# Optimal Control of Linear Discrete-Time Systems with Quantization Effects

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**Abstract:** This paper studies optimal control designs for networked linear discrete-time systems with quantization effects and/or fading channel. The quantization errors and/or fading channels are modeled as multiplicative noises. The $H_2$ optimal control in mean-square sense is formulated. The necessary and sufficient condition to the existence of the mean-square stabilizing solution to a modified algebraic Riccati equation (MARE) is presented. The optimal $H_2$ control via state feedback for the systems is designed by using the solution to the MARE. It is a nature extension for the result in standard optimal discrete-time $H_2$ state feedback design. It is shown that this optimal state feedback design problem is eigenvalue problem (EVP) and the optimal design algorithm is developed.

**Key Words:** Quantization error, Multiplicative noise, Optimal control, Algebraic Riccati equation.

## 1 INTRODUCTION

During last two decades, networked control system has attracted many research interests. In these works, the main issues include modeling communication channel uncertainties such as data rate limits, quantization errors, channel fading, data package drop, channel delays etc., analyzing design constraints on feedback systems caused by these uncertainties and developing feedback control design methods for networked systems. It is shown that the multiplicative noise model may be an efficient way in modeling uncertainties which appear in communication channels in feedback systems, such as, packet loss ([23], [24] and [32]), quantization errors ([20], [26]), fading channels [7] and etc. These motive further research in networked control and stochastic control areas. Since linear time invariant systems with multiplicative noises involve nonlinearities, several issues in stabilization and optimal control problems for the systems are still opened meanwhile LQG control theory has been well established for LTI systems with additive noises.

The studies for LTI systems with multiplicative noises can be traced back to the later of 1960's. In 1971, Willems and Blankenship [28] formulated the mean-square stability problem for linear time-invariant (LTI) SISO feedback systems with multiplicative noises, respectively, and presented the sufficient and necessary condition of the stability for the systems. These results are extended to MIMO systems in [16]. In [13], [14] and [17], the criterion of mean-square stability is studied for LTI systems with multiplicative noises. And then, the mean-square stabilization via state feedback is studied for the systems (for example see [1] and [29]). The optimal control is another issue which has been studied widely. The earlier works were reported in [30] and [31] where optimal regulation problem is formulated based on a quadratic cost function. It is shown in these works that the optimal state feedback is determined by a positive semi-definite solution to a modified algebraic Riccati equation (MARE). After then, the necessary and sufficient condition for this optimal control design problem is presented in terms of the solvability of a linear matrix inequality (LMI) in [19]. In [6] and [34], the relation between the existence of the solution to the optimal regulation problem and the mean-square stabilizability, observability is studied. The sufficient conditions are presented for the optimal design in several cases.

In [7], Elia revisited the mean-square stabilization problem for a MIMO system with multiplicative noises which are used to model communication channel uncertainties in networked systems. Sinopoli *et. al.* [23], [24] studied Kalman filtering with packet loss in communication channels, where the channel uncertainty caused by packet loss is modeled as a multiplicative noise. Sufficient conditions are presented for the convergence of the Kalman filter with intermittent observations. Xiao *et. al.* [32] studied the stabilization problem for networked systems with packet loss and presented an explicit connection between mean-square stabilization and signal-to-noise ratios of control communication channels. In networked setting, we [20], [22], [26] formulated the optimal tracking problem with quantization error by applying multiplicative noise models and solved optimal tracking problem via output feedback for a mini-

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mum phase LTI system.

In this paper, networked feedback systems with quantization effects and communication channels are considered. The quantization errors and/or uncertainties in communication channels are modeled by multiplicative noises. By this model, we formulates optimal $H_2$ control problem in mean-square sense for the networked systems. Then the formulations for optimal regulation and optimal tracking problems are considered. Following the stochastic small gain theorem [16], the existence of the optimal controller is studied for the optimal $H_2$ control in mean-square sense. It is turned out that the optimal state feedback gain in this problem is determined by a positive semi-definite solution to a MARE. And the necessary and sufficient condition for the existence of this solution is presented. It extends the standard optimal $H_2$ state feedback design for LTI systems to LTI systems with multiplicative noises. Moreover, it is found that the optimal $H_2$ design problem is a generalized eigenvalue problem, i.e., the global optimal solution exists and can be solved by a set of linear matrix inequalities (L-MI) and line search technique. The remainder of this paper is organized as follows. In Section 2, we formulate the optimal design problems for an LTI system with multiplicative noises. In Section 3, optimal $H_2$ design via state feedback is discussed. Section 4 concludes the paper.

The notation used throughout this paper is fairly standard. We denotes the set of real n-dimensional vector space by $R^n$. Denote the transpose of a matrix by $(\cdot)^T$, and rank of a matrix by Rank($\cdot$), respectively. Denote the mathematical expectation operator by E($\cdot$) and the spectral radius by $\rho(\cdot)$, respectively. We denote block-diagonal matrix formed from the arguments by diag$\{\cdots\}$. For conjugate symmetric matrix X, Y, the notation $X > Y$(respectively, X$\geq Y$) is used to denote $X-Y$ is positive definite (positive semidefinite).

## 2 Problem Formulation

This paper study optimal control and optimal tracking problems for networked LTI systems with quantization effects (or input multiplicative noises).

We first consider the networked feedback system shown in Fig. 1. In the system, $P$ is a plant, $K$ is a feedback controller and the channel is modeled as a quantization law and noise free communication channel. The control signal $u_q$ from $R^m$ is quantized in the sending side and the quantized control signal $u_q$ is received at the receiving side perfectly. The signal $y$ from $R^l$ is the measurement. The model of

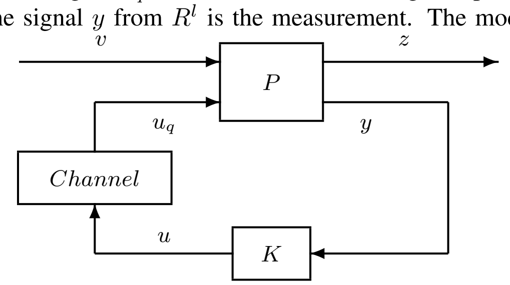


Figure 1: The networked system with quantization effects and a state noise

the plant with quantizers (or input multiplicative noises) is given as below:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 v(k) + B_2 u_q(k) \\ z(k) &= C_1 x(k) \\ y(k) &= C_2 x(k) \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state, $u_q(k) \in R^m$ is the quantized control input (or control input corrupted by multiplicative noises). The input $v$ is independent and identically distributed (*i.i.d*) process with zero mean. The covariance of $v$ is an identity matrix. The quantization errors $d(k)$ are modeled with multiplicative noises $\omega_1, \cdots, \omega_m$, i.e.,

$$d(k) = u_q(k) - u(k) = \omega(k)u(k) \tag{2}$$

where $\omega = \text{diag}\{\omega_1, \cdots, \omega_m\}$, diag $\{*, \cdots, *\}$ is a diagonal matrix and $*$ is an entry in the diagonal of the matrix. $\omega$ is referred to as the relative quantization error.

*Assumption 1 The noises $\omega_i(k), i \in \{1, \cdots, m\}$ are mutually independent white noise processes with*

$$E\{\omega_i(k)\} = 0,\ E\{\omega_i(k_1)\omega_i(k_2)\} = \begin{cases} \sigma_i^2, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases},$$

*for $i \neq j$,*

$$E\{\omega_i(k_1)\omega_j(k_2)\} = 0.$$

Now, we formulate optimal $H_2$ control problem for the system in Fig. 1. It is assumed that the *i.i.d.* process $\{v(k)\}$ is independent of $\omega$. The averaged power of the output signal $z$ with regard to the noises $v$ and $\omega$ is considered:

$$\mathbf{E}\|z\|_p^2 = \mathbf{E} \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} z(i)^T z(i). \tag{3}$$

Compared with the standard $H_2$ norm of linear time invariant systems (for example see [33]), $\mathbf{E}\|z\|_p^2$ is a generalized $H_2$ norm in mean-square sense for the closed-loop system with multiplicative noises. The optimal control design is to find an optimal control law $K_{1,opt}$ to minimize this norm, i.e.

$$K_{1,opt} = arg \inf_K \mathbf{E}\|z\|_p^2. \tag{4}$$

This problem is referred as to optimal $H_2$ control in mean-square sense. The $H_2$ norm in mean-square sense of the system is denoted by $J_{H_2}$.

In general, the closed-loop systems of the plants with feedback controllers and quantized control inputs are diagrammed as the system shown in Fig. 2 where $\omega$ is the relative quantization error given by the model (2). The nominal closed-loop system $G_e$ is partitioned as below

$$G_e = \begin{bmatrix} G_{z0} & G_{z1} & \cdots & G_{zm} \\ G_{10} & G_{11} & \cdots & G_{1m} \\ & \cdots & \cdots & \\ G_{m0} & G_{m1} & \cdots & G_{mm} \end{bmatrix} \tag{5}$$

which is compatible to the sizes of signals $v(k), d(k), z(k)$ and $u(k)$ in the closed-loop system.

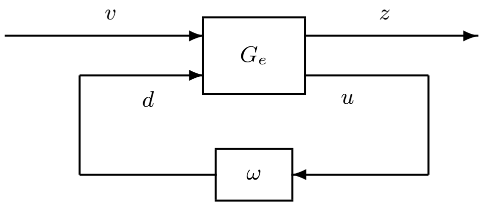


Figure 2: The closed-loop system with multiplicative noises

Now, define the mean-square stability and mean-square stabilizibility for the closed-loop systems and the plants with quantized control inputs of which the quantization errors are modeled as multiplicative input noises by Assumption 1, respectively.

*Definition 1 It is referred to as that the closed-loop system shown in Fig. 2 is mean-square stable if, for any bounded initial state of the system with zero inputs, the covariance of the state is convergent to zero.*

*Definition 2 It is referred to as that a plant above is mean-square stabilizable, if there exists a state control law $u(k) = Fx(k)$ such that the resultant closed-loop system is mean-square sable, i.e., for any bounded initial state $x(0)$ with zero input, $\lim_{k\to\infty} E\{x(k)^T x(k)\} = 0$.*

To study the mean-square stability of the closed-loop system, let

$$G = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ & \cdots & \\ G_{m1} & \cdots & G_{mm} \end{bmatrix}. \tag{6}$$

Then, let

$$\hat{G} \triangleq \begin{bmatrix} \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \ddots & \\ \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix} \tag{7}$$

and

$$\Sigma \triangleq \mathrm{diag}\left\{\sigma_1^2, \cdots, \sigma_m^2\right\}. \tag{8}$$

*Lemma 1 (see [16]) Suppose that the nominal closed-loop system $G_e$ of the system shown in Fig. 2 is stable. The system is mean-square stable if and only if the spectral radius of the matrix $\hat{G}\Sigma$ is less than one, i.e.,*

$$\rho(\hat{G}\Sigma) < 1. \tag{9}$$

## 3 Optimal $H_2$ Control in Mean-Square Sense

In this section, the optimal $H_2$ control in mean-square sense is studied for the plant (1). From the quantization error model (2), the plant is decomposed to the nominal plant

$$x(k+1) = Ax(k) + B_1 v(k) + B_2 d(k) + B_2 u(k) \tag{10}$$
$$z(k) = C_1 x(k)$$
$$y(k) = C_2 x(k)$$

and the multiplicative noise part

$$d(k) = \omega(k) u(k). \tag{11}$$

Following the diagram in Fig. 2, we can see that the nominal part $G_e$ in the closed-loop system of the plant (1) and a feedback controller $K$ is determined by the nominal plant (10) and the feedback controller $K$. Now, the $H_2$ norm in mean-square sense of the closed-loop system is studied in terms of $G_e$.

*Lemma 2 The $H_2$ norm in mean-square sense of the closed-loop system is given by*

$$J_{H_2} = \|G_{z0}\|_2^2 + \begin{bmatrix} \|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2 \end{bmatrix} \Sigma (I - \hat{G}\Sigma)^{-1} \times \begin{bmatrix} \|G_{10}\|_2^2 \\ \vdots \\ \|G_{m0}\|_2^2 \end{bmatrix}. \tag{12}$$

Proof is presented in Appendix .
Let

$$\hat{G}_e = \begin{bmatrix} \|G_{z0}\|_2^2 & \|G_{z1}\|_2^2 & \cdots & \|G_{zm}\|_2^2 \\ \|G_{10}\|_2^2 & \|G_{11}\|_2^2 & \cdots & \|G_{1m}\|_2^2 \\ & \cdots & \cdots & \\ \|G_{m0}\|_2^2 & \|G_{m1}\|_2^2 & \cdots & \|G_{mm}\|_2^2 \end{bmatrix}. \tag{13}$$

*Lemma 3 Consider the closed-loop system shown in Fig. 2 with given $\sigma_1, \cdots, \sigma_m$. For any given $\sigma_0 > 0$, it holds that*

$$J_1 < \frac{1}{\sigma_0^2} \quad \text{and} \quad \rho(\hat{G}\Sigma) < 1 \tag{14}$$

*if and only if there exists a $\sigma_0 > 0$ so that*

$$\rho(\hat{G}_e \Sigma_e) < 1$$

*where $\Sigma_e \triangleq \mathrm{diag}\left\{\sigma_0^2, \sigma_1^2, \cdots, \sigma_m^2\right\}$.*

*Remark 1 Lemma 3 is a general version of Theorem 4.1 in [16] which is available in the case when $v$ and $z$ are scalars.*

Consider the plant (10-11) with a state feedback control law $u(k) = Fx(k)$. The nominal transfer function $G_e$ in the closed-loop system is given by

$$G_e = \begin{bmatrix} C_1(zI - A_F)^{-1}B_1 & C_1(zI - A_F)^{-1}B_{21} & \cdots & \cdots & C_1(zI - A_F)^{-1}B_{2m} \\ F_1(zI - A_F)^{-1}B_1 & F_1(zI - A_F)^{-1}B_{21} & \cdots & \cdots & F_1(zI - A_F)^{-1}B_{2m} \\ & \cdots & \cdots & \cdots & \\ F_m(zI - A_F)^{-1}B_1 & F_m(zI - A_F)^{-1}B_{21} & \cdots & \cdots & F_m(zI - A_F)^{-1}B_{2m} \end{bmatrix} \tag{15}$$

where $A_F = A + B_2 F$, $B_{2i}$ and $F_i$ are $i$-th column of $B_2$ and $i$-th row of $F$, respectively.
From Lemma 3, we can see that the spectral radius $\rho(\hat{G}_e\Sigma_e)$ of the matrix $\hat{G}_e\Sigma_e$ plays key role in the optimal state feedback design. Hence, the feature of positive matrix is needed.

*Lemma 4 (see [12]) For any square matrix $T$, if all entries of the matrix are greater than zero, then its spectral radius $\rho(T)$ (or largest eigenvalue) is given as follows*

$$\rho(T) = \inf_{\Gamma} \|\Gamma T \Gamma^{-1}\|_\infty \triangleq \inf_{\Gamma} \max_{j} \sum_{i} \gamma_i^2 t_{ij} \frac{1}{\gamma_j^2} \quad (16)$$

*where* $\Gamma = \mathrm{diag}\left\{\gamma_1^2, \cdots, \gamma_m^2\right\} > 0$.

To study the optimal $H_2$ state feedback design in mean-square sense for the system, the standard optimal $H_2$ state feedback design is reviewed for the plant (17)

$$\begin{aligned} x(k+1) =& Ax(k) + B_1 v(k) + B_2 u(k) \quad (17) \\ z(k) =& \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \Gamma^{\frac{1}{2}} \end{bmatrix} u(k). \end{aligned}$$

*Lemma 5 The discrete algebraic Riccati equation (18) has the unique stabilizing solution $X \geq 0$ (i.e., all eigenvalues of $A + B_2 F$ with $F = -(\Gamma + B_2^T X B_2)^{-1} B_2 X A$ are in the open unit disc) if and only if $(A, B_2)$ is stabilizable and $(A, C_1)$ has no unobservable pole in the unit circle,*

$$X = A^T X A - A^T X B_2 (\Gamma + B_2^T X B_2)^{-1} B_2 X A + C_1 C_1^T. \quad (18)$$

The proof follows [18] and [27].

*Lemma 6 The minimum $H_2$ norm of the closed-loop system (17) via state feedback and optimal state feedback gain are given by*

$$\min_F \left\| \begin{bmatrix} C_1 \\ \Gamma^{\frac{1}{2}} F \end{bmatrix} (zI - A - B_2 F)^{-1} B_1 \right\|_2^2 = \mathrm{tr}\left\{B_1^T X B_1\right\}$$

*and*

$$F = -(\Gamma + B_2^T X B_2)^{-1} B_2 X A,$$

*respectively, where $X$ is a positive semi-definite solution to (18).*

The proof follows that of Theorem 6.4.1 in [3].

*Theorem 1 For given $\sigma_1, \cdots, \sigma_m$, the modified algebraic Riccati equation (19)*

$$\begin{aligned} X =& A^T X A + C_1^T C_1 \\ & - A^T X B_2 \left(\Phi + B_2^T X B_2\right)^{-1} B_2^T X A \quad (19) \end{aligned}$$

*where $\Phi = \mathrm{diag}\left\{\sigma_1^2 B_{21}^T X B_{21}, \cdots, \sigma_m^2 B_{2m}^T X B_{2m}\right\}$ has a mean-square stabilization solution if and only if the plant (10-11) is mean-square stabilizable, $(A, C_1)$ has no unobservable in the unit circle.*
*The minimum $H_2$ norm in mean-square sense of the plant (10-11) via state feedback is*

$$J_1 = \mathrm{tr}\left\{B_1^T X B_1\right\}$$

*and the optimal state feedback gain is given by*

$$F = -\left(\Phi + B_2^T X B_2\right)^{-1} B_2^T X A$$

*where $X$ is a positive semi-definite solution to (19).*

Sketch of proof: Following Lemmas 2 and 6, the optimal $H_2$ design problem in mean-square sense is a standard $H_2$ optimal design problem with $m$ linear inequalities. According to Lemma 5, the standard $H_2$ optimal design problem is solved by the discrete algebraic Riccati equation (18). Combining (18) with the $m$ linear inequalities, we can obtain a sequence of discrete algebraic Riccati equation. This sequence converges to the modified algebraic Riccati equation (19) if and only only the plant (10-11) is mean-square stabilizible. Applying Lemmas 5 and 6 to the sequence of discrete algebraic Riccati equation leads to this theorem.
Moreover, this theorem can be extended to the case where the plant (1) has state dependent multiplicative noises $\omega_{si}$, $i = 1, \cdots, m_s$ as below:

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{i=1}^{m_s} A_i \omega_{si}(k) + B_1 v(k) + B_2 u_q(k) \\ & \qquad (20) \\ z(k) &= C_1 x(k) \end{aligned}$$

where the matrixes $A_i$, $i = 1, \cdots, m_s$ have rank one. The noises $\omega_{si}$, $i = 1, \cdots, m_s$ are *i.i.d.* processes with zero mean and variances $\sigma_{si}$, respectively. Moreover, $\omega_{si}$, $i = 1, \cdots, m_s$, $\omega_i$, $i = 1, \cdots, m$ and $v$ are mutually independent.

*Corollary 1 The optimal $H_2$ state feedback design is solvable if and only if the plant is mean-square stabilizable via a state feedback and $\left(A, \begin{bmatrix} C_1 \\ A_1 \\ \vdots \\ A_{m_s} \end{bmatrix}\right)$ has no unobservable pole in the unit circle.*
*Furthermore, the MARE (21) has a positive semi-definite solution*

$$\begin{aligned} X =& A^T X A + \sum_{i=1}^{m_s} \sigma_{si}^2 A_i^T X A_i + C_1^T C_1 \\ & - A^T X B_2 \left(\Phi + B_2^T X B_2\right)^{-1} B_2^T X A. \quad (21) \end{aligned}$$

*The minimum $H_2$ norm in mean-square sense is given by $J_{H_2} = \mathrm{tr}\left\{B_1^T X B_1\right\}$.*

## 4 Conclusions

In this paper, the optimal $H_2$ control via state feedback for discrete-time systems with quantization effects has been studied. Under the multiplicative noise model, the necessary and sufficient condition is presented for the solvability of this optimal control problem. Furthermore, this optimal control result is generalized to a more version for plants with state-dependent noises.

## 5 Appendix: Proof of Lemma 2

Let $\{G_e(0), G_e(1), G_e(2), \cdots\}$ be impulse response associated with the transfer function $G_e$ in (5). Following the partition of $G_e$ in (5), we write $G_e(k)$, $k = 0, 1, 2, \cdots$ as below:

$$G_e(k) = \begin{bmatrix} g_{z0}(k) & g_{z1}(k) & \cdots & g_{zm}(k) \\ g_{10}(k) & g_{11}(k) & \cdots & g_{1m}(k) \\ \vdots & \vdots & \ddots & \vdots \\ g_{m0}(k) & g_{m1}(k) & \cdots & g_{mm}(k) \end{bmatrix}.$$

Note the fact that the nominal plant (10) has a strict proper function from input $(v, d, u)$ to output $(z, y)$. The transfer function $G_e$ of the plant with a proper feedback control law is also strict proper. Hence, it holds that $g_{ij}(0) = 0$, $i = 1, \cdots, m$, $j = 0, \cdots, m$.
When the initial state of the system in Fig. 2 is at rest, the output of the system is given by

$$\begin{bmatrix} z(k) \\ u(k) \end{bmatrix} = \sum_{i=0}^{k} G_e(i) \begin{bmatrix} v(k-i) \\ d(k-i) \end{bmatrix} \tag{22}$$

with $d(k-i) = \omega(k-i)u(k-i)$.
Subsequently, the output $z(k)$ is written a summation as below:

$$z(k) = z_v(k) + \sum_{j=1}^{m} z_{d_j}(k) \tag{23}$$

where

$$z_v(k) = \sum_{i=0}^{k} G_{zd}(i)v(k-i) \tag{24}$$

$$z_{d_j}(k) = \sum_{i=0}^{k} G_{zj}(i)d_j(k-i),\ j = 1, \cdots, m.$$

Following Assumption 1 and assumption on $v$, we obtain that

$$\mathbf{E}z^T(k)z(k) = \mathbf{E}z_v^T(k)z_v(k) + \sum_{j=1}^{m} \mathbf{E}z_{d_j}^T(k)z_{d_j}(k). \tag{25}$$

Denote the autocorrelation matrixes and power spectral density of $z_v$ by $R_{zv}(\tau)$ and $S_{zv}$, i.e.

$$R_{zv}(\tau) = \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} \mathbf{E}z_v(i+\tau)z_v^T(i).$$

From the definition of the averaged power, it holds that

$$\mathbf{E}\|z_v\|_p^2 = \mathrm{tr}\{R_{zv}(0)\} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left\{S_{zv}(e^{j\omega})\right\} d\omega. \tag{26}$$

Noting (24), we rewrite (26) as below:

$$\mathbf{E}\|z_v\|_p^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left\{G_{z0}(e^{j\omega})S_v(e^{j\omega})G_{z0}^*(e^{j\omega})\right\} d\omega.$$

Since the power spectral density $S_v$ of the signal $v$ is the identity matrix, it holds that

$$\mathbf{E}\|z_v\|_p^2 = \|G_{z0}\|_2^2. \tag{27}$$

Denote the power spectral densities of $z_{d_j}$ and $u_j$ by $S_{d_j}$ and $S_{u_j}$, respectively. Following the process on the discussion above, we obtain that

$$\mathbf{E}\|z_{d_j}\|_p^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left\{G_{zj}(e^{j\omega})S_{d_j}(e^{j\omega})G_{zj}^*(e^{j\omega})\right\} d\omega. \tag{28}$$

Note the fact that

$$S_{d_j}(e^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_{d_j}(\tau)e^{-j\tau\omega} \quad \text{and} \quad d_j(k) = \omega_j(k)u_j(k). \tag{29}$$

It follows from Assumption 1 that

$$R_{d_j}(0) = \lim_{k\to\infty} \frac{1}{k+1}\sigma_j^2 \mathbf{E}\sum_{i=0}^{k} u_j^2(i) = \sigma_j^2 \mathbf{E}\|u_j\|_p^2 \tag{30}$$

and

$$R_{d_j}(\tau) = 0,\ \tau = 1, 2, \cdots. \tag{31}$$

Substituting (30) and (31) into (29) yields that

$$S_{d_j}(e^{j\omega}) = \sigma_j^2 \mathbf{E}\|u_j\|_p^2. \tag{32}$$

Subsequently, (28) is rewritten as

$$\mathbf{E}\|z_{d_j}\|_p^2 = \sigma_j^2 \|G_{zj}\|_2^2 \mathbf{E}\|u_j\|_p^2. \tag{33}$$

With (25), (27) and (33), we have that

$$\mathbf{E}\|z\|_p^2 = \|G_{z0}\|_2^2 + \sum_{i=1}^{m} \sigma_i^2 \|G_{zi}\|_2^2 \mathbf{E}\|u_i\|_p^2. \tag{34}$$

In the light of the discussion above, we can obtain straightforwardly that

$$\mathbf{E}\|z_{u_j}\|_p^2 = \|G_{j0}\|_2^2 + \sum_{i=1}^{m} \sigma_i^2 \|G_{ji}\|_2^2 \mathbf{E}\|u_i\|_p^2, \quad j = 1, \cdots, m \tag{35}$$

or

$$\begin{bmatrix} \mathbf{E}\|z_{u_1}\|_p^2 \\ \vdots \\ \mathbf{E}\|z_{u_m}\|_p^2 \end{bmatrix} = \begin{bmatrix} \mathbf{E}\|G_{10}\|_p^2 \\ \vdots \\ \mathbf{E}\|G_{m0}\|_p^2 \end{bmatrix} + \hat{G}\Sigma \begin{bmatrix} \mathbf{E}\|z_{u_1}\|_p^2 \\ \vdots \\ \mathbf{E}\|z_{u_m}\|_p^2 \end{bmatrix}. \tag{36}$$

From (34) and (36), we obtain (12).

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