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LQG Control with Fixed-Rate Uniform Quantization for SISO Systems

Li Chai[1], Minyue Fu[2]
1. School of Information Science and Engineering
Wuhan University of Science and Technology, Wuhan 430081, China
E-mail: chaili@wust.edu.cn
2. Department of Control Science and Engineering
Zhejiang University, Hangzhou 310027, China
E-mail: myfu@iipc.zju.edu.cn

Abstract: This paper considers the quantized infinite-horizon LQG control system, of which the measurement signal is quantized by a fixed-rate quantizer before going into the controller. It turns out that only weak separation principle holds for the LQG control system with communication channels. We study the problem of quantized LQG for a single-input-single-output (SISO) system. An adaptive fixed-rate quantizer is designed to achieve the mean-square stability and the optimal distortion performance. For a quantizer with a fixed bit rate of R (per sample), we show that the quantization distortion order is $R2^{-2R}$ for a large R. Simulation examples are given to demonstrate the effectiveness of the proposed methods.

Key Words: Quantized feedback control, linear quadratic Gaussian control, fixed-rate quantization, uniform quantizer.

1 Introduction

Recently, networked control systems (NCSs) have drawn a great deal of attention from control scientists and engineer [7, 12, 14]. As an important component in wireless communication, quantization may cause serious problems to feedback systems [13]. Lots of results have been reported to address the stability and stabilization problems with logarithm quantization in a determinate setup [7], some researchers studied the problem of linear quadratic Gaussian control (LQG) with quantization data [2, 4, 6, 11, 14]. The weak separation principle has established for finite-horizon LQG control, where a linear predictive code (LPC) with memoryless fixed-rate quantizer is given and separation principle is shown to hold under a high resolution quantization assumption and a mild rank condition [6]. For the infinite-horizon LQG control, however, the LPC with memoryless fixed-rate quantizer can not guarantee the stochastic stability of the closed-loop system, let alone the performance. This is caused by the saturation effect of the finite-support of the quantizer, as shown in [12].

There is extensive literature about quantization for autoregressive sources [1, 5, 8, 10]. A systematic analysis of the optimal fixed-rate uniform scalar quantization is given for a class of memoryless distributions in [9]. Explicit asymptotic formulas are presented for the distortion and optimal quantizer length approximation, about Gamma distribution, of which Gaussian is a special case. However the results can not be used directly to the quantized LQG control problem since they are based on a key assumption that $|A| \leq 1$. In the control problem, we usually consider unstable systems, i.e. $|A| > 1$.

Based on results in [9], a simple LPC scheme with *adaptive* fixed-rate quantizer has been proposed for infinite-horizon quantized LQG control of scalar systems [3]. It has been shown that the mean-square stability of the quantized feedback system is achieved, and the average distortion is in the order of $N^{-2}\ln N$, where $N = 2^R$, and R is the quantization bit rate.

In this paper, we extend the result in [3] to general SISO high order systems. We will show that the LPC scheme with the *adaptive* scalar fixed-rate quantizer can preserve the mean-square stability, and achieve a small distortion, which is in the order of $N^{-2}\ln N$, where $N = 2^R$, and R is the quantization bit rate.

2 Preliminaries and problem statement

2.1 Fixed-rate uniform scalar quantization

In this paper, we consider the fixed-rate uniform scalar quantizer, which is the simplest and most common form of quantizer, and of which the asymptotic behavior has been understood recently for a class of source densities with infinite support, such as Gaussian [9]. In [9], explicit asymptotic formulas are presented for the distortion and optimal quantizer length approximation, about Gamma distribution, of which Gaussian is a special case. To introduce the result, we need some notations. Consider an $N = 2^R$ level symmetric uniform scalar quantizer with step size Δ. Let $(-L_N, L_N]$ be the support of this quantizer, where $L_N = N\Delta/2$. Define $y_i = -N\Delta/2+(i-1/2)\Delta$ and $S_i = (y_i-\Delta/2, y_i+\Delta/2]$ for $i = 1, \ldots, N$. The quantizer is defined as

$$Q_\Delta(x) = \begin{cases} y_0 \text{ if } x \leq -L, \\ y_i \text{ if } x \in S_i, \\ y_N \text{ if } x > L. \end{cases}$$

Then the MSE granular and overload distortions are defined as follows

$$\begin{aligned} D^{gran} &= \sum_{i=1}^{N} \int_{S_i} (x-y_i)^2 p(x)dx \\ D^{over} &= 2\int_{L}^{\infty} (x-y_N)^2 p(x)dx, \end{aligned}$$

where $p(x)$ is the source density function.

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We summarized some results in [9] by Lemma 1-3, which will be used in this paper. For notational simplicity, we denote the PDF of a Gaussian random variable as

$$\rho(\mu,\sigma^2,x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \qquad (1)$$

Lemma 1 *For a Gaussian distribution with zero mean and variance σ, the optimal quantization length for the uniform fixed-rate quantizer is given by $L \approx 2\sigma\sqrt{\ln N}$. Moreover, the distortions satisfy*

$$\lim_{N\to\infty}\frac{D^{over}}{D^{gran}} = 0 \qquad (2)$$

$$D\approx D^{gran} \approx \frac{\Delta_N^2}{12}=\frac{L^2}{3N^2}. \qquad (3)$$

Lemma 2 *For $\rho(0,\sigma^2,x)$, define $W_{\sigma^2}(y)$ as*

$$W_{\sigma^2}(y)=\frac{1}{\sqrt{2\pi}\sigma}\int_y^\infty (x-y)^2 e^{-\frac{x^2}{2\sigma^2}}dx.$$

Then

$$\lim_{N\to\infty}\frac{D_L^{over}}{2W(L)} = 1 \qquad (4)$$

$$W_{\sigma^2}(y) = \frac{2\sigma^5}{\sqrt{2\pi}}y^{-3}e^{-\frac{y^2}{2\sigma^2}}(1+o_y(1)), \qquad (5)$$

where $o_y(1)$ tends to zero as y tends to infinity.

Lemma 3 *For any source density whatsoever*

$$\lim_{N\to\infty}\frac{D^{gran}}{\Delta_N^2/12}=1.$$

2.2 Problem statement

Consider the following SISO discrete-time system

$$\begin{aligned} x_{t+1} &= Ax_t+Bu_t+w_t \\ y_t &= Cx_t+v_t, \end{aligned} \qquad (6)$$

where $x_t\in\mathbb{R}^n$ is the state, $u,y\in\mathbb{R}$ are the control input and the measured output respectively. Assume that x_0, $w_t\in\mathbb{R}^n$ and $v_t\in\mathbb{R}$ are mutually independent Gaussian with zero mean and covariances $\Sigma_0>0$, $W>0$ and $V>0$, respectively. The cost function is defined as

$$J=\lim_{T\to\infty}\sup\frac{1}{T}\mathcal{E}\left(\sum_{t=0}^{T-1}\begin{bmatrix}x_t\\u_t\end{bmatrix}^T\begin{bmatrix}Q & H\\ H^T & S\end{bmatrix}\begin{bmatrix}x_t\\u_t\end{bmatrix}\right), \qquad (7)$$

where $S>0$, $Q\geq 0$, and $Q-HS^{-1}H'\geq 0$.

The problem is to design an observer based controller, and an R-bit uniform quantizer to minimize the cost J. Let P be the solution of the following Ricatti equation

$$P=Q+A'PA-(B'PA+H)'(S+B'PB)^{-1}(B'PA+H). \qquad (8)$$

Define

$$K=-(S+B'PB)^{-1}(B'PA+H). \qquad (9)$$

Let the optimal observer based controller is given by $K\mathcal{Q}(\hat{x}_t)$, where K is the feedback gain matrix, and $\mathcal{Q}(\hat{x}_t)$ is the quantized value of the estimated state from the following Kalman filter

$$\begin{aligned} \hat{x}_t &= \hat{x}_{t|t-1}+\Gamma(y_t-C\hat{x}_{t|t-1}) \\ \hat{x}_{t+1|t} &= A\hat{x}_t+Bu_t. \end{aligned} \qquad (10)$$

where $\Gamma=EC^T(CEC^T+V)^{-1}$ and E is the solution of the following Ricatti equation

$$E=AEA^T-AEC^T(CEC^T+V)^{-1}CEA^T+W. \qquad (11)$$

The weak separation principle stated below [6] suggests that optimal quantized LQG control can be achieved by first constructing the optimal estimate $\hat{x}_t$, which is independent of the cost function, then quantizing it and the optimal control is given by

$$u_t=K\mathcal{Q}(\hat{x}_t). \qquad (12)$$

Lemma 4 *Consider the system (6), the cost function (7), the quantized feedback controller $K\mathcal{Q}(\hat{x}_t)$, with K given by (9), $\hat{x}_t$ given by (10), and the R-bit fixed-rate quantization $\mathcal{Q}(\cdot)$. Then, the quantized LQG controller is optimal if $\mathcal{Q}(\hat{x}_t)$ is obtained by the quantizer that minimizes the following distortion function*

$$D=\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathcal{E}[(\hat{x}_t-\mathcal{Q}(\hat{x}_t))'\Omega(\hat{x}_t-\mathcal{Q}(\hat{x}_t))] \qquad (13)$$

where $\Omega=K'(S+B'PB)K$. The corresponding cost function is given by

$$J=J_{LQG}+\min D=tr(PW)+tr(\Omega E)+\min D.$$

An LPC type quantization scheme is proposed in [6] as below

$$\begin{aligned} \mathcal{Q}(\hat{x}_t)&=(A+BK)\mathcal{Q}(\hat{x}_{t-1})+\mathcal{Q}(\varepsilon_t) \\ \varepsilon_t&=\Gamma(y_t-c\hat{x}_{t|t-1})+A(\hat{x}_{t-1}-\mathcal{Q}(\hat{x}_{t-1})) \end{aligned} \qquad (14)$$

where $\mathcal{Q}(\hat{x}_{-1})=0$, $\varepsilon_0=\Gamma v_0$, and $\mathcal{Q}(\varepsilon_t)$ is the quantized value of ε_t with the distortion function of $\mathcal{E}[(\varepsilon_t-\mathcal{Q}(\varepsilon_t))'\Omega(\varepsilon_t-\mathcal{Q}(\varepsilon_t))]$.

Lemma 5 *Let $\mathcal{Q}(\hat{x}_t)$ and $\mathcal{Q}(\varepsilon_t)$ be defined as (14), then we have*

$$\begin{aligned} &\mathcal{E}[(\varepsilon_t-\mathcal{Q}(\varepsilon_t))'\Omega(\varepsilon_t-\mathcal{Q}(\varepsilon_t))] \\ &\quad=\mathcal{E}[(\hat{x}_t-\mathcal{Q}(\hat{x}_t))'\Omega(\hat{x}_t-\mathcal{Q}(\hat{x}_t))]. \end{aligned}$$

Proof. Combining (10), (12) and (14), we have

$$\begin{aligned} &(\hat{x}_t-\mathcal{Q}(\hat{x}_t)) \\ =\;& A\hat{x}_t+Bu_t++\Gamma(y_t-C\hat{x}_{t|t-1})-\mathcal{Q}(\hat{x}_t)) \\ =\;& \varepsilon_t+A\mathcal{Q}(\hat{x}_(t-1))-(A+BK)\mathcal{Q}(\hat{x}_{t-1})-\mathcal{Q}(\varepsilon_t) \\ =\;& \varepsilon_t-\mathcal{Q}(\varepsilon_t)+BKu_{t-1}-BK\mathcal{Q}(\hat{x}_{t-1}) \\ =\;& \varepsilon_t-\mathcal{Q}(\varepsilon_t). \end{aligned}$$

This completes the proof.

3 Quantizer design

In this section, we will design the LPC type quantizer as (14) so that not only (13) is minimized, but also the mean-square stability is achieved. For high-resolution quantizers, we show that the separation principle holds when A is stable, which means the optimal distortion can be achieved by a memoryless quantizer. However, it is shown in [12] that the feedback system is unstable in probability 1 when A is unstable if the LPC with memoryless quantizer is used. In this paper, we choose the LPC quantization with adaptive quantizer.

Define $\eta_t = Ax_t - A\hat{x}_t$. Combining the system (6), the controller (12), the state estimator (10) and the quantizer (14) together, we obtain the following equations

$$\eta_{t+1} = (A - A\Gamma C)\eta_t + \Gamma C w_t + \Gamma v_{t+1} + w_t \quad (15)$$

$$\hat{x}_{t+1} = (A + BK)\hat{x}_t - BK(\varepsilon_t - \mathcal{Q}(\varepsilon_t)) + \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} \quad (16)$$

$$\varepsilon_{t+1} = \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} + A(\varepsilon_t - \mathcal{Q}(\varepsilon_t)). \quad (17)$$

Remark 1 *1) Since $\eta_0 = (A - A\Gamma C)x_0 - A\Gamma v_0$, then η_t is Gaussian for any t if the initial state x_0, w_t and v_t are all Gaussian.*
2) Define $= \mathcal{E}(\eta_t\eta_t^{'})$. Since $A - A\Gamma C$ is stable, we have

$$\Sigma_\eta = (A - A\Gamma C)\Sigma_\eta(A - A\Gamma C)^{'} + (\Gamma C + I)W(C^{'}\Gamma' + I) + V\Gamma\Gamma^{'},$$

where $\Sigma_\eta = \lim_{t\to\infty}\Sigma_{\eta_t}$.
3) Denote $z_{t+1} = C\eta_t + Cw_t + v_{t+1}$, then z_{t+1} is Gaussian with zero mean and variance σ_{t+1}^2, where

$$\sigma_{t+1}^2 = C(\Sigma_{\eta_t} + W)C' + V. \quad (18)$$

Denote σ_z^2 as

$$\sigma_z^2 := \lim_{t\to\infty}\sigma_t^2 = C(\Sigma_\eta + W)C' + V. \quad (19)$$

With the above notations, we consider the following system in the rest of this paper

$$\varepsilon_{t+1} = \Gamma z_{t+1} + A(\varepsilon_t - \mathcal{Q}(\varepsilon_t)) \quad (20)$$

$$\xi_{t+1} = \sqrt{S + B'PB}K\varepsilon_{t+1} \quad (21)$$

The quantized LQG control problem becomes to design a quantizer to ε_t given by (20) such that the distortion

$$D = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathcal{E}[(\varepsilon_t - \mathcal{Q}(\varepsilon_t))^{'}\Omega(\varepsilon_t - \mathcal{Q}(\varepsilon_t))] \quad (22)$$

is minimized. This is equivalent to design a quantizer to ξ_{t+1} such that the distortion

$$D = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathcal{E}[\xi_{t+1}'\xi_{t+1}] \quad (23)$$

is minimized if the system is well-behaved, for example, the variance of ε_{t+1} is bounded for unstable A.

When A is small, a memoryless quantizer can be designed to achieve the optimal distortion as stated below.

Theorem 1 *For the SISO system (20-21), assume that the system is stable and $K\Gamma \neq 0$. Denote*

$$\theta = (K\Gamma)^2(S + B^{'}PB). \quad (24)$$

Let the step size Δ_t of the R-bit uniform fixed-rate quantizer $\mathcal{Q}(\xi_t)$ is given by $\Delta_t = 2L_t/N$, where $N = 2^R$ and $L_t \approx 2\sigma_t\sqrt{\theta\ln N}$. When R is large, the optimal quantizer $\mathcal{Q}(\varepsilon_t)$ is given by

$$\mathcal{Q}(\varepsilon_t) = \theta^{-\frac{1}{2}}\Gamma\mathcal{Q}(\xi_t). \quad (25)$$

Moreover, the optimal performance is

$$J = J_{LQG} + D_o, \quad (26)$$

where $D_o \approx \frac{4\theta\sigma_z^2\ln N}{3N^2}$ and σ_z^2 is defined by (19).

Remark 2 *Theorem 1 is a direct conclusion from [6]. The reason we present here is to show that for unstable A, the same performance can be achieved by an adaptive quantizer which guarantees the mean square stability.*

Remark 3 *For the SISO system (6), $K\Gamma$ and $(S + B^{'}PB)$ are scalars. ξ_t is a one-dimensional random variable and the quantizer $\mathcal{Q}(\xi_t)$ is a scalar uniform fixed-rate quantizer. Note that ε_t is a random vector. Theorem 1 tells us that the optimal $\mathcal{Q}(\varepsilon_t)$ minimizing (22) can be achieved by the memoryless quantizer that only minimizes $\mathcal{E}[(\varepsilon_t - \mathcal{Q}(\varepsilon_t))^{'}\Omega(\varepsilon_t - \mathcal{Q}(\varepsilon_t))$ at each step.*

Remark 4 *When A is unstable, two problems arise. The first one is that the quantization scheme (14) with memoryless quantizers can not guarantee stability of the whole feedback system if A is unstable [12]. The second one is that a small distortion $\mathcal{E}[(\varepsilon_t - \mathcal{Q}(\varepsilon_t))'\Omega(\varepsilon_t - \mathcal{Q}(\varepsilon_t))]$ may result a large element of $\varepsilon_t - \mathcal{Q}(\varepsilon_t)$, because Ω is not full rank (it is actually rank 1 for SISO systems).*

For scalar systems with $|A| > 1$, the first problem in Remark 4 has been solved in [3]. An adaptive quantization scheme is proposed to keep the mean square stability and maintain the same LQG performance (26). The basic idea is that one can enlarge the step size of the quantizer once saturation happens. Although this may increase the distortion, the whole distortion is not changed in the sense that the probability of enlarging the step size is very small. For a scalar system, K, Γ and A are all real numbers. There is no difference between the quantization of ε_t and quantization of ξ_t. For a high-order system when A is a matrix, the quantization of ε_t and quantization of ξ_t are different, and we have to deal with the null-space of $K\varepsilon_t$.

Now we are ready to state the main results of this paper.

Theorem 2 *For the system (20-21), assume that A is unstable and $K\Gamma \neq 0$. Let $a = \bar{\lambda}(A'A)$ be the largest singular value and θ be given by (24). Let the step size Δ_t of the R-bit uniform fixed-rate quantizer $\mathcal{Q}(\xi_{t+1})$ is given by $\Delta_t = 2L_{t+1}/N$, where L_{t+1} is chosen as follows*

$$L_{t+1} = \begin{cases} L_{t+1,1} \approx (4\theta\sigma_{t+1}^2\ln N + a^n L_t^2)^{\frac{1}{2}} \\ \quad \text{if } |\xi_t - \mathcal{Q}(\xi_t)| > \frac{\Delta_t}{2} \\ L_{t+1,2} \approx (4\theta\sigma_{t+1}^2\ln N + N^{-2}L_t^2)^{\frac{1}{2}} \\ \quad \text{if } |\xi_t - \mathcal{Q}(\xi_t)| \leq \frac{\Delta_t}{2} \end{cases} \quad (27)$$

with $L_0 \approx 2\sigma_0\sqrt{\theta \ln N}$. Assume $N \gg a^{n/2}$. Then the distortion satisfies

$$D_{t+1} \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2} + \frac{a^n}{N^2} D_t. \tag{28}$$

Theorem 3 *Consider the system (6), the cost function (7), the quantized feedback controller $K\mathcal{Q}(\hat{x}_t)$ with K given by (9). Let $\hat{x}_t$ and $\mathcal{Q}(\hat{x}_t)$ be given by (10) and (14) respectively, where the quantizer $\mathcal{Q}(\varepsilon_t)$ is defined by (27), and $\mathcal{Q}(\xi_t)$ is defined in Theorem 2. When $N = 2^R \gg a^{n/2}$, the whole cost function J is given by (26).*

Proof. When $N = 2^R \gg a^{n/2}$, it follows from (28) that

$$\lim_{t\to\infty} D_t \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2}.$$

Therefore the quantization distortion is given by

$$D_o = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} D_t \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2}.$$

Using Lemma 4, we know that the cost function J is given by (26).

4 Proof of Theorem 2

Theorem 2 can be proved step by step by the following lemmas.

Lemma 6 *For the system (20-21), assume that A is unstable and $K\Gamma \neq 0$. let the quantizer be defined as (27), the PDF $p_{s_t}(s)$ of $s_t = \xi_t - \mathcal{Q}(\xi_t)$ is given by*

$$p_{s_t}(s) = \begin{cases} h_t(-s + L_t + \frac{\Delta_t}{2}) & s < -\frac{\Delta_t}{2} \\ p_{s_k}(s) & |s| \leq \frac{\Delta_t}{2} \\ h_t(-s - L_t - \frac{\Delta_t}{2}) & s > \frac{\Delta_t}{2}, \end{cases} \tag{29}$$

where $h_t(s)$ is the PDF of ξ_t. Furthermore, when $|s| > \frac{\Delta_t}{2}$, $p_{s_t}(s)$ satisfies

$$p_{s_t}(s) \leq \frac{1}{N^2}\rho(0, \frac{L_t^2}{4\ln N}, s) \tag{30}$$

for both $L_t = L_{t,1}$ and $L_t = L_{t,2}$, where $\rho(0, \frac{L_t^2}{4\ln N}, s)$ is defined as (1)

Proof. The main difficulty is the fact that $\varepsilon_{t+1} - \mathcal{Q}(\varepsilon_{t+1})$ may be large although

$$\mathcal{E}[\xi_{t+1}'\xi_{t+1}] = \mathcal{E}[(\varepsilon_{t+1} - \mathcal{Q}(\varepsilon_{t+1}))'\Omega(\varepsilon_{t+1} - \mathcal{Q}(\varepsilon_{t+1}))]$$

is small, because Ω is not full-rank. Note that we assume the feedback system is stabilizable, which means $\begin{bmatrix} K \\ KA \\ \vdots \\ KA^{n-1} \end{bmatrix}$ is of full-rank. Therefore, $\mathcal{E}[(\varepsilon_{t+n} - \mathcal{Q}(\varepsilon_{t+n}))'(\varepsilon_{t+n} - \mathcal{Q}(\varepsilon_{t+n}))]$ does not depend on $\mathcal{E}[(\varepsilon_t - \mathcal{Q}(\varepsilon_t))'(\varepsilon_t - \mathcal{Q}(\varepsilon_t))]$. Hence it is bounded by $\beta a^n \frac{\ln N}{N^2}$, where β is a constant not depending on a, n and N. The remaining procedure is similar to [3].

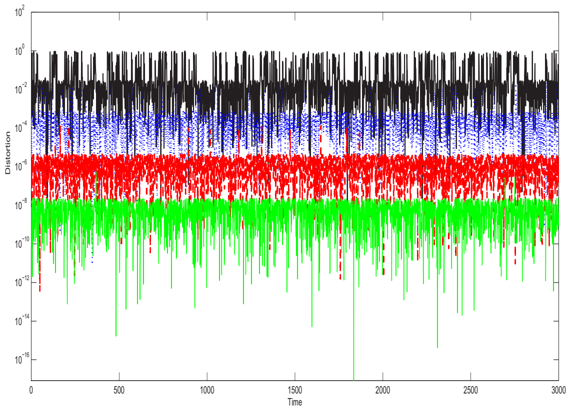

Fig. 1 Distortion for different R.

Lemma 7 *For the scalar system (20), let the quantizer be defined as (27), the saturation probability satisfies*

$$\Pr(|\varepsilon_t - \varepsilon_t^q| > \frac{\Delta_t}{2}) \leq N^{-2} \tag{31}$$

for any $t \geq 0$.

Lemma 8 *For the scalar system (20), let the quantizer be defined as (27), we have*

$$D_{t+1}^{gran} \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2} + \frac{a^n}{N^2} D_t^{gran}, \ \forall t \geq 0. \tag{32}$$

and

$$\lim_{N\to\infty} \frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = 0, \ \forall t \geq 0 \tag{33}$$

The proof of Lemma 7 and Lemma 8 are similar to [3]. Theorem 2 follows from Lemma 8 directly by using the fact that $D_{t+1} = D_{t+1}^{over} + D_{t+1}^{gran}$.

5 Numerical example

Consider the system (20) with $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 & 1 \end{bmatrix}'$ and $S + B'PB = 1$. Then we can compute $\theta = 9$. We use the quantization scheme (27) with $R = 6$, $R = 8$, $R = 12$ and $R = 16$. For each case, D_t is plotted from $t = 1$ to $t = 3000$ in Fig. 1. Table 1 shows the total times that $\mathcal{Q}(\xi_t)$ achieves the bound (i.e. it saturates) for different R. We see that the larger R is, the smaller the number of saturation times. Table 1 also shows the estimated distortion, which is consistent with the simulation result in Fig. 1.

6 Conclusion

We have studied the infinite-horizon quantized LQG control problem for general SISO systems. Under high resolution quantization framework, an adaptive fixed-rate quantization scheme is proposed to achieve the stochastic stability and the LQG performance. We have shown that the average quantization distortion has the order of $R2^{-2R}$ under high resolution quantization, which is the same with that of LPC scheme with memoryless quantizer. Numerical examples have been given to illustrate the main results.

Table 1: Saturation times of different R.

	R=6	R=8	$R = 12$	$R = 16$
$\mathcal{Q}(\xi_t)$ ♯ saturation	182	54	11	3
$\mathcal{Q}(\xi_t)$ ♯ not saturation	2818	2946	2989	2997
$R2^{-2R}$	0.018	0.0015	$8 * 10^{-6}$	$4 * 10^{-8}$

References

[1] D. S. Arnstein, “Quantization errors in predictive coders,” *IEEE Trans. Commun.*, vol. COM-26, no. 4, pp. 423–429, 1975.

[2] V. Borkar and S. Mitter, “LQG control with communication constraints,” in *Communications, Computation, Control and Signal Processing: A Tribute to Thomas Kailath,*, Norwell, MA:Kluwer, pp. 365-373, 1997.

[3] L. Chai and M. Fu, “Infinite horizon LQG control with fixed-rate quantization for scalar systems,” *Proc. 8th Would Congress Intelligent Control and Automation (WCICA)*, Jinan, China, Jul., 2010, pp. 894-899.

[4] T. R. Fischer, “Optimal quantized control,” *IEEE Trans. Automat. Contr.*, vol. AC-27, no. 4, pp. 996-998, 1982.

[5] N. Farvardin and J. W. Modestino, “Rate-distrotion performance of DPCM schemes for autoregressive sources,” *IEEE Trans. Inform. Theory*, vol. IT-31, no. 3, pp. 402–418, 1985.

[6] M. Fu, “Quantized linear quadratic Gaussian control,” *Proc. American Control Conf.*, St. Louis, USA, Jun., 2009, pp. 2172-2177.

[7] M. Fu and L. Xie, “The sector bound approach to quantized feedback control,” *IEEE Trans. Automat. Contr.*, vol. 50, no. 11, pp. 1698-1711, 2005.

[8] O. G. Guleryuz and M. T. Orchard, “On the DPCM compression of Gaussian autoregressive sequences,” *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp. 945–956, 2001.

[9] D. Hui and D. L. Neuhoff, “Asymptotic analysis of optimal fixed-rate uniform scalar quantization,” *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp. 957–977, 2001.

[10] J. C. Kieffer, “Stochastic stability for feedback quantization schemes,” *IEEE Trans. Inform. Theory*, vol. IT-28, no. 2, pp. 248–254, 1982.

[11] A. S. Matveev and A. V. Savkin, “The problem of LQG optimal control via a limited capacity communication channel,” *Systems and Control Letters,*, vol. 53, no. 1, pp. 51-64, 2004.

[12] G. N. Nair and R. J. Evans, “Stabilizability of Stochastic linear systems with finite feedback data rates,” *SIAM J. Contr. Optim.*, vol. 43, no. 2, pp. 413–436, 2004.

[13] G. N. Nair, F. Fagnani, S. Aampieri, and R. J. Evans, “Feedback control under data rate constraints: An overview,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[14] S. Tatikonda, A. Sahai and S. Mitter, “Stochastic linear control over a communication channel,” *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1549–1561, 2004.