Proceedings of the 34th Chinese Control Conference
July 28-30, 2015, Hangzhou, China

Feedback Stabilization of MIMO Systems in the Presence of Stochastic Network Uncertainties and Delays

Tian Qi[1,2], Jie Chen[2], Weizhou Su[1], and Minyue Fu[3,4]

1. School of Automation Science and Engineering, South China University of Technology, Guangzhou, P. R. China
E-mail: auqt@scut.edu.cn, wzhsu@scut.edu.cn

2. Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong
E-mail: jichen@cityu.edu.hk

3. School of Electrical and Computer Engineering, University of Newcastle, NSW 2308, Australia
E-mail: minyue.fu@newcastle.edu.au

4. State Key Laboratory of Industrial Control Technology and Department of Control Science and Engineering, Zhejiang University, Hangzhou, P. R. China

Abstract: The purpose of this paper is to study stabilization problem of linear time-invariant systems subject to stochastic multiplicative uncertainties and time delays. We consider a structured multiplicative perturbation which consists of static, zero-mean stochastic processes and we assess the stability of system based on mean-square criteria. The mean-square stabilization problem for multi-input multi-output systems generally requires solving an optimization problem involving the spectral radius of a certain closed loop transfer function matrix. This problem in general is non-convex and by and large unresolved, only approximate solutions are available based on numerical algorithms resembling to the *D-K iteration* for μ-synthesis. Our main contributions include the fundamental conditions, both necessary and sufficient, which insure that the multi-input multi-output minimum phase systems can be stabilized by output feedback in the mean-square sense. We provide a complete, computationally efficient solution in the form of a generalized eigenvalue problem readily solvable by means of linear matrix inequality optimization. For conceptual insights, limiting cases are analyzed in depth to characterize and quantify explicitly how the directions of unstable poles may affect the mean-square stabilizability of multi-input multi-output systems.

Key Words: Mean square stability and stabilization, constant time-delay, stochastic multiplicative uncertainty.

1 Introduction

For well over four decades, control of linear time-invariant (LTI) systems under stochastic multiplicative noise effect has attracted a great deal of research interest (see, e.g., [6], [7], [10], [14], [19], [20], [17], [26], [35]). Recent development in networked control shows that stochastic multiplicative noises can be used to effectively model uncertainties in communication channels, including packet loss ([10], [11], [30], [36], [38]), quantization error ([25], [24], [33]) and channel fading ([2], [10], [12]). Thus, while an age-old problem of fundamental interest by itself, the study of control under stochastic multiplicative uncertainties has a direct linkage to networked control problems, which, undoubtedly, has prevailed in the recent control literature (see, e.g., [21], [28], [29], [31], [2], [25], [39] and the references therein).

Time-delay is an enduring subject in the studies of dynamical systems, which may arise from sources ranging from signal transmission delay, computation delay, to physical transport delay. It has been known for long that delay can lead to degraded performance, poor robustness and even instability of systems. Contemporary studies reveal that such adversary effects can be particularly conspicuous in networked feedback systems. For both its intrinsic and renewed interest, there have been considerable advances in the study of time-delay systems, which, most notably, have made available various time- and frequency-domain stability analysis approaches (see, e.g., [22], [3], [27], [13], [23]). Despite the developments in stability analysis, however, the stabilization of time-delay systems proves fundamentally more difficult and remains to be a daunting task.

In this paper we study stabilization problems for discrete-time LTI systems subject to stochastic multiplicative uncertainties and time delays. In its full generality, we model the system uncertainty as a structured multiplicative stochastic perturbation, which, unlike in robust control theory (see, e.g., [40]), consists of static, zero-mean stochastic processes. Under this formulation, the uncertainty can be interpreted as state- or input-dependent random noises [20, 35], while in the networked control setting, as parallel memoryless noisy communication channels [10, 37]. We assess the system's stability based on *mean-square* criteria; in other words, the stability is evaluated statistically using the system's state variance. We derive necessary and sufficient *stabilizability* conditions for multi-input multi-output (MIMO) systems to be mean-square stabilizable in the presence of structured stochastic multiplicative uncertainties and time delays.

Our approach is inspired by the pioneering work of Willems and Blankenship [35] on stochastic multiplicative noises, who studied the closed loop stability of single-input, single-output (SISO) systems and obtained a necessary and sufficient condition for mean-square stability. In subsequent works, Hinrichsen and Pritchard [14], and Lu and Skelton [20] formulated the mean-square stability problem as one of robust stability against stochastic multiplicative uncertainties, which allowed them to obtain necessary and sufficient mean-square stability conditions for MIMO systems. In much the same spirit, Elia [10], and Xiao *et al.* [37] developed similar conditions for networked control problems. With the distinctive feature of a frequency-domain, input-

The work was performed while the first author was a postdoctoral fellow at City University of Hong Kong, Hong Kong, and is supported in part by the Hong Kong RGC under Projects CityU 111613, CityU 111511, in part by the City University of Hong Kong under Project 9380054 and in part by the National Nature Science Foundation under Grant 91273109.

output based approach, these developments share much in common with robust stability analysis and lead to stability results reminiscent of small gain conditions, herein dubbed as *mean-square small gain theorems*. We employ the mean-square small gain approach to tackle our stabilization problems.

The mean-square stabilization problem for MIMO systems generally requires solving an optimization problem involving the spectral radius of a certain closed loop transfer function matrix. This problem in general is non-convex and consequently poses a formidable technical barrier. Similar problems of minimizing the spectral radius have been widely known for their difficulties in robust synthesis [16, 40], and are also found in networked feedback stabilization problems [10, 34]. Problems in this category are by and large unresolved, and only approximate solutions are available based on e.g., numerical algorithms resembling to the *D-K iteration* for μ-synthesis [10, 40]. As a main contribution of this paper, nonetheless, we resolve this problem for minimum phase systems. Specifically, we show that for an MIMO minimum phase plant with time delays, it is both necessary and sufficient to establish the mean-square stabilizability by solving a *generalized eigenvalue problem (GEVP)*, which can be solved using *linear matrix inequality* (LMI) optimization methods. Further investigation into limiting cases shows that the stabilizability condition depends on not only the locations of the plant unstable poles, but also the directions associated with the poles. Furthermore, the delays, which may result from the plant itself, or be conjured as network delays in the networked control setting, are seen to have a direct, monotonically increasing effect on the mean-square stabilizability.

We now collect the notation throughout this paper. For any complex number z, any vector u, and any matrix A, we denote by z^*, u^*, and A^* their conjugate and conjugate transposes, respectively. For any square matrix A, we denote its spectral radius by $\rho(A)$. The Hölder ℓ_2, ℓ_1, and ℓ_∞ induced norms of a matrix $A = [a_{ij}]$ are denoted by $\|A\|$, $\|A\|_1$, and $\|A\|_\infty$, respectively, i.e.,

$$\|A\|_1 = \max_j \sum_i |a_{ij}|, \|A\|_\infty = \max_i \sum_j |a_{ij}|.$$

For any transfer function matrix $G(z)$, we represent a state-space realization of $G(z)$ by $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Let the open unit disc be denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the closed unit disc by $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, the unit circle by $\partial\mathbb{D}$, and the complements of $\mathbb{D}$ and $\bar{\mathbb{D}}$ by $\mathbb{D}^c$ and $\bar{\mathbb{D}}^c$, respectively. With respect to the unit circle $\partial\mathbb{D}$, we shall frequently encounter the Hilbert space

$$\mathcal{L}_2 := \Bigg\{ F : F(z) \text{ measurable in } \partial\mathbb{D}, \\ \|F\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|F(e^{j\theta})\|_F^2 d\theta \right)^{\frac{1}{2}} < \infty \Bigg\},$$

in which the inner product is defined as

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr}\left(F^H(e^{j\theta}) G(e^{j\theta})\right) d\theta.$$

It is well known [40] that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspace of $\mathcal{L}_2$, and they constitute orthogonal complements in $\mathcal{L}_2$. For any $F \in \mathcal{H}_2^\perp$ and $G \in \mathcal{H}_2$, $\langle F, G \rangle = 0$. Note that we use the same notation $\| \cdot \|_2$ for the spaces $\mathcal{L}_2$, $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$, but the the distinction will be self-evident from the context. Define also the Hardy space $\mathcal{H}_\infty = \{F : F(z) \text{ bounded and analytic in } \mathbb{D}^c\}$. A subset of $\mathcal{H}_\infty$, $\mathcal{RH}_\infty$, is the set of all proper stable rational transfer function matrices.

2 Problem Formation and Preliminary Results

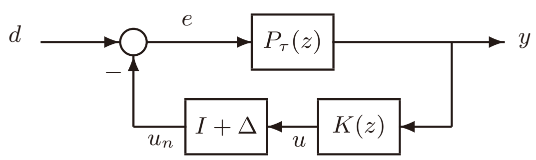


Fig. 1: Feedback system with multiplicative uncertainties and delays

We consider the feedback control system depicted in Fig. 1. Let $P_\tau(z)$ be a family of plants subject to delays:

$$P_\tau(z) = P_0(z)\Lambda(z),$$

where $P_0(z)$ is the transfer function matrix of the delay free part, and $\Lambda(z)$ is a diagonal transfer function matrix consisting of input delays:

$$\Lambda = \text{diag}\left(z^{-\tau_1}, \ \cdots \ , \ z^{-\tau_m}\right), \tau_i > 0, i = 1, \ldots, m. \quad (1)$$

The control signal $u(k)$, which is produced by LTI controller $K(z)$, is corrupted by the perturbation Δ which arises in different channels and assumes the form:

$$\begin{aligned} u_n(k) &= (I + \Delta(k))u(k), \\ \Delta(k) &= \text{diag}\left(\Delta_1(k), \ \cdots \ , \ \Delta_m(k)\right). \end{aligned} \quad (2)$$

Throughout this paper we make the following assumptions which are standard in the earlier studies of random multiplicative noises (see, e.g.,[20]):

Assumption 1 $\{\Delta_i(k), i = 1, \ldots, m\}$*, is a white noise with variance* σ_i^2.

Assumption 2 $\{\Delta_i(k)\}$ *and* $\{\Delta_j(k)\}$ *are uncorrelated processes for* $i \neq j$*, i.e.,*

$$E\{\Delta_i(k_1)\Delta_j(k_2)\} = 0, \quad \forall k_1, \ k_2 \text{ and } i \neq j.$$

Assumption 3 $\{\Delta_i(k), i = 1, \ldots, m\}$*, is uncorrelated with* $d(k)$.

Together with Assumptions 1-3, perturbation Δ in (2) renders the system in Fig. 1 as one subject to a structured stochastic multiplicative uncertainty. We focus on stabilization of the system in the mean-square sense, which means that for any bounded initial states of the plant and controller, the variances of these states will converge asymptotically to the zero matrix when $k \to \infty$. The following definition gives an equivalent notion of internal stability from an input-output perspective, appropriately tailored from [35], [20].

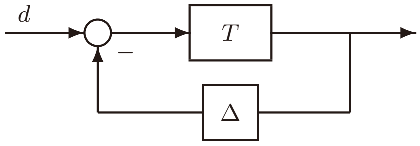


Fig. 2: Mean square small gain theorem: Linear systems with stochastic multiplicative uncertainty

Definition 1 The system in Fig. 2 is said to be mean-square input-output stable if for any input sequence $\{d(k)\}$ with bounded variance $E\{d(k)d^*(k)\} < \infty$, the variances of the error and output sequences $\{e(k)\}$, $\{y(k)\}$ are also bounded, i.e., $E\{e(k)e^*(k)\} < \infty$ and $E\{y(k)y^*(k)\} < \infty$.

The following result, herein referred to as the mean-square small gain theorem, is adapted from [20] (see also [35], [14]), which provides a necessary and sufficient condition for the mean-square input-output stability. This result will play a pivotal role in our subsequent development.

Lemma 1 (Mean-Square Small Gain Theorem) Let T be a stable LTI system, and $\Delta(k)$ be given by (2). Then under Assumptions 1-3, the system in Fig. 2 is mean-square stable if and only if

$$\rho\left(\begin{bmatrix} \|T_{11}\|_2^2 & \cdots & \|T_{1m}\|_2^2 \\ \vdots & \cdots & \vdots \\ \|T_{m1}\|_2^2 & \cdots & \|T_{mm}\|_2^2 \end{bmatrix}\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}\right) < 1. \tag{3}$$

It is straightforward to show via direct manipulation that the system in Fig. 1 can be rearranged to that in Fig. 2, with the transfer function matrix $T(z)$ given by the complementary sensitivity function of systems $P_\tau(z)$:

$$T(z) = K(z)P_\tau(z)\left[I + K(z)P_\tau(z)\right]^{-1}.$$

Thus, under Assumptions 1-3, Lemma 1 can be applied at once to determine the mean-square stability of the system. Let a right and left coprime factorization of the plant transfer function matrix $P_\tau(z)$ be given by

$$P_\tau = LM^{-1} = \tilde{M}^{-1}\tilde{L},$$

where $L, M, \tilde{L}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{L} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & Y \\ L & X \end{bmatrix} = \begin{bmatrix} M & Y \\ L & X \end{bmatrix}\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{L} & \tilde{M} \end{bmatrix} = I \tag{4}$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. It is well-known that every stabilizing controller K can be parameterized as [40]

$$\begin{aligned} K &= (Y - MR)(LR - X)^{-1} \\ &= (R\tilde{L} - \tilde{X})^{-1}(\tilde{Y} - R\tilde{M}), \quad R \in \mathbb{R}\mathcal{H}_\infty. \end{aligned} \tag{5}$$

In turn, every stable complementary sensitivity function $T(z)$ can be found as

$$T = -(Y - MR)\tilde{L}, \quad R \in \mathbb{R}\mathcal{H}_\infty. \tag{6}$$

In light of Lemma 1, the following condition for mean-square stabilizability is immediate.

Lemma 2 Let Δ be given by (2). Then under Assumptions 1-3, the system in Fig. 1 is mean-square stabilizable if and only if

$$\rho_{\min} = \inf_{R \in \mathcal{RH}_\infty} \rho(W) < 1 \tag{7}$$

where

$$W = \begin{bmatrix} \sigma_1^2\|[(Y-MR)\tilde{L}]_{11}\|_2^2 & \cdots & \sigma_m^2\|[(Y-MR)\tilde{L}]_{1m}\|_2^2 \\ \vdots & \ddots & \vdots \\ \sigma_1^2\|[(Y-MR)\tilde{L}]_{m1}\|_2^2 & \cdots & \sigma_m^2\|[(Y-MR)\tilde{L}]_{mm}\|_2^2 \end{bmatrix} \tag{8}$$

3 Main Results

For an MIMO plant, let a complex number $p \in \bar{\mathbb{D}}^c$ be a unstable pole with an output direction vector η, $\|\eta\| = 1$ if $\eta^H M(p) = 0$. In the sequel, for an all-pass transfer function matrix $M_{in}(z) = \left[\begin{array}{c|c} A_{in} & B_{in} \\ \hline C_{in} & D_{in} \end{array}\right]$, we use the familiar realization of

$$M_{in}^{-1}(z) = \left[\begin{array}{c|c} A_{in} - B_{in}D_{in}^{-1}C_{in} & -B_{in}D_{in}^{-1} \\ \hline D_{in}^{-1}C_{in} & D_{in}^{-1} \end{array}\right]. \tag{9}$$

We shall also write

$$\hat{A} = A_{in} - B_{in}D_{in}^{-1}C_{in}. \tag{10}$$

It is useful to point out that the eigenvalues of $\hat{A}$ coincide with the zeros of $M_{in}(z)$.

The following lemma (see, e.g., [32], [1]) will serve as the foundation in our subsequent developments which recast the mean-square stability problem as a scaled norm minimization.

Lemma 3 For any nonnegative matrix W,

$$\rho(W) = \inf_{\Gamma}\|\Gamma W \Gamma^{-1}\|_1 = \inf_{\Gamma}\|\Gamma W \Gamma^{-1}\|_\infty, \tag{11}$$

where the infimum is taken over the set of positive diagonal matrices $\Gamma = \operatorname{diag}\left(\gamma_1^2, \ \cdots, \ \gamma_m^2\right)$.

We shall present a general stabilizability condition for minimum phase systems with relative degree zero under input delays. For this purpose, we consider delays described in transfer function matrix $\tilde{L}$ for $\tau_i \geq 0, \ i = 1, \ \cdots, \ m$ as

$$\tilde{L}(z) = \tilde{L}_{out}(z)\begin{bmatrix} z^{-\tau_1} & & \\ & \ddots & \\ & & z^{-\tau_m} \end{bmatrix}. \tag{12}$$

Here we assume that $\tilde{L}_{out}^{-1}(z) \in \mathbb{R}\mathcal{H}_\infty$.

Theorem 1 Suppose that $P_0(z)$ is minimum phase and has relative degree zero, and that $\tilde{L}$ is given in (12). Let $M_{in}(z) = \left[\begin{array}{c|c} A_{in} & B_{in} \\ \hline C_{in} & D_{in} \end{array}\right]$ be an inner of $M(z)$. Then under Assumptions 1-3,

$$\begin{aligned} \rho_{\min} = \inf \Big\{ \mu : \ & \sigma_i^2 e_i^* D_{in}^{*-1} B_{in}^* \left(\hat{A}^*\right)^{\tau_i - 1} X \hat{A}^{\tau_i - 1} B_{in} D_{in}^{-1} e_i \\ & < \mu e_i^* \Gamma e_i, \ i = 1, \ \cdots, \ m \Big\}, \end{aligned} \tag{13}$$

where $X > 0$ is the solution to the ARE

$$A_{in}^* X A_{in} - X + C_{in}^* \Gamma C_{in} - (A_{in}^* X B_{in} + C_{in}^* \Gamma D_{in}) \times (B_{in}^* X B_{in} + D_{in}^* \Gamma D_{in})^{-1} (B_{in}^* X A_{in} + D_{in}^* \Gamma C_{in}) = 0. \tag{14}$$

Furthermore,

$$\rho_{\min} = \inf \left\{ \mu : \left(\sum_{i=1}^m \gamma_i X_i \right) - \frac{1}{\mu} \gamma_i \sigma_i^2 \hat{A}^{\tau_i - 1} B_{in} D_{in}^{-1} e_i \cdot e_i^* D_{in}^{*-1} B_{in}^* \left(\hat{A}^* \right)^{\tau_i - 1} > 0,\ \gamma_i > 0,\ i = 1,\ \cdots,\ m \right\}, \tag{15}$$

where X_i is the solution to the Lyapunov equation

$$X_i - \hat{A} X_i \hat{A}^* + B_{in} D_{in}^{-1} e_i e_i^* D_{in}^{*-1} B_{in}^* = 0. \tag{16}$$

The system in Fig. 1 is mean-square stablizable for any $\left\{\sigma_1^2,\ \cdots,\ \sigma_m^2\right\}$ if and only if $\rho_{min} < 1$.

Proof. Using Lemma 2-3, we first note that

$$\rho(W) = \inf_{\Gamma} \|\Gamma W \Gamma^{-1}\|_1 = \inf_{\Gamma} \max_i\ \sigma_i^2 \left\| \Gamma^{\frac{1}{2}} (Y - MR) \tilde{L} e_i \right\|_2^2 \gamma_i^{-2}. \tag{17}$$

Let $M = M_{in} M_{out}$ be an inner-outer factorization of M. Then for any $\Gamma = \operatorname{diag}\left(\gamma_1^2,\ \cdots,\ \gamma_m^2\right)$, $\gamma_i > 0$, $i = 1,\ \cdots,\ m$, the transfer function matrix $\Gamma^{\frac{1}{2}} M_{in}$ can be factorized as $\Gamma^{\frac{1}{2}} M_{in} = M_{\Gamma in} M_{\Gamma out}$, where $M_{\Gamma in} = \left[\begin{array}{c|c} A_{\Gamma in} & B_{\Gamma in} \\ \hline C_{\Gamma in} & D_{\Gamma in} \end{array}\right]$ is inner with the realization [18]

$$M_{\Gamma in} = \left[\begin{array}{c|c} A_{in} + B_{in} F & B_{in} (D_{in}^* \Gamma D_{in} + B_{in}^* X B_{in})^{-\frac{1}{2}} \\ \hline \Gamma^{\frac{1}{2}} C_{in} + \Gamma^{\frac{1}{2}} D_{in} F & \Gamma^{\frac{1}{2}} D_{in} (D_{in}^* \Gamma D_{in} + B_{in}^* X B_{in})^{-\frac{1}{2}} \end{array}\right],$$

with $X > 0$ being the solution to the ARE (14) and

$$F = -(B_{in}^* X B_{in} + D_{in}^* \Gamma D_{in})^{-1} (B_{in}^* X A_{in} + D_{in}^* \Gamma C_{in}).$$

By using the Bezout identity (4), it follows that for any $i = 1,\ \cdots,\ m$,

$$\begin{aligned} &\left\| \Gamma^{\frac{1}{2}} (Y - MR) \tilde{L} e_i \right\|_2^2 \gamma_i^{-2} \\ &= \left\| \Gamma^{\frac{1}{2}} (I - M\tilde{X} - MR) e_i \right\|^2 \gamma_i^{-2} \\ &= \left\| (M_{\Gamma in}^{-1} \Gamma^{\frac{1}{2}} - M_{\Gamma out} M_{out} \tilde{X} - M_{\Gamma out} M_{out} R \tilde{L}) e_i \right\|^2 \gamma_i^{-2}. \end{aligned}$$

It is readily calculated that

$$M_{\Gamma in}^{-1}(\infty) \Gamma^{\frac{1}{2}} - M_{\Gamma out}(\infty) M_{out}(\infty) \tilde{X}(\infty) = 0.$$

Thus,

$$\begin{aligned} z(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty)) \Gamma^{\frac{1}{2}} &\in \mathcal{H}_2^{\perp}, \\ z(M_{\Gamma in}^{-1}(\infty) \Gamma^{\frac{1}{2}} - M_{\Gamma out} M_{out} \tilde{X}(\infty)) &\in \mathcal{H}_2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left\| \Gamma^{\frac{1}{2}} (Y - MR) \tilde{L} e_i \right\|_2^2 \gamma_i^{-2} \\ &= \left\| z \left(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty) \right) e_i \right\|_2^2 + \Big\| \Big(\gamma_i M_{\Gamma in}^{-1}(\infty) \\ &\quad - M_{\Gamma out} M_{out} \tilde{X} + M_{\Gamma out} M_{out} R \tilde{L} \Big) e_i \Big\|_2^2 \gamma_i^{-2}. \end{aligned}$$

We proceed to calculate the $\mathcal{L}_2$ norm of $z\left(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty)\right) e_i$, which, according to (9), has the realization such that

$$\begin{aligned} &\left(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty) \right) e_i \\ &= \left[\begin{array}{c|c} A_{\Gamma in} - B_{\Gamma in} D_{\Gamma in}^{-1} C_{\Gamma in} & -B_{\Gamma in} D_{\Gamma in}^{-1} e_i \\ \hline D_{\Gamma in}^{-1} C_{\Gamma in} & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \hat{A} & -B_{in} D_{in}^{-1} \Gamma^{-\frac{1}{2}} e_i \\ \hline D_{\Gamma in}^{-1} C_{\Gamma in} & 0 \end{array}\right]. \end{aligned}$$

It follows from an standard exercise [40] that

$$\left\| z \left(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty) \right) e_i \right\|_2^2 = e_i^* D_{in}^{*-1} B_{in}^* X B_{in} D_{in}^{-1} e_i \gamma_i^{-2},$$

where X is the solution to the

$$X - \hat{A}^* X \hat{A} + C_{\Gamma in}^* D_{\Gamma in}^{*-1} D_{\Gamma in}^{-1} C_{\Gamma in} = 0. \tag{18}$$

It is readily recognized that this equation coincides with the ARE (14). Let the impulse response sequence of $M_{\Gamma out} M_{out} \tilde{X} e_i$ be denoted as $\{f_k\}$; that is,

$$M_{\Gamma out}(z) M_{out}(z) \tilde{X}(z) e_i = \sum_{k=0}^{\infty} f_k z^{-k}.$$

Since $\tilde{L} e_i$ has relative degree τ_i, the impulse response sequence of $M_{\Gamma out} M_{out} R \tilde{L} e_i$ is equal to zero for $k = 0,\ 1,\ \cdots,\ \tau_i - 1$. Furthermore, we have

$$\begin{aligned} &\left\| \left(\gamma_i M_{in}^{-1}(\infty) - M_{\Gamma out} M_{out} \tilde{X} + M_{\Gamma out} M_{out} R \tilde{L} \right) e_i \right\|_2^2 \\ &= \sum_{k=1}^{\tau_i - 1} \|f_k\|^2 + \left\| \sum_{k=\tau_i}^{\infty} f_k z^{-k} - M_{\Gamma out} M_{out} R \tilde{L} e_i \right\|_2^2. \end{aligned}$$

Evidently,

$$\inf_{R \in \mathbb{R}\mathcal{H}_\infty} \left\| \sum_{k=\tau_i}^{\infty} f_k z^{-k} - M_{\Gamma out} M_{out} R \tilde{L} e_i \right\|_2^2 = 0.$$

As such,

$$\begin{aligned} &\inf_{R \in \mathbb{R}\mathcal{H}_\infty} \left\| \Gamma^{\frac{1}{2}} (Y - MR) \tilde{L} e_i \right\|_2^2 \gamma_i^{-2} \\ &= \sum_{k=1}^{\tau_i - 1} \|f_k\|^2 \gamma_i^{-2} + e_i^* D_{in}^{*-1} B_{in}^* X B_{in} D_{in}^{-1} e_i \gamma_i^{-2}. \end{aligned} \tag{19}$$

We next seek to determine the impulse response sequence $\{f_k\}$ for $k = 1,\ \cdots,\ \tau_i - 1$. For this purpose, denote the impulse response sequences of $M_{\Gamma in}(z)$ and $M_{\Gamma in}^{-1}(z) e_i$

by $\{g_k\}$ and $\{h_k\}$, respectively. From the Bezout identity $M_{\Gamma in}M_{\Gamma out}M_{out}\tilde{X}e_i - \Gamma^{\frac{1}{2}}Y\tilde{L}e_i = \gamma_i e_i$, and the fact that $\tilde{L}e_i$ has relative degree τ_i, it follows at once that

$$\begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ \vdots & \vdots & & \\ g_{\tau_i-1} & g_{\tau_i-2} & \cdots & g_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{\tau_i-1} \end{bmatrix} = \begin{bmatrix} \gamma_i e_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

On the other hand, since $M_{\Gamma in}M_{\Gamma in}^{-1}e_i = e_i$, we have $f_k = \gamma_i h_k$, $k = 0,\ 1,\ \cdots,\ \tau_i - 1$, where $\{h_k\}$, as the impulse response sequence of $M_{\Gamma in}^{-1}(z)e_i$, is found to be

$$\begin{aligned} h_k &= -D_{\Gamma in}^{-1}C_{\Gamma in}\hat{A}^{k-1}B_{\Gamma in}D_{\Gamma in}^{-1}e_i \\ &= -D_{\Gamma in}^{-1}C_{\Gamma in}\hat{A}^{k-1}B_{in}D_{in}^{-1}e_i\gamma_i^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^{\tau_i-1}\|f_k\|^2\gamma_i^{-2} =& \gamma_i^{-2}\sum_{k=1}^{\tau_i-1} e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*k-1}C_{\Gamma in}^* D_{\Gamma in}^{*-1} \\ & \cdot D_{\Gamma in}^{-1}C_{\Gamma in}\hat{A}_{in}^{k-1}B_{in}D_{in}^{-1}e_i. \end{aligned} \quad (20)$$

In view of the Lyapunov equation (18), we then obtain

$$\begin{aligned} &\sum_{k=1}^{\tau_i-1}\|f_k\|^2\gamma_i^{-2} \\ &= \gamma_i^{-2}\sum_{k=1}^{\tau_i-1} e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*k-1}\left(\hat{A}^*X\hat{A} - X\right) \\ &\quad \cdot \hat{A}_{in}^{k-1}B_{in}D_{in}^{-1}e_i \\ &= \gamma_i^{-2}\sum_{k=1}^{\tau_i-1} e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*k}X\hat{A}_{in}^{k}B_{in}D_{in}^{-1}e_i \\ &\quad - \gamma_i^{-2}\sum_{k=1}^{\tau_i-1} e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*k-1}X\hat{A}_{in}^{k-1}B_{in}D_{in}^{-1}e_i \\ &= \gamma_i^{-2}e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*\tau_i-1}X\hat{A}_{in}^{\tau_i-1}B_{in}D_{in}^{-1}e_i \\ &\quad - \gamma_i^{-2}e_i^* D_{in}^{*-1}B_{in}^*XB_{in}D_{in}^{-1}e_i. \end{aligned}$$

As a consequence,

$$\begin{aligned} &\inf_{R\in\mathbb{R}\mathcal{H}_\infty}\rho(W) \\ &= \inf_{R\in\mathbb{R}\mathcal{H}_\infty}\inf_{\Gamma}\max_i\ \sigma_i^2\left\|\Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} \\ &= \inf_{\Gamma}\inf_{R\in\mathbb{R}\mathcal{H}_\infty}\max_i\ \sigma_i^2\left\|\Gamma^{\frac{1}{2}}(Y - MR)\tilde{L}e_i\right\|_2^2\gamma_i^{-2} \\ &= \inf_{\Gamma}\max_i\ \sigma_i^2\gamma_i^{-2}e_i^* D_{in}^{*-1}B_{in}^*\hat{A}_{in}^{*\tau_i-1}X\hat{A}_{in}^{\tau_i-1}B_{in}D_{in}^{-1}e_i. \end{aligned}$$

Alternatively, we may write the last equality as

$$\begin{aligned} \rho_{\min} =& \inf_{\Gamma}\Big\{\mu:\ \sigma_i^2 e_i^* D_{in}^{*-1}B_{in}^*\hat{A}^{*\tau_i-1}X\hat{A}^{\tau_i-1}B_{in}D_{in}^{-1}e_i\gamma_i^{-2} \\ &\qquad \le \mu,\ i = 1,\ \cdots,\ m\Big\} \\ =& \inf_{\Gamma}\Big\{\mu:\ \sigma_i^2 e_i^* D_{in}^{*-1}B_{in}^*\hat{A}^{*\tau_i-1}X\hat{A}^{\tau_i-1}B_{in}D_{in}^{-1}e_i \\ &\qquad \le \mu e_i^*\Gamma e_i,\ i = 1,\ \cdots,\ m\Big\}. \end{aligned}$$

This establishes (13). To prove (15), we calculate $D_{\Gamma in}^{-1}C_{\Gamma in}$, which is found to be

$$D_{\Gamma in}^{-1}C_{\Gamma in} = -(B_{in}^*XB_{in} + D_{in}^*\Gamma D_{in})^{-1/2}B_{in}^*X\hat{A}.$$

Thus, the Lyapunov equation (18) can be rewritten as

$$\begin{aligned} &X - \hat{A}^*X\hat{A} + \hat{A}^*XB_{in} \\ &\qquad \times (B_{in}^*XB_{in} + D_{in}^*\Gamma D_{in})^{-1}B_{in}^*X\hat{A} = 0, \end{aligned}$$

and further as

$$\begin{aligned} &X - \hat{A}^*(X - XB_{in}(B_{in}^*XB_{in} \\ &\qquad + D_{in}^*\Gamma D_{in})^{-1}B_{in}^*X)\hat{A} = 0. \end{aligned}$$

Employing the *Sherman-Morrison-Woodbury formula* [15], the Lyapunov equation (18) can be written as

$$X - \hat{A}^*\left(X^{-1} + B_{in}\left(D_{in}^*\Gamma D_{in}\right)^{-1}B_{in}^*\right)^{-1}\hat{A} = 0,$$

or equivalently,

$$X^{-1} - \hat{A}X^{-1}\hat{A}^* + B_{in}D_{in}^{-1}\Gamma^{-1}D_{in}^{*-1}B_{in}^* = 0. \quad (21)$$

Let X_i be the solution to (16). Then it is readily seen that the solution to (21), and equivalently that to the ARE (14), is given by

$$X = \left(\sum_{i=1}^{m}\gamma_i^{-2}X_i\right)^{-1}.$$

Substitute X into the inequalities in (13). Then by a repeated use of *Schur complement* [15], the inequalities in (13) are found to be equivalent to

$$\left(\sum_{i=1}^{m}\gamma_i^{-2}X_i\right) - \frac{1}{\mu}\gamma_i^{-2}\sigma_i^2\hat{A}^{-1}B_{in}D_{in}^{-1}e_i e_i^* D_{in}^{*-1}B_{in}^*\hat{A}^{*-1} > 0.$$

The proof is now completed by setting γ_i^{-2} to γ_i. ■

We note that the conditions (15) constitute a GEVP problem, which can be efficiently solved using LMI optimization techniques [4, 5]. Thus, these results furnish a computational and effective necessary and sufficient condition for mean square stabilizability. Thanks to the fact that the solutions depend only on the matrices $A_{in},\ B_{in},\ D_{in}$, state-feedback mean square stabilizability problem has been solved as well. It is worth noting that since the eigenvalues of $\hat{A}$ coincide with the plant unstable poles, Theorem 1 captures the effect of delay on conditions for mean square stabilizability which becomes proportionally more demanding as the delays in the plant increase. It is also important to point out that in addition to the locations of the plant unstable poles, the realizations of the inner factor M_{in} also depend on their directions, and as such, so do the stabilizability conditions. In particular, when $B_{in}D_{in}^{-1}e_i = 0$, the ith inequality in (13) is rendered moot.

In what follows we study two limiting cases of multiple poles to analyze in further depth the dependence of the stabilizability on pole direction. For simplicity, we restrict our attention to proper plants subject to equal delay length of each input channel. In the case of multiple poles $p_j \in \bar{\mathbb{D}}^c$

with pole direction vectors η_j, $j = 1, \cdots, n$, we define three sets with respect to η_j and e_i, respectively

$$I_j = \{1 \le i \le m : \eta_j^* e_i \ne 0\},$$

$$J_i = \{1 \le j \le n : \eta_j^* e_i \ne 0\},$$

$$\mathcal{I} = \{1 \le i \le m : \eta_i = \eta_j = \eta, \eta^* e_i \ne 0\}.$$

Corollary 1 Suppose that $P_\tau(z)$ has no zero in $\mathbb{D}^c$ and be subject to delays of same length $\tau_i = \tau = 0,\ 1$.
(i) Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \cdots, n$, be the unstable poles of $P_\tau(z)$ with parallel directions spanned by a pole direction vector η. Then

$$\rho_{\min} = \frac{1}{\sum_{i\in\mathcal{I}} \frac{1}{\sigma_i^2}} \left(\prod_{i=1}^n |p_i|^{2(\tau-1)}\right)\left(\prod_{i=1}^n |p_i|^2 - 1\right). \quad (22)$$

(ii) Let $p_i \in \bar{\mathbb{D}}^c$, $i = 1, \cdots, n, n \le m$, be the unstable poles of $P_\tau(z)$ with orthogonal directions spanned by η_i. Then

$$\rho_{\min} = \max_i \frac{\sum_{j\in J_i} |p_j|^{2(\tau-1)}\left(|p_j|^2 - 1\right)}{\sum_{i\in I_j} \frac{1}{\sigma_i^2}}. \quad (23)$$

Proof. We prove the corollary for $\tau = 1$; the case $\tau = 0$ follows analogously and hence is omitted. It follows from [9] that

$$M_{in}(z) = \prod_{i=1}^n M_i(z) = \prod_{i=1}^n \left(I - \eta_i\eta_i^* + \frac{z - p_i}{1 - p_i^* z}\eta_i\eta_i^*\right)$$

where $M_i(z) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$ can be constructed as

$$M_i(z) = \left[\begin{array}{c|c} \frac{1}{p_i^*} & \frac{\sqrt{|p_i|^2-1}}{p_i^*}\eta_i^* \\ \hline \frac{\sqrt{|p_i|^2-1}}{p_i^*}\eta_i & I - \left(1 + \frac{1}{p_i^*}\right)\eta_i\eta_i^* \end{array}\right].$$

For the n parallel pole directions, we may assume that the pole direction vectors $\eta_i = \eta$. In this case, the solution of ARE (14) is given as follows

$$X = \begin{bmatrix} X_\Sigma & \\ & \frac{1}{\eta^*\Gamma^{-1}\eta} \end{bmatrix}, \quad X_\Sigma = \begin{bmatrix} \frac{1}{\eta^*\Gamma^{-1}\eta} & & \\ & \ddots & \\ & & \frac{1}{\eta^*\Gamma^{-1}\eta} \end{bmatrix}.$$

Let $\prod_{i=1}^{n-1} M_i(z) = \left[\begin{array}{c|c} A_\Sigma & B_\Sigma \\ \hline C_\Sigma & D_\Sigma \end{array}\right]$, for $\tau = 1$, it follows from the proof of Theorem 1 that

$$\rho_{\min} = \inf_\Gamma \max_i \sigma_i^2 \Bigg\{\gamma_i^{-2} e_i^* D_\Sigma^{*-1} B_\Sigma^* X_\Sigma B_\Sigma D_\Sigma^{-1} e_i + \gamma_i^{-2} e_i^* D_\Sigma^{*-1} D_n^{*-1} B_n{}^* \frac{1}{\eta^*\Gamma^{-1}\eta} B_n D_n^{-1} D_\Sigma^{-1} e_i\Bigg\}$$

Consequently

$$\begin{aligned}\rho_{\min} &= \inf_\Gamma \max_i \sigma_i^2 \Bigg\{\gamma_i^{-2} e_i^* D_\Sigma^{*-1} B_\Sigma^* X_\Sigma B_\Sigma D_\Sigma^{-1} e_i \\ &\quad + \frac{|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i=1}^n |\eta^* e_i|^2 \gamma_i^{-2}} \prod_{i=1}^{n-1} |p_i|^2 (|p_n|^2 - 1)\Bigg\} \\ &= \inf_\Gamma \max_i \sigma_i^2 \left(\prod_{i=1}^n |p_i|^2 - 1\right) \frac{|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i=1}^n |\eta^* e_i|^2 \gamma_i^{-2}}.\end{aligned}$$

The infimum is found at such γ_i, $i \in \mathcal{I}$ that for $i \ne j$, $i,\ j \in \mathcal{I}$,

$$\begin{aligned}&\sigma_i^2 \left(\prod_{i=1}^n |p_i|^2 - 1\right) \frac{|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i=1}^n |\eta^* e_i|^2 \gamma_i^{-2}} \\ &\qquad = \sigma_i^2 \left(\prod_{i=1}^n |p_i|^2 - 1\right) \frac{|\eta^* e_i|^2 \gamma_i^{-2}}{\sum_{i=1}^n |\eta^* e_i|^2 \gamma_i^{-2}},\end{aligned}$$

which gives rise to the solution (22). To establish Corollary 1-(ii), it suffices to note that with mutually orthogonal pole directions η_i, $i = 1, \cdots, n$, an all-pass factor of $\Gamma^{\frac{1}{2}} M(z)$ can be constructed as [8]

$$M_{\Gamma in}(z) = \begin{bmatrix} \hat{\eta}^* \\ \vdots \\ \hat{\eta}_n^* \\ \hat{U}^* \end{bmatrix}^* \begin{bmatrix} \frac{z-p_1}{1-p_1^* z} & & & \\ & \ddots & & \\ & & \frac{z-p_n}{1-p_n^* z} & \\ & & & I \end{bmatrix} \begin{bmatrix} \hat{\eta}^* \\ \vdots \\ \hat{\eta}_n^* \\ \hat{U}^* \end{bmatrix},$$

where $\hat{\eta}_i = \Gamma^{-\frac{1}{2}}\eta_i / \|\Gamma^{-\frac{1}{2}}\eta_i\|$, and $\begin{bmatrix}\hat{\eta}_1 & \cdots & \hat{\eta}_n & \hat{U}\end{bmatrix}$ is a unitary matrix. Thus, we obtain

$$\begin{aligned}\rho_{\min} &= \inf_\Gamma \max_i\ \sigma_i^2 \left\|\left(M_{\Gamma in}^{-1}(z) - M_{\Gamma in}^{-1}(\infty)\right) e_i\right\|_2^2 \\ &= \inf_\Gamma \max_i \sigma_i^2 \sum_{j=1}^n |\hat{\eta}_j^* e_i|^2 \left\|\frac{1 - p_j^* z}{z - p_j} + p_j^*\right\|_2^2 \\ &= \inf_\Gamma \max_i \sigma_i^2 \sum_{j\in J_i} \left(|p_j|^2 - 1\right) \frac{|\eta_j^* e_i|^2 \gamma_i^{-2}}{\sum_{i\in I_j} |\eta_j^* e_i|^2 \gamma_i^{-2}}. \quad (24)\end{aligned}$$

Define

$$x_i = \sigma_i^2 \sum_{j\in J_i} \left(|p_j|^2 - 1\right) \frac{|\eta_j^* e_i|^2 \gamma_i^{-2}}{\sum_{i\in I_j} |\eta_j^* e_i|^2 \gamma_i^{-2}}.$$

It follows that the minimax problem in (24) achieves the minimum at $x_i = x_k$ for $i \ne k$, $i,\ k \in I_j$, subject to the constraint

$$\sum_{i\in I_j} \frac{x_i}{\sigma_i^2} = \sum_{j\in J_i} \left(|p_j|^2 - 1\right).$$

This leads to the solution (23), thus completing the proof. ∎

Thus, when in the extreme their directions are parallel, the unstable poles contribute to the difficulty to stabilization *collectively*. In contrast, when the directions are orthogonal, the poles tend to affect the stabilizability *individually* in an additive manner. This latter scenario is particularly complex and varies widely depending on how the individual pole directions are aligned with the Euclidean basis.

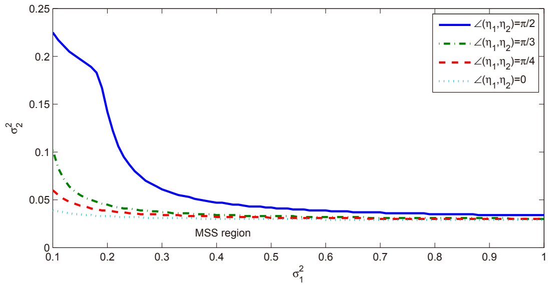


Fig. 3: Pole effect on MMS: direction.

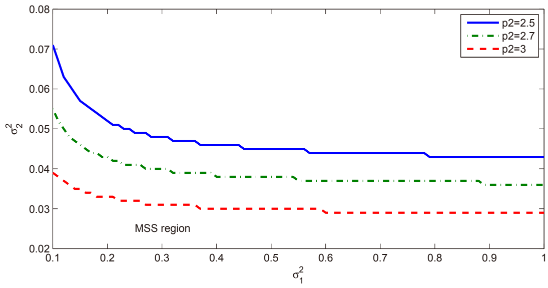
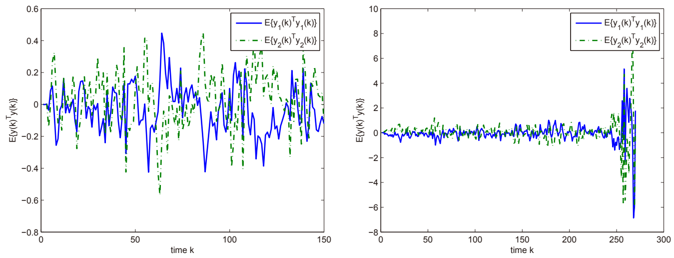
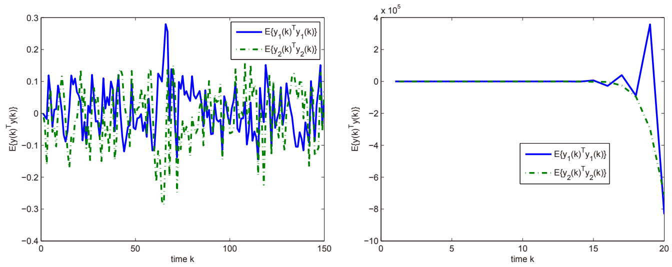


Fig. 4: Pole effect on MMS: location.



(a) $\sigma_2^2 = 0.038$ (b) $\sigma_2^2 = 0.043$

Fig. 5: Sensitivity of stabilizability bound



(a) $\tau_2 = 1$ (b) $\tau_2 = 2$

Fig. 6: Delay effort

4 Example

In this section, we use an example to illustrate our results. Consider a 2×2 plant:

$$P_\tau(z) = \begin{bmatrix} \frac{z}{z-2} & \frac{z}{z-2} \\ \frac{(a+3)z^2+(a-3)z}{(z-2)(z-p_1)} & \frac{(3a+1)z^2-(3a-1)z}{(z-2)(z-p_1)} \end{bmatrix} \begin{bmatrix} z^{-\tau_1} & \\ & z^{-\tau_2} \end{bmatrix}.$$

For simplicity, let $\tau_1 = \tau_2 = 1$. It is easy to find that the plant has two unstable poles, p_1 with output direction vector $\eta_1 = \frac{1}{\sqrt{a^2+1}}[a, 1]^*$ and $p_2 = 2$ with output direction vector $\eta_2 = \frac{1}{\sqrt{2}}[1, 1]^*$. As such

$$\cos \angle(\eta_1, \eta_2) = \frac{a+1}{\sqrt{2a^2+2}}.$$

We will illustrate how the unstable pole affect mean-square stability in terms of location and direction. To this end, we first fix $p_1 = 3$. Fig. 3 shows the regions of (σ_1^2, σ_2^2) given by the condition (13) for different $\angle(\eta_1, \eta_2)$. Note that any values of (σ_1^2, σ_2^2) under curves guarantee closed loop mean square stability, and is indicated as the mean-square stability (MMS) region. It is clear that the MMS region is shrink gradually with the decrease of angle $\angle(\eta_1, \eta_2)$. Second, we fix $a = 1$ and allow p_1 to vary. Fig. 4 demonstrates the effect of pole location on MSS region. One can see that the farther is the distance from the unit circle, the smaller is the MMS region.

Next, we fix $a = 1$ and $p_1 = 3$, and show the necessity and sufficiency of condition (13). We use MATLAB to generate a zero-mean white process $\{\Delta_1(k)\}$ so that with $\sigma_1^2 = 0.1$. The processes $\{d_i(k), i = 1, 2\}$ are generated to meet $E\{d_i^2(k)\} = 0.1$ similarly. In this case, the maximal variance σ_2^2 of $\Delta_2(k)$ allowed for mean-square stabilization is found as $\sigma_{2max}^2 = 0.04$. Fig. 5 shows that for σ_2^2 marginally less ($\sigma_2^2 = 0.038$) and greater ($\sigma_2^2 = 0.043$) than 0.04, the output variance $E(y_i^2(k)), i = 1, 2$ converges and diverges, respectively; in other words, the closed loop system is and is not mean-square stable for these values of σ_2^2.

Finally, consider $\sigma_2^2 = 0.0017$, the variances pair $(\sigma_1^2, \sigma_2^2) = (0.1, 0.0017)$ belongs to MMS region; indeed, it is well in the MMS region. Then, there is a optimal controller to stable the plant $P_\tau(z)$ in mean-square sense. Fig. 6(a) shows the stable responses of the system. With the same controller, however, if we change $\tau_2 = 1$ to $\tau_2 = 2$. Fig. 6(b) shows that the mean-square stability is lost.

5 Conclusion

This paper investigates the mean-square stabilization problem for LTI systems subject to delays and stochastic multiplicative uncertainties. Our contributions include computationally efficient necessary and sufficient conditions to insure a general minimum phase MIMO systems to be mean square stabilized over output feedback, as well as some explicit analytic conditions for limiting cases. The condition for general case amounts to solving a GEVP problem which is readily solvable using LMI optimization techniques, and coincidentally furnishes a solution to state-feedback mean square stabilizability problem as well. Deeper investigation into limiting cases shows that for MIMO systems, the stabilizability condition is sensitive to the directions of the unstable poles. Furthermore, the delays are seen to have a direct, monotonically increasing effect on the mean-square stabilizability.

References

[1] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical sciences*, New York: Academic, 1979.

[2] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Trans. Automat. Contr.*, vol. 52, no. 8, pp. 1391-1403, Aug. 2007.

[3] E. K. Boukas and Z. K. Liu, *Deterministic and Stochastic Time-Delay System*, Boston: Birkhäuser, 2002.

[4] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix Inequalities in system and Control Theory*, SIAM, 1994.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge Univ. Press, Cambridge, UK: 2004.

[6] R. Brockett and D. Liberzon, "Quantized feedback stabilizalion of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1279-1289, Jun. 2000.

[7] B. S. Chen and W. Zhang, "Stochastic $\mathcal{H}_2/H_\infty$ control with state-dependent noise," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 45-57, Jan. 2004.

[8] J. Chen, "Sensitivity integral relations and design tradeoffs in linear multivariable feedback systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1700-1716, Oct. 1995.

[9] J. Chen, "Integral constraints and performance limits on complementary sensitivity," *Syst. Control Lett.*, vol. 39, no. 1, pp. 45-53, Jan. 2000.

[10] N. Elia, "Remote stabilization over fading channels," *Syst. Control Lett.*, vol. 54, No. 3, pp. 237-249, Mar. 2005.

[11] N. Elia and J. N. Eisenbeis, "Limitations of linear control over packet drop networs," *IEEE Trans. Automat. Contr.*, Vol. 56, No. 4, pp. 826-841, Apr. 2011.

[12] J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky, "Minimum variance control over a gaussian communication channel," *IEEE Trans. Automat. Contr.*, vol. 56, no. 8, pp. 1751-1765, Aug. 2011.

[13] K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay System*, Boston: Birkhäuser, 2003.

[14] D. Hinrichsen and A. Pritchard, "Stability radii of systems with stochastic uncertainty and their optimization by output feedback," *SIAM J. Control Optim.*, vol. 34, pp. 1972-1998, 1996.

[15] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge, UK: Cambridge University Press, 1985.

[16] M. Khammash, M.V. Salapaka, and T. Van Voorhis, "Robust synthesis in ℓ_1: A globally optimal solution," *IEEE Trans. Automat. Contr.*, vol. 46, no. 11, pp. 1744-1754, Nov. 2001.

[17] L. Li and H. Zhang. "Optimal control for stochastic systems with multiple input delays," in *Proc. 2013 Chinese Control Conf.*, pp.2281-2286, XiAn, China, Jul. 2013.

[18] Z. Lin, B. M. Chen, A. Saberi, and Y. A. Shamash, "Inner-outer factorization of discrete-time transfer function matrices," *IEEE Trans. Circ. Sys., Part I: Fundamental Theory and Applications*, vol. 43, pp. 941-945, Nov. 1996.

[19] J. Lu, and R. Skelton, "Robust variance control for systems with finite-signal-to-noise uncertainty," *Automatica*, vol. 36, no. 4, pp. 511-525, Apr. 2000.

[20] J. Lu and R. E. Skelton, "Mean-square small gain theorem for stochastic control: discrete-time case," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 490-494, Mar. 2002.

[21] G. N. Nair, F. Fagnani, S. Sampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview," *Proc. IEEE*, vol. 95, pp. 108-137, 2007.

[22] S. I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*, Heidelberg, Germany: Spring-Verlag, 2001.

[23] S. I. Niculescu and K. Gu, *Advances in Time-Delay Systems*, Heidelberg, Germany: Spring-Verlag, 2004.

[24] T. Qi and W. Su, "Optimal tracking performance of a unstable linear system with a quantized control input," in *Proc. 2008 Chinese Control Conf.*, Kunming, China, Jul. 2008, pp. 437-441.

[25] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation," *IEEE Trans. Automat. Contr.*, Vol. 58, No. 3, pp. 554-568, Mar. 2013.

[26] M. Rami, X. Chen, J. Moore, and X. Zhou, "Solvability and asymptotic behavior of genralized riccati equations arising in indefinite stochastic LQ controls," *IEEE Trans. Automat. Contr.*, vol. 46, no. 3, pp. 428-440, Mar. 2001.

[27] J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[28] *Special Issue on Networked Control System, IEEE Trans. Automat. Contr.*, vol. 49, Sep. 2004.

[29] *Special Issue on Technology of Networked Control Systems, Proc. IEEE*, vol. 95, no. 1, Jan. 2007.

[30] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, I. Jordan, and S. Sastry, "Kalman filtering with intermittent observations,"*IEEE Trans. Automat. Contr.*, vol. 49, pp. 1453-1464, Sep. 2004.

[31] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proc. the IEEE*, vol. 95, pp. 163-187, Jan. 2007.

[32] J. Stoer and C. Witzgall, "Thansformations by deagonal matrices in a normed spaces," *Numerische Math.*, vol. 3, pp. 158-171, Apr. 1962.

[33] W. Su, T. Qi, J. Chen, and M. Fu, "Optimal tracking design of an MIMO linear system with quantization effects," *Proc. 2011 IFAC World Congress*, Milano, Italy, Aug. 2011, pp. 3268-3273.

[34] F. Vargas, J. Chen, and E. Silva, "Necessary and sufficient conditions for mean square stabilization over MIMO SNR-constrained channels," *Proc. 51st IEEE Conf. Decision and Control*, Maui, Hawaii, December 2012.

[35] J. Willems and G. Blankenship, "Frequency domain stability criteria for stochastic systems," *IEEE Trans. Automat. Contr.*, vol. 16, pp. 292-299, Aug. 1971.

[36] J. Wu, T. Chen, "Design of networked control systems with packet dropouts," *IEEE Trans. Automat. Contr.*, vol. 52, no. 7, pp.1314-1319, Jul. 2007

[37] N. Xiao, L. Xie, and L. Qiu, "Mean square stabilization of multi-input systems over stochastic multiplicative channels," *Proc. Joint 48th IEEE Conf. Decision and Control and 2009 Chinese Control Conf.*, Shanghai, China, Dec. 2009, pp. 6893-6898.

[38] N. Xiao, L. H. Xie and L. Qiu, "Feedback stabilization of discrete-Time networked systems over fading channels," *IEEE Trans. Automat. Contr.*, Vol. 57, No. 9, pp. 2167-2189, Sep. 2012.

[39] K. Y. You and L. H. Xie, "Minimum data rate for mean square stabilizability of linear systems with Markovian packet losses," *IEEE Trans. Automatic Control*, Vol. 56, No. 4, pp. 772-785, Apr. 2011.

[40] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, New Jersey: Prentice-Hall, 1986.