalling the definition of γ_s^k in (9), (10), the LMMSE estimation $\hat{x}_o(k|k)$ is given by*

$$\hat{x}_o(k|k) = \hat{x}_o(k|k-1) + \phi_{k,0}K(k)[z(k) - H\hat{x}_o(k|k-1)], \quad (14)$$

where $K(k)$ is the solution to the following equation

$$K(k) = P(k|k-1)H'(HP(k|k-1)H' + GG')^{-1}, \quad (15)$$

and the estimation $\hat{x}_o(k|k-1)$ is computed by the following steps.

- *Step 1 Calculate $\hat{x}_o(s|s-1), 0 \le s \le k - \bar{r}$ by the following Kalman recursion*

$$\hat{x}_o(s+1|s) = A\hat{x}_o(s|s-1) + \sum_{i=0}^{\bar{r}} \phi_{s,i}K(s) \times[z(s) - H\hat{x}_o(s|s-1)], \quad (16)$$

where

$$K(s) = AP(s|s-1)H'(HP(s|s-1)H' + GG')^{-1},$$

with

$$P(s+1|s) = AP(s|s-1)A' - \sum_{i=0}^{\bar{r}} \phi_{s,i}K(s) \times HP(s|s-1)A' + CC'. \quad (17)$$

- *Step 2 Calculate $\hat{x}_o(s+1|s), k - \bar{r} < s \le k$ by the following recursion*

$$\hat{x}_o(s+1|s) = A\hat{x}_o(s|s-1) + \sum_{i=0}^{k-s} \phi_{s,i}K(s) \times[z(s) - H\hat{x}_o(s|s-1)], \quad (18)$$

where

$$K(s) = AP(s|s-1)H'(HP(s|s-1)H' + GG')^{-1},$$

with

$$P(s+1|s) = AP(s|s-1)A' - \sum_{i=0}^{k-s} \phi_{s,i}K(s) \times HP(s|s-1)A' + CC'. \quad (19)$$

- *Step 3 Set $s+1 = k$ in (18), then $\hat{x}_o(k|k-1)$ is obtained directly.*

Proof. As in the derivation of the Kalman filter, we first define the innovation sequences

$$e_s^k = y_s^k - \hat{y}_s^k, 0 \le s \le k, \quad (20)$$

where $\hat{y}_s^k$ is the LMMSE estimation of y_s^k given the observations

$$\{y_0^k, \cdots, y_{s-1}^k, \gamma_0^k, \cdots, \gamma_{s-1}^k\}.$$

In view of the projection formula and the definition of the innovation sequences (20), we will obtain (14)-(19) immediately. □

Remark 1 *In practice, the optimal estimator corresponds to a time-varying Kalman filter for the system (1) and (7), since all the values of the modes of operations $\gamma_0^k, \cdots, \gamma_k^k$ are known at time k. The recursive equation for the covariance error matrix $P(k|k-1)$ and the gain of the filter $K(k)$ are sample path dependent, and the number of sample paths grows exponentially subject to $\bar{r}$. Up to time k, it would be necessary to pre-computed $2^{\bar{r}}$ gains. Thus, in the next section, we will present an alternative optimal filter $\hat{x}_e(k|k)$ which just depends on the present value of the γ_k^k rather than on the entire past history of modes $\gamma_0^k, \cdots, \gamma_k^k$. Then the optimal filter in this form requires much less pre-computed gains ($\frac{\bar{r}(\bar{r}+1)}{2}$ instead of $2^{\bar{r}}$).*

4 Optimal Markov Jump Linear Filter

We consider in this section the optimal Markov jump linear filter in the recursive form for systems (1) and (7). First, we present the following definition.

Definition 2 *Consider the given time instant k, the optimal Markov jump linear filter $\hat{x}_e(k|k,k)$ of $x(k)$ is defined as*

$$\hat{x}_e(k|k,k) \triangleq E\{x(k)|y_0^k, \cdots, y_k^k; \gamma_k^k\}, \quad (21)$$

and for $0 \leq s \leq k$, the optimal Markov jump linear filter $\hat{x}_e(s|s-1,s-1)$ of $x(s)$ is defined as

$$\hat{x}_e(s|s-1,s-1) \triangleq E\{x(s)|y_0^k, \cdots, y_{s-1}^k; \gamma_{s-1}^k\}. \quad (22)$$

In view of Definition 2, the optimal Markov jump filters (21) and (22) can be written as

$$\begin{aligned} \hat{x}_e(k|k) &= \hat{x}_e(k|k-1) - F(k,\gamma_k^k)[\gamma_k^k z(k) \\ &\quad -\gamma_k^k H\hat{x}_e(k|k-1)], \quad (23) \\ \hat{x}_e(s+1|s) &= A\hat{x}_e(s|s-1) - F(s,\gamma_s^k)[\gamma_s^k z(s) \\ &\quad -\gamma_s^k H\hat{x}_e(s|s-1)], 0 \leq s \leq k. \quad (24) \end{aligned}$$

where $F(k,\gamma_k^k)$ and $F(s,\gamma_s^k)$ are the filter gains which need to be determined.

It follows from (9) and (10) that the Markov jump linear filter in the form of (23)-(24) can be rewritten recursively as

$$\begin{aligned} \hat{x}_e(k|k) &= \hat{x}_e(k|k-1) - F(k,\phi_{k,0})[\phi_{k,0} z(k) \\ &\quad -\phi_{k,0} H\hat{x}_e(k|k-1)], \quad (25) \\ \hat{x}_e(s+1|s) &= A\hat{x}_e(s|s-1) - F(s,\phi_{s,i})[\phi_{s,i} z(s) \\ &\quad -\phi_{s,i} H\hat{x}_e(s|s-1)], \quad (26) \\ &\quad 0 \leq s \leq k-\bar{r}, i = 0,1,\cdots,\bar{r}, \\ \hat{x}_e(s+1|s) &= A\hat{x}_e(s|s-1) - F(s,\phi_{s,i})[\phi_{s,i} z(s) \\ &\quad -\phi_{s,i} H\hat{x}_e(s|s-1)], \quad (27) \\ &\quad k-\bar{r} < s \leq k, i = 0,1,\cdots,k-s, \end{aligned}$$

The goal is to determine the filter gains $F(k,\gamma_k^k)$, $F(s,\gamma_s^k)(0 \leq s \leq k)$, such that its estimation error covariance is minimized.

Define

$$\begin{aligned} \tilde{x}_e(k|k) &= x(k) - \hat{x}_e(k|k), \quad (28) \\ Y_0^{(0)}(k|k) &= \mathrm{E}\{\tilde{x}_e(k|k)\tilde{x}_e(k|k)'\phi_{k,0}\}, \quad (29) \\ \tilde{x}_e(s|s-1) &= x(s) - \hat{x}_e(s|s-1), 0 \leq s \leq k, \quad (30) \\ Y_i^{(\bar{r})}(s) &= \mathrm{E}\{\tilde{x}_e(s|s-1)\tilde{x}_e(s|s-1)'\phi_{s,i}\}, \quad (31) \\ &\quad 0 \leq s \leq k-\bar{r}, i = 0,1,\cdots,\bar{r}, \\ Y_i^{(k-s)}(s) &= \mathrm{E}\{\tilde{x}_e(s|s-1)\tilde{x}_e(s|s-1)'\phi_{s,i}\} \quad (32) \\ &\quad k-\bar{r} < s \leq k, i = 0,1,\cdots,k-s. \end{aligned}$$

Then the desired filter $\hat{x}_e(k|k)$ is obtained in the following result.

Theorem 2 *Consider the system (1) and (7), and given time $k > \bar{r}$, the minimum mean square error solution to the Markov jump linear filter $\hat{x}_e(k|k)$ is given by*

$$\begin{aligned} \hat{x}_e(k|k) &= \hat{x}_e(k|k-1) - F_0^{(0)}(k)[\phi_{k,0} z(k) \\ &\quad -\phi_{k,0} H\hat{x}_e(k|k-1)], \quad (33) \end{aligned}$$

where the filter gain $F_0^{(0)}(k) = F^{(0)}(k,\phi_{k,0})$ for $\phi_{k,0} = 1$, which is determined by

$$F_0^{(0)}(k) = -Y_0^{(0)}(k)H(HY_0^{(0)}(k)H' + \pi_0(k)GG')^{-1}, \quad (34)$$

and $\hat{x}_e(k|k-1)$ and $Y_0^{(0)}(k)$ are computed by the following steps:

- *Step 1 Calculate $\hat{x}_e(s|s-1)$ and $Y_i^{(\bar{r})}(s)$ for $0 \leq s \leq k-\bar{r}$, $i = 0,1,\cdots,\bar{r}$*

$$\begin{aligned} \hat{x}_e(s+1|s) &= A\hat{x}_e(s|s-1) - F_i^{(\bar{r})}(s)[\phi_{s,i} z(s) \\ &\quad -\phi_{s,i} H\hat{x}_e(s|s-1)], \quad (35) \end{aligned}$$

where the filter gain $F^{(\bar{r})}(s,\phi_{s+i,i}) = F_i^{(\bar{r})}(s)$ for $\phi_{s,i} = 1$, which is calculated by

$$\begin{aligned} F_i^{(\bar{r})}(s) &= -AY_i^{(\bar{r})}(s)H'(HY_i^{(\bar{r})}(s)H' \\ &\quad +\pi_i(s)GG')^{-1}, \quad (36) \end{aligned}$$

and $Y_i^{(\bar{r})}(s)$ satisfies the following coupled Riccati difference equation

$$\begin{aligned} &Y_j^{(\bar{r})}(s) \\ =& \sum_{i=0}^{\bar{r}} \lambda_{ij}\{AY_i^{(\bar{r})}(s)A' + \pi_i(s)CC' - AY_i^{(\bar{r})}(s)H' \\ &\times(HY_i^{(\bar{r})}(s)H' + \pi_i(s)GG')^{-1}HY_i^{(\bar{r})}(s)A'\} \quad (37) \end{aligned}$$

- *Step 2 Calculate $\hat{x}_e(s+1|s)$ and $Y_i^{(k-s)}(s)$ for $k-\bar{r} < s \leq k$, $i = 0,1,\cdots,k-s$*

$$\begin{aligned} \hat{x}_e(s+1|s) &= A\hat{x}_e(s|s-1) - F_i^{(k-s)}(s)[\phi_{s,i} z(s) \\ &\quad -\phi_{s,i} H\hat{x}_e(s|s-1)], \quad (38) \end{aligned}$$

where the filter gain $F^{(k-s)}(s,\phi_{s,i}) = F_i^{(k-s)}(s)$ for $\phi_{s,i} = 1$, which is calculated by

$$\begin{aligned} F_i^{(k-s)}(s) &= -AY_i^{(k-s)}(s)H'(HY_i^{(k-s)}(s)H' \\ &\quad +\pi_i(s)GG')^{-1}, \quad (39) \end{aligned}$$

and $Y_i^{(k-s)}(s)$ satisfies the following coupled Riccati difference equation

$$\begin{aligned} &Y_j^{(k-s-1)}(s+1) \\ =& \sum_{i=0}^{k-s} \lambda_{ij}\{AY_i^{(k-s)}(s)A' + \pi_i(s)CC' - A \\ &\times Y_i^{(k-s)}(s)H'(HY_i^{(k-s)}(s)H' + \pi_i(s)GG')^{-1} \\ &\times HY_i^{(k-s)}(s)A'\}. \quad (40) \end{aligned}$$

- *Step 3 Set $s+1 = k$ in Step 2, then $\hat{x}_e(k|k-1)$ and $Y_0^{(0)}(k)$ is obtained from (38) and (40), respectively.*

Proof: First, for $s = k$, from (1) and (33), and recalling the definition of $\tilde{x}_e(k|k)$, we have that

$$\begin{aligned} \tilde{x}_e(k|k) &= [I + \phi_{k,0}F^{(0)}(k,\phi_{k,0})H]\tilde{x}_e(k|k-1) \\ &\quad +\phi_{k,0}F^{(0)}(k,\phi_{k,0})Gv(k), \tilde{x}_e(0|0) = x_0, \end{aligned}$$

so the estimation error covariance matrix

$$\begin{aligned} Y_0^{(0)}(k|k) &\triangleq \mathrm{E}\{\tilde{x}_e(k|k)\tilde{x}_e(k|k)'\phi_{k,0}\} \\ &= [I + F_0^{(0)}(k)H]Y_0^{(0)}(k)[I + F_0^{(0)}(k)H]' \\ &\quad +\pi_0(k)F_0^{(0)}(k)GG'F_0^{(0)}(k)'. \quad (41) \end{aligned}$$

Seen from (41) that $Y_0^{(0)}(k)$ will be minimized if and only if

$$F_0^{(0)}(k) = -Y_0^{(0)}(k)H(HY_0^{(0)}(k)H' + \pi_0(k)GG')^{-1},$$

the minimum mean square jump filter gain (34) is obtained.

Next, for $0 \leq s \leq k-\bar{r}$, we have from (1), (35) and note the definition of $\tilde{x}_e(s|s-1)$ that

$$\begin{aligned}\tilde{x}_e(s|s-1) &= (A+\phi_{s+i,i}F^{(\bar{r})}(s,\phi_{s+i,i})H)\tilde{x}_e(s|s-1)\\ &\quad +Cw(s)+\phi_{s+i,i}F^{(\bar{r})}(s,\phi_{s+i,i})Gv(s).\end{aligned} \tag{42}$$

In view of (31) and (42), we get that

$$\begin{aligned}Y_j^{(\bar{r})}(s+1) &\triangleq \mathrm{E}\{\tilde{x}_e(s+1|s)\tilde{x}_e(s+1|s)\phi_{s+1,j}\}\\ &= \sum_{i=0}^{\bar{r}}\lambda_{ij}\{(A+F_i^{(\bar{r})}(s)H)Y_i^{(\bar{r})}(s)\\ &\quad (A+F_i^{(\bar{r})}(s)H)'+\pi_i(s)CC'\\ &\quad +\pi_i(s)F_i^{(\bar{r})}(s)GG'F_i^{(\bar{r})}(s)'\}.\end{aligned} \tag{43}$$

Obviously, the minimum mean square error filter gain subject to (35) is that

$$F_i^{(\bar{r})}(s) = -AY_i^{(\bar{r})}(s)H'(HY_i^{(\bar{r})}(s)H'+\pi_i(s)GG')^{-1},$$

and thus (43) becomes as in (37).

Finally, following the similar procedure as in the determination of $F_i^{(\bar{r})}(s)$ and $Y_i^{(\bar{r})}(s)$ for $0 \leq s \leq k-\bar{r}$, we will obtain (39) and (40)immediately. This completes the proof of Theorem 2. □

In the following, we will show that the filter presented in Theorem 2 is the optimal realization of the general recursive Markov jump linear filters. We assume the general Markov jump linear filter is with the form

$$\hat{x}_u(k|k) = \hat{A}(k,\phi_{k,0})\hat{x}_u(k|k-1)+\hat{B}(k,\phi_{k,0})z(k), \tag{44}$$

$$\begin{aligned}\hat{x}_u(s+1|s) &= \hat{A}(s,\phi_{s,i})\hat{x}_u(s|s-1)+\hat{B}(s,\phi_{s,i})z(s),\\ \hat{x}_u(0|0) &= 0, 0\leq s\leq k-\bar{r}, i=0,1,\cdots,\bar{r},\end{aligned} \tag{45}$$

$$\begin{aligned}\hat{x}_u(s+1|s) &= \hat{A}(s,\phi_{s,i})\hat{x}_u(s|s-1)+\hat{B}(s,\phi_{s,i})z(s),\\ &\quad k-\bar{r}<s\leq k, i=0,1,\cdots,k-s.\end{aligned} \tag{46}$$

Denote

$$\tilde{x}_u(k|k) = x(k)-\hat{x}_u(k|k), \tag{47}$$
$$\tilde{x}_u(s|s-1) = x(s)-\hat{x}_u(s|s-1), 0\leq s\leq k, \tag{48}$$

then the following result will be obtained.

Theorem 3 *Let $\tilde{x}_u(s|s-1) (0 \leq s \leq k)$, $Y_i^{(\bar{r})}(s), (0 \leq s \leq k-\bar{r}, i=0,1,\cdots,\bar{r})$ and $Y_i^{(k-s)}(s), (k-\bar{r} < s \leq k, i = 0,1,\cdots,k-s)$ be as in (48), (37), and (40), respectively. Then for $0 \leq s \leq k-\bar{r}$,*

$$E\{\|\tilde{x}_u(s|s-1)\|^2\} \geq \sum_{i=0}^{\bar{r}} tr\{Y_i^{(\bar{r})}(s)\}, \tag{49}$$

and for $k-\bar{r} < s \leq k$,

$$E\{\|\tilde{x}_u(s|s-1)\|^2\} \geq \sum_{i=0}^{k-s} tr\{Y_i^{(k-s)}(s)\}. \tag{50}$$

Proof: As in the proof of Theorem 5.3 in [24], we can show that

$$\begin{aligned}&\mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_u(s|s-1)'\phi_{s,i}\} = 0, &(51)\\ &0\leq s\leq k-\bar{r}, i=0,1,\cdots,\bar{r},\\ &\mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_e(s|s-1)'\phi_{s,i}\} = 0, &(52)\\ &0\leq s\leq k-\bar{r}, i=0,1,\cdots,\bar{r},\\ &\mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_u(s|s-1)'\phi_{s,i}\} = 0, &(53)\\ &k-\bar{r}<s\leq k, i=0,1,\cdots,k-s,\\ &\mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_e(s|s-1)'\phi_{s,i}\} = 0, &(54)\\ &k-\bar{r}<s\leq k, i=0,1,\cdots,k-s.\end{aligned}$$

Then for $0 \leq s \leq k-\bar{r}$, we have from (51) that

$$\begin{aligned}&\mathrm{E}\{\|\tilde{x}_u(s|s-1)\|^2\}\\ =\ &\mathrm{E}\{\|x(s)-\hat{x}_e(s|s-1)+\hat{x}_e(s|s-1)\\ &-\hat{x}_u(s|s-1)\|^2\}\\ =\ &\sum_{i=0}^{\bar{r}}\mathrm{E}\{\|(\tilde{x}_e(s|s-1)+\hat{x}_e(s|s-1)\\ &-\hat{x}_u(s|s-1))\phi_{s,i}\|^2\}\\ =\ &\sum_{i=0}^{\bar{r}}\mathrm{tr}\{Y_i^{(\bar{r})}(s)\}+\mathrm{E}\{\|\hat{x}_e(s|s-1)-\hat{x}_u(s|s-1)\|^2\}\\ &-\sum_{i=0}^{\bar{r}}\mathrm{tr}\{\mathrm{E}(\tilde{x}_e(s|s-1)\hat{x}_u(s|s-1)'\phi_{s,i})\}\\ &+\sum_{i=0}^{\bar{r}}\mathrm{tr}\{\mathrm{E}(\tilde{x}_e(s|s-1)\hat{x}_e(s|s-1)'\phi_{s,i})\},\\ \geq\ &\sum_{i=0}^{\bar{r}}\mathrm{tr}\{Y_i^{(\bar{r})}(s)\},\end{aligned}$$

since

$$\begin{aligned}\mathrm{E}\{\tilde{x}_e(s|s-1)\tilde{x}_e(s|s-1)'\phi_{s,i}\} &= Y_i^{(\bar{r})}(s),\\ \mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_e(s|s-1)'\phi_{s,i}\} &= 0,\\ \mathrm{E}\{\tilde{x}_e(s|s-1)\hat{x}_u(s|s-1)'\phi_{s,i}\} &= 0,\\ \mathrm{E}\{\|\hat{x}_e(s|s-1)-\hat{x}_u(s|s-1)\|^2\} &\geq 0.\end{aligned}$$

The similar reasoning shows that $\mathrm{E}\{\|\tilde{x}_u(s|s-1)\|^2\} \geq \sum_{i=0}^{\bar{r}}\mathrm{tr}\{Y_i^{(k-s)}(s)\}$. Completing the proof of the theorem. □

Theorem 3 shows that the optimal solution to the Markov jump linear filtering problem can be obtained from the filtering recursive equations $Y_i^{(\bar{r})}(s)(i = 0,\cdots,\bar{r})$ and $Y_i^{(k-s)}(s)(i = 0,\cdots,k-s)$ as in (37) and (40), respectively.

Remark 2 *For the case in which A, C, H, G and λ_{ij} in (1) and (2) are time invariant and $\{r(k)\}$ satisfies the ergodic assumption, so that $\pi_i(k)$ converges to $\pi_i > 0$ as k goes to infinity, the filtering coupled Riccati difference equations (37) and (40) leads to the following coupled algebraic Ric-*

cati equations

$$\begin{aligned} Y_j^{(\bar r)} &= \sum_{i=0}^{\bar r}\lambda_{ij}\{AY_i^{(\bar r)}A' + \pi_i CC' - AY_i^{(\bar r)}H' \\ &\quad \times(HY_i^{(\bar r)}H' + \pi_i GG')^{-1}HY_i^{(\bar r)}A'\}, \\ &\quad j = 0,1,\cdots,\bar r, \qquad (55) \\ Y_j^{(l)} &= \sum_{i=0}^{l+1}\lambda_{ij}\{AY_i^{(l+1)}A' + \pi_i CC' - AY_i^{(l+1)}H' \\ &\quad \times(HY_i^{(l+1)}H' + \pi_i GG')^{-1}HY_i^{(l+1)}A'\}, \\ &\quad l = \bar r - 1,\cdots,0, j = 0,1,\cdots,l. \qquad (56) \end{aligned}$$

while the corresponding filter gains in Theorem 2 become as

$$\begin{aligned} F_i^{(\bar r)} &= -AY_i^{(\bar r)}H'(HY_i^{(\bar r)}H' + \pi_i GG')^{-1}, \qquad (57) \\ &\quad i = 0,1,\cdots,\bar r, \\ F_i^{(l)} &= -AY_i^{(l)}H'(HY_i^{(l)}H' + \pi_i GG')^{-1}, \qquad (58) \\ &\quad l = \bar r - 1,\cdots,0, i = 0,1,\cdots,l. \end{aligned}$$

Thus the constant-gain filter just requires us to keep in memory the gains $F^{(l)} = (F_0^{(l)},\cdots,F_l^{(l)})$, $l = \bar r,\cdots,0$. *In the next section, we will give the conditions for the existence of the constant-gain filter, and shows that the optimal Markov jump linear filter (33), (35) and (38) converge to the constant-gain filter.*

5 Conclusion

This paper has addressed the optimal filtering problems for the Markovian transmission delayed systems. Two kinds of optimal filters have been developed. The first one is the optimal Kalman filtering, which requires high computation and doesn't converge to a steady state in general. The second one is an alternative Markov jump linear filter which just depends on the present value of the Markov chain, and thus requires less pre-computed gains. It can be shown that, under standard assumptions, this filter is convergent to a constant-gain filter which is viewed as the third designed filter.

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