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Sampled-Data Based Average Consensus With Logarithmic Quantizers

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Abstract: This paper considers the sampled-data average consensus problem for multi-agent systems with first order continuous dynamics. The communication channels among the agents are constrained in which the exchanged information is digital rather than analogue. In this paper, the logarithmic quantizer is applied to the communication channels. A distributed consensus protocol is proposed based on sampled measurements. It is proved that as long as the quantization levels are dense enough, the proposed protocol is robust to the logarithmic quantization, i.e. all the states of the agents are uniformly bounded and the gap between the state of each agent and the average value of the initial conditions converges to zero as the density of quantization levels goes to infinity. An example is given to demonstrate the effectiveness of the protocol.

Key Words: Consensus, multi-agent systems, quantization, sampled-data system, logarithmic quantizer

1 Introduction

Distributed consensus becomes a hot research topic in recent years [12], [20], [1], [2], [21]. The problem is widely encountered in the real world, for example, in distributed computation, flocking, traffic control, networked control, formation flight, etc.

The average consensus problem means to design a networked interaction protocol such that the states of all the agents converge to the average of their initial states asymptotically or in finite time. However, when we consider that each agent can be able to communicate with only a set of discrete signals, the problem becomes complicated. Based on the gossip algorithm, [14] designs an average consensus protocol under the assumption that the states of agents are integer-valued. Under the same assumption, [19] analyzes the quantization effect on the average consensus and gives an upper bound for the consensus errors. [6] considers three kinds of update strategies including totally quantized, partially quantized and compensating based on both deterministic quantization and probabilistic quantization. Based on the assumption that the quantization errors are white noises, two coding schemes are provided by [22] and conditions under which the consensus is achieved are obtained. Under the same assumption, [7] analyzes the average consensus problem under Cayley graph. Other works for consensus problem with additive noises in channels can be found in [11], [16], [13] and [18]. In [8], an upper bound for the consensus errors is derived. Although the quantization level in the above works can be finite, the average consensus problem is not exactly solved. Given a logarithmic quantizer, [4] and [5] solve the average consensus problem when the quantization level is unlimited. By introducing a scaling function in coding and decoding, [3] and [15] obtain a lower bound for the number of quantization levels such that the average consensus problem is solved. Moreover, [15] provides a way to reduce the number of transmitting bits to one by properly selecting the control parameters. The results is generalized to multi-agent systems with communication delays [17]. Other works about choosing scaling function in communication constraint control are referred in [10].

Comparing with [4] and [5], we propose an average consensus protocol based on sampled measurement of each agent in this paper. The logarithmic quantization is also applied to the communication channels. The proposed protocol is much simpler than the ones in [4] and [5]. Moreover, we prove that the proposed protocol is robust with respect to the quantizer and the consensus error is asymptotically convergent to 0 as the quantization density goes to infinity.

Some remarks on notation are given as follows. $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of real numbers, real n-dimensional column vectors and real $m \times n$ matrices, respectively. For a vector or matrix A, A' denotes its transpose. $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the Euclidian norm and infinity norm, respectively. $\mathbf{1}$ stands for a column vector with every element of 1. Given a square matrix M with all the eigenvalues real, $\lambda_i(M)$ and $\lambda_{\max}(M)$ are the ith smallest eigenvalue and the largest eigenvalue of M, respectively. The floor function is denoted by $\lfloor\cdot\rfloor$.

2 Preliminaries

In this section, we shall review some basics of graph theory and logarithmic quantization which are fundamental to the later development.

2.1 Concepts in Graph Theory

A directed graph is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_\mathcal{G}, A_\mathcal{G}\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of node with i representing the

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ith agent; $\mathcal{E}_{\mathcal{G}}$ is the set of edges which are represented by a pair of node indices (i, j). We consider that $(i, j) \in \mathcal{E}$ if and only if node i can send its information to node j. In this case, node i is called the parent node and node j is called the child node. The set of neighbors of the ith agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. If $(i, i) \in \mathcal{E}$, we say that node i has self-loop. In this paper, we assume that no self-loop exists. $A_{\mathcal{G}} = [a_{i,j}] \in \mathbb{R}^{N \times N}$ is the the adjacency matrix associated with $\mathcal{G}$. If $j \in \mathcal{N}_i$, $a_{i,j} > 0$, otherwise $a_{i,j} = 0$. If matrix A is symmetric, then the corresponding graph is called undirected graph.

A graph is balanced if the in-degree $deg_{in}(i) \triangleq \sum_{j \in \mathcal{V} \setminus \{i\}} a_{i,j}$ and the out-degree $deg_{out}(i) \triangleq \sum_{j \in \mathcal{V} \setminus \{i\}} a_{j,i}$ are equal for all $i \in \mathcal{V}$. For example, an undirected graph is a kind of balanced graph.

There is a path from node i to node j if there exists a sequence $l_1, \ldots, l_p \in \mathcal{V}$ satisfying $(i, l_1), (l_1, l_2), \ldots, (l_p, j) \in \mathcal{E}_{\mathcal{G}}$ where $i, l_1, \ldots, l_p, j$ are distinct vertices. Given a graph $\mathcal{G}$, it contains a spanning tree if there exists at least one node i such that for any other node j, there is a path from i to j. If an undirected graph contains a spanning tree, it is connected.

The Laplacian matrix $L = [l_{i,j}]$ of the graph $\mathcal{G}$ is defined as that for any $i, j \in \mathcal{V}$ and $i \neq j$, $l_{i,j} = -a_{i,j}$ and $l_{i,i} = \sum_{j \in \mathcal{V} \setminus \{i\}} a_{i,j}$. By denoting all the eigenvalues of L as λ_i, $i = 1, 2, \ldots, n$, some properties of the Laplacian matrix are recalled below [21], [20]:

Lemma 2.1 *For an undirected graph $\mathcal{G}$ with Laplacian matrix L, we have the following properties:*

1) $\lambda_1(L) = 0$.

2) $\lambda_2(L) > 0$ if and only if $\mathcal{G}$ is connected.

Lemma 2.2 *Let L be the Laplacian matrix of a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$. Then $\hat{L} \triangleq (L + L')/2$ is a valid Laplacian matrix for its mirror graph[1] $\hat{\mathcal{G}}$ if and only if $\mathcal{G}$ is balanced.*

2.2 Concepts in logarithmic quantization

A quantizer $q(\cdot) : \mathbb{R} \to \Gamma$ is a map from $\mathbb{R}$ to the set Γ of quantized levels. Γ is finite or denumerable. $q(\cdot)$ is called logarithmic if it has the form

$$\Gamma = \{\pm y_{(i)} : y_{(i)} = \rho^i y_{(0)}, i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\},$$
$$0 < \rho < 1,\ y_{(0)} > 0.$$

The associated quantizer $q(\cdot)$ is defined as follows:

$$q(x) = \begin{cases} y_{(i)}, & \text{if} \frac{1}{1+\beta} y_{(i)} < x \leq \frac{1}{1-\beta} y_{(i)} \\ 0, & \text{if } x = 0 \\ -q(-x), & \text{if } x < 0 \end{cases} \tag{1}$$

where $\beta = \frac{1-\rho}{1+\rho} \in (0, 1)$. The quantization density[9] for the quantizer (1) is $\frac{-2}{\ln \rho}$. It is straightforward that the smaller β is, the more the quantization levels we have in any given subset of $\mathbb{R}$.

From (1) we can see that a logarithmic quantizer has the following properties:

$$q(x) = (1 + \Delta)x,\ \ \exists \Delta \in (-\beta, \beta),\ \forall x \in \mathbb{R}. \tag{2}$$

[1] See [20] Definition 2 for the definition of mirror graph.

3 Problem Statement

In this section, we shall formulate the consensus problem to be studied for multi-agent systems. The agent i is assumed to have the following dynamics

$$\dot{x}_i(t) = u_i(t),\ i = 1, 2, \ldots, N, \tag{3}$$

where $x_i(t) \in \mathbb{R}$ is the state information of agent i, $u_i(t) \in \mathbb{R}$ is the control input. The communication graph is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ and the corresponding Laplacian matrix is L.

Assume that agent i can receive its neighbors' quantized state information

$$y_{j,i}(t) = q(x_j(t)),\ j \in \mathcal{N}_i,\ i = 1, 2, \ldots, N, \tag{4}$$

where $q(\cdot)$ is as defined in (1). According to (2), $y_{j,i}(t)$ can be written as

$$y_{j,i}(t) = (1 + \Delta_j(t)) x_j(t),\ \Delta_j \in (-\beta, \beta). \tag{5}$$

We call a group of controls $\mathcal{U} = \{u_i, i = 1, 2, \ldots, N\}$ a distributed protocol if $u_i(t)$ is a function of $\{x_i(s), y_{j,i}(s), 0 \leq s \leq t, j \in \mathcal{N}_i\}$. The objective is to design a distributed protocol for $\mathcal{G}$ such that for any initial conditions $x_i(0)$, $i = 1, 2, \ldots, N$

$$\lim_{t \to \infty} x_i(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(0),$$

i.e. average consensus control.

Since the quantization is involved in the communication, the exact average consensus cannot be easily achieved. As such, we will introduce the concept of robust average consensus with respect to the logarithmic quantization. Denote

$$X(t) = [x_1(t), \ldots, x_N(t)]',\ \Delta(t) = [\Delta_1(t), \ldots, \Delta_N(t)]'.$$

Then the closed-loop system can be modeled as the following uncertain system

$$\dot{X}(t) = F\big(\{X(s), \Delta(s), 0 \leq s \leq t\}\big), \tag{6}$$

where $\Delta(t)$ satisfies $\Delta = \{\Delta(t), t \geq 0\} \in Q(\beta)$ and $Q(\cdot)$ is defined as

$$Q(\gamma) = \Big\{\Lambda = \{\Lambda(t) \in \mathbb{R}^N, t \geq 0\} \mid \sup_{t \geq 0} \|\Lambda(t)\|_\infty \leq \gamma\Big\},$$
$$0 < \gamma \leq 1.$$

Define $J = \frac{1}{N}\mathbf{1}\mathbf{1}'$, $\delta(t) = X(t) - JX(t)$, then we have the following definition.

Definition 3.1 *A distributed protocol $\mathcal{U}$ is robust with respect to the logarithmic quantization if there exist $\beta^* > 0$ such that for any $0 < \beta < \beta^*$, the closed-loop uncertain system* (6) *satisfies*

$$\sup_{\Delta \in Q(\beta)} \sup_{t \geq 0} \|X(t)\|_2 < \infty,\ \ \forall X(0) \in \mathbb{R}^N. \tag{7}$$

Moreover, if

$$\lim_{\beta \to 0} \sup_{\Delta \in Q(\beta)} \limsup_{t \to \infty} \|\delta(t)\|_2 = 0, \tag{8}$$

$\mathcal{U}$ is called a robust average consensus protocol with respect to the logarithmic quantization.

4 Sample-Data Based Protocol

In this paper, we consider a sampled-data setting where the measurements are made at discrete sampling times and the control inputs are based on zero-order hold. We assume that the sampling interval is h, then we propose the distributed protocol as

$$u_i(t) = \begin{cases} 0, & \mathcal{N}_i = \emptyset, \\ \sum\limits_{j \in \mathcal{N}_i} [q(x_j(kh)) - q(x_i(kh))], & \mathcal{N}_i \neq \emptyset, \end{cases} \quad t \in [kh, (k+1)h). \tag{9}$$

The discretized model with zero-order hold can be written as

$$x_i(k+1) = x_i(k) + hu_i(k), \ k = 0, 1, \dots \tag{10}$$

where we omit the sampling time interval for simplifying the notation.

Lemma 4.1 *If $\mathcal{G}$ is balanced and contains a spanning tree, there exists a set of vectors $\{\phi_1, \phi_2, \dots, \phi_{N-1}\}$ as a standard orthogonal basis of the column space of L. Denote $\phi = [\phi_1, \phi_2, \dots, \phi_{N-1}]$, then*

1) *the matrix $\Phi = [\frac{\mathbf{1}}{\sqrt{N}}, \phi]$ is a standard orthogonal matrix;*
2) *$\Phi' L \Phi = diag(0, \tilde{L})$, where $\tilde{L}$ is an $(N-1) \times (N-1)$ matrix with all its eigenvalues having positive real parts.*

By defining $\hat{L} = (L + L')/2$, we have the following result.

Theorem 4.1 *The protocol* (9) *is applied to the system* (3)-(4)*. If $\mathcal{G}$ is balanced and contains a spanning tree and the sampling interval satisfies $0 < h < \frac{2\lambda_2(\hat{L})}{\|L\|_2^2}$, then protocol* (9) *is a robust average consensus protocol with respect to the logarithmic quantization. Moreover, β^* can be chosen as*

$$\beta^* = \frac{1 - \|I - h\tilde{L}\|_2}{h\|L\|_2},$$

where $\tilde{L}$ is as defined in Lemma 4.1.

Proof. Substituting (9) to the discrete-time system (10) leads to that

$$X(k+1) = (I - hL - hL\bar{\Delta}(k))X(k), \tag{11}$$

where $\bar{\Delta}(k) = diag\{\Delta_1(k), \Delta_2(k), \dots, \Delta_N(k)\}$. Since $\mathcal{G}$ is balanced, we have $\mathbf{1}'L = 0$ and

$$JX(k) \equiv JX(0).$$

Therefore

$$X(k) = \delta(k) + JX(k) = \delta(k) + JX(0). \tag{12}$$

Substituting (12) into (11) and noting the fact that $J\mathbf{1} = 0$, we have

$$\delta(k+1) = (I - hL - hL\bar{\Delta}(k))\delta(k) - hL\bar{\Delta}(k)JX(0). \tag{13}$$

Next, we introduce the linear transformation $\tilde{\delta}(k) = \Phi'\delta(k)$, where Φ is as defined in Lemma 4.1. According to Lemma 4.1 and (13), we can see that $\tilde{\delta}(k) = [0, \tilde{\delta}_2'(k)]'$, where $\tilde{\delta}_2(k)$ satisfies

$$\tilde{\delta}_2(k+1) = (I - h\tilde{L} - h\phi' L\bar{\Delta}(k)\phi)\tilde{\delta}_2(k) - h\phi' L\bar{\Delta}(k)JX(0). \tag{14}$$

Then we shall prove that the above system is exponentially stable. Now, we consider the autonomous system

$$z(k+1) = (I - h\tilde{L} - h\phi' L\bar{\Delta}(k)\phi)z(k). \tag{15}$$

Since $\mathcal{G}$ is balanced and contains a spanning tree, according to Lemma 2.1 and Lemma 2.2 we know that $\hat{L}$ is positive semi-definite. From Lemma 4.1 we have

$$\Phi'\hat{L}\Phi = \frac{\Phi' L\Phi + \Phi' L'\Phi}{2} = diag\Big(0, \frac{\tilde{L} + \tilde{L}'}{2}\Big),$$

where $\frac{\tilde{L}+\tilde{L}'}{2}$ is positive definite and

$$\lambda_2(\hat{L}) = \lambda_2(\Phi' L\Phi) = \lambda_1\Big(\frac{\tilde{L} + \tilde{L}'}{2}\Big) > 0, \tag{16}$$

$$\|L\|_2^2 = \lambda_{\max}(\Phi' L'\Phi\Phi' L\Phi) = \lambda_{\max}(\tilde{L}'\tilde{L}). \tag{17}$$

Considering (16) and (17), we have

$$\begin{aligned} \|I - h\tilde{L}\|_2^2 &= \lambda_{\max}\big[(I - h\tilde{L}')(I - h\tilde{L})\big] \\ &= \lambda_{\max}\Big[I - 2h\frac{\tilde{L} + \tilde{L}'}{2} + h^2\tilde{L}'\tilde{L}\Big] \\ &\leq \lambda_{\max}\Big[I - 2h\frac{\tilde{L} + \tilde{L}'}{2}\Big] + h^2\lambda_{\max}(\tilde{L}'\tilde{L}) \\ &\leq 1 - 2h\lambda_2(\hat{L}) + h^2\|L\|_2^2, \end{aligned} \tag{18}$$

which together with $0 < h < \frac{2\lambda_2(\hat{L})}{\|L\|_2^2}$ leads to

$$\|I - h\tilde{L}\|_2 < 1.$$

By (15), for any $n \geq k$ we have

$$\begin{aligned} \|z(n)\|_2 \leq{} & \|I - h\tilde{L}\|_2^{n-k}\|z(k)\|_2 \\ & + h\sum_{i=k}^{n-1} \|I - h\tilde{L}\|_2^{n-i-1}\big\|\phi' L\bar{\Delta}(i)\phi\big\|_2\|z(i)\|_2. \end{aligned} \tag{19}$$

For any $0 < \beta < \beta^*$, there must exist $0 < \varepsilon < 1$ such that

$$\beta = \frac{(1-\varepsilon)\big(1 - \|I - h\tilde{L}\|_2\big)}{h\|L\|_2}. \tag{20}$$

From the definition of $\bar{\Delta}(i)$, we know that $\forall i \geq 0$, $\|\bar{\Delta}(i)\|_2 \leq \beta$. Then by (19) and (20) we have

$$\begin{aligned} \|z(n)\|_2 \leq{} & \|I - h\tilde{L}\|_2^{n-k}\|z(k)\|_2 \\ & + h\beta\|L\|_2\sum_{i=k}^{n-1} \|I - h\tilde{L}\|_2^{n-i-1}\|z(i)\|_2 \\ \leq{} & \|I - h\tilde{L}\|_2^{n-k}\|z(k)\|_2 + (1-\varepsilon) \\ & \times\big(1 - \|I - h\tilde{L}\|_2\big)\sum_{i=k}^{n-1} \|I - h\tilde{L}\|_2^{n-i-1}\|z(i)\|_2, \end{aligned} \tag{21}$$

where we use the fact that $\|\phi\|_2 = \|\phi'\|_2 = 1$. According to Bellman inequality, we arrive at the following inequality

$$\begin{aligned}
\|z(n)\|_2 \le& \|I - h\tilde{L}\|_2^{n-k}\|z(k)\|_2 \prod_{i=0}^{n-k-1}\Big[1 + (1-\varepsilon) \\
&\times\big(1 - \|I - h\tilde{L}\|_2\big)\|I - h\tilde{L}\|_2^i\Big] \\
\le& \|z(k)\|_2 \prod_{i=1}^{n-k}\Big[\|I - h\tilde{L}\|_2 + (1-\varepsilon) \\
&\times\big(1 - \|I - h\tilde{L}\|_2\big)\|I - h\tilde{L}\|_2^i\Big] \\
\le& \|z(k)\|_2 \prod_{i=1}^{n-k}\Big[\|I - h\tilde{L}\|_2 + (1-\varepsilon) \\
&\times\|I - h\tilde{L}\|_2\big(1 - \|I - h\tilde{L}\|_2\big)\Big]. \qquad (22)
\end{aligned}$$

Denoting $\gamma = \|I - h\tilde{L}\|_2 + (1-\varepsilon)\|I - h\tilde{L}\|_2\big(1 - \|I - h\tilde{L}\|_2\big) \ge 0$, it follows that

$$\begin{aligned}
\gamma &< \|I - h\tilde{L}\|_2 + (1-\varepsilon)\big(1 - \|I - h\tilde{L}\|_2\big) \\
&< \|I - h\tilde{L}\|_2 + \big(1 - \|I - h\tilde{L}\|_2\big) = 1. \qquad (23)
\end{aligned}$$

Then we have

$$\|z(n)\|_2 \le \|z(k)\|_2 \gamma^{n-k}. \qquad (24)$$

Define

$$\Psi(n,k) = \begin{cases} \big(I - h\tilde{L} - h\phi' L\bar{\Delta}(n-1)\phi\big)\cdots \\ \big(I - h\tilde{L} - h\phi' L\bar{\Delta}(k)\phi\big), & n > k, \\ I, & n = k. \end{cases}$$

According to (15) and (24) we have

$$\|\Psi(n,k)z(k)\|_2 \le \|z(k)\|_2\gamma^{n-k}.$$

The arbitrariness of $z(k)$ leads to that

$$\sup_{\|z(k)\|_2 \ne 0} \frac{\|\Psi(n,k)z(k)\|_2}{\|z(k)\|_2} \le \gamma^{n-k},$$

which means

$$\|\Psi(n,k)\|_2 \le \gamma^{n-k}. \qquad (25)$$

By considering (14) we have

$$\tilde{\delta}_2(k) = \Psi(k,0)\tilde{\delta}_2(0) - h\sum_{i=0}^{k-1}\Psi(k,i+1)\phi' L\bar{\Delta}(i)JX(0),$$

which together with (25) results in that

$$\begin{aligned}
\|\tilde{\delta}_2(k)\|_2 \le& \gamma^k\|\tilde{\delta}_2(0)\|_2 + h\beta\|L\|_2\|X(0)\|_2\sum_{i=0}^{k-1}\gamma^{k-i-1} \\
\le& \gamma^k\|\tilde{\delta}_2(0)\|_2 + \big(1-\gamma^k\big)\frac{h\beta\|L\|_2\|X(0)\|_2}{1-\gamma} \\
\le& \gamma^k\|\tilde{\delta}_2(0)\|_2 + \big(1-\gamma^k\big)\frac{h\beta\|L\|_2\|X(0)\|_2}{\big(1 - \|I - h\tilde{L}\|_2\big)^2}. \qquad (26)
\end{aligned}$$

Then we have

$$\sup_{\Delta\in Q(\beta)}\sup_{k\ge 0}\|\tilde{\delta}_2(k)\|_2 \le \max\left\{\|\tilde{\delta}_2(0)\|_2, \frac{h\beta\|L\|_2\|X(0)\|_2}{\big(1 - \|I - h\tilde{L}\|_2\big)^2}\right\},$$

and

$$\sup_{\Delta\in Q(\beta)}\limsup_{k\to 0}\|\tilde{\delta}_2(k)\|_2 \le \frac{h\beta\|L\|_2\|X(0)\|_2}{\big(1 - \|I - h\tilde{L}\|_2\big)^2}.$$

Note that $\forall k \ge 0$, $\|\delta(k)\|_2 = \|\tilde{\delta}_2(k)\|_2$ and $\|\delta(k)\|_2 \le \|(I-J)X(k)\|_2 \le \|X(k)\|_2$. We can get

$$\sup_{\Delta\in Q(\beta)}\sup_{k\ge 0}\|\delta(k)\|_2 \le M_1(h,\beta,\mathcal{G})\|X(0)\|_2, \qquad (27)$$

and

$$\sup_{\Delta\in Q(\beta)}\limsup_{k\to 0}\|\delta(k)\|_2 \le M_2(h,\beta,\mathcal{G})\|X(0)\|_2, \qquad (28)$$

where

$$M_1(h,\beta,\mathcal{G}) = \max\{1, M_2(h,\beta,\mathcal{G})\},$$

$$M_2(h,\beta,\mathcal{G}) = \frac{h\beta\|L\|_2}{\big(1 - \|I - h\tilde{L}\|_2\big)^2}.$$

Similar to (13), we can get the continuous dynamics of $\delta(t)$ during the sampling time interval which is given below:

$$\dot{\delta}(t) = -L(I + \bar{\Delta}(t_s))\delta(t_s) - L\bar{\Delta}(t_s)JX(0), \qquad (29)$$

where $t_s = \lfloor \frac{t}{h} \rfloor h$ is the last sampling time instant. Since the right hand side of (29) is constant within $[t_s, t_s + h)$, we then have

$$\begin{aligned}
\delta(t) =& \big[I - (t-t_s)L - (t-t_s)\bar{\Delta}(t_s)L\big]\delta(t_s) \\
&-(t-t_s)L\bar{\Delta}(t_s)JX(0).
\end{aligned}$$

Then according to (27) and (28) we can get

$$\begin{aligned}
\sup_{\Delta\in Q(\beta)}\sup_{t\ge 0}\|\delta(t)\|_2 \le& \big[1 + (t-t_s)\|L\|_2 + (t-t_s)\beta\|L\|_2\big] \\
&\times\|\delta(t_s)\|_2 + (t-t_s)\beta\|L\|_2\|X(0)\|_2 \\
\le& \Big[\big(1 + h\|L\|_2 + h\beta\|L\|_2\big) \\
&\times M_1(h,\beta,\mathcal{G}) + h\beta\|L\|_2\Big]\|X(0)\|_2, \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\sup_{\Delta\in Q(\beta)}\limsup_{t\to 0}\|\delta(t)\|_2 \le& \Big[\big(1 + h\|L\|_2 + h\beta\|L\|_2\big) \\
&\times M_2(h,\beta,\mathcal{G}) + h\beta\|L\|_2\Big]\|X(0)\|_2. \qquad (31)
\end{aligned}$$

Note that

$$\lim_{\beta\to 0} M_2(h,\beta,\mathcal{G}) = 0.$$

Then we have

$$\lim_{\beta\to 0}\sup_{\Delta\in Q(\beta)}\limsup_{t\to\infty}\|\delta(t)\|_2 = 0,$$

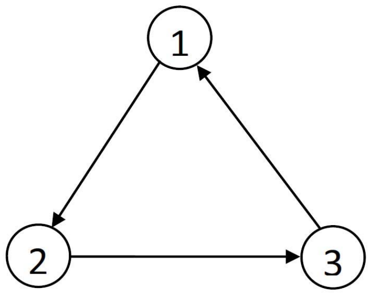

Fig. 1: Communication graph $\mathcal{G}$.

which is (8). On the other hand, according to $X(t) = \delta(t) + JX(0)$ and (30), we arrive at

$$\sup_{\Delta \in Q(\beta)} \sup_{t \geq 0} \|X(t)\|_2 \leq \Big[\big(1 + h\|L\|_2 + h\beta\|L\|_2\big) M_1(h, \beta, \mathcal{G})$$

$$+ h\beta\|L\|_2 + 1\Big] \|X(0)\|_2,$$

which completes the proof. □

5 Numerical Example

In this section, we shall give an example to verify the protocol. We consider system (3) with 3 agents connecting end to end. The communication graph is shown in Fig. 1. The corresponding adjacency matrix A, Lalacian matrix L and Laplacian matrix $\hat{L}$ of mirror graph are given below:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\hat{L} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}.$$

It is clear that the graph $\mathcal{G}$ is balance and contains a spanning tree. We choose

$$\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-2}{\sqrt{6}} \end{bmatrix},$$

then according to Lemma 4.1, the corresponding $\tilde{L}$ is

$$\tilde{L} = \begin{bmatrix} \frac{3}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix}.$$

The initial values of the 3 agents are $x_1(0) = 1.46$, $x_2(0) = 1.84$ and $x_3(0) = 1.2$. It is clear that the average is $\frac{1}{3}\big[x_1(0) + x_2(0) + x_3(0)\big] = 1.5$. The consensus protocol (9) is applied to system (3). By Theorem 4.1, we can calculate the upper bound of sampling interval h which is $\frac{2\lambda_2(\hat{L})}{\|L\|_2^2} = 1$. So that we choose $h = 0.5$. β^* can be calculated accordingly, which is $\beta^* = 0.57735$. The smaller β is, the closer the states is to the average. We then select different β and compare the consensus errors. The state trajectories are shown in Fig. 2 and Fig. 3. In Fig. 2, $\beta = 0.2$, the states are uniformly bounded and the consensus errors are relatively large comparing with the ones in Fig. 3, in which $\beta = 0.02$. The Fig. 2 and Fig. 3 also verify that when β goes to 0, the average consensus errors also goes to 0.

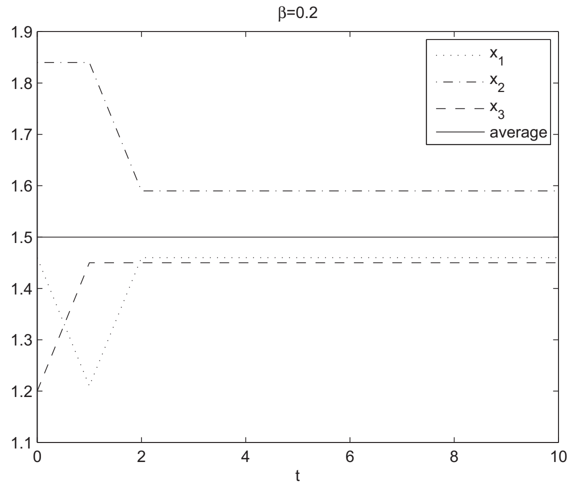

Fig. 2: State trajectories with $\beta = 0.2$.

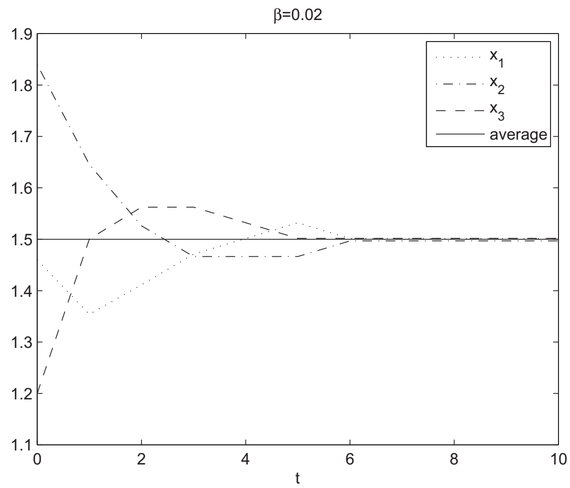

Fig. 3: State trajectories with $\beta = 0.02$.

6 Conclusion

In this paper we have considered average consensus problem for multi-agent systems with logarithmic quantization in communication channels. The agents are homogeneous and with first order continuous dynamics. A protocol has been proposed based on sampled measurements. It has been proved that when the sampling rate is high enough and the quantization levels are dense enough, all the states of the agents are uniformly bounded and the average consensus error will converge to 0 as the quantization density goes to infinity. A numerical example has been provided to demonstrate the results.

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