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Stability of Kalman Filters subject to Intermittent Observations

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Abstract— **This paper addresses the stability of a Kalman filter when measurements are intermittently available due to the non-transparent communication channel between sensor and estimator. More specifically, we present a method to determine whether the expected value of the estimation error covariance is bounded for a given stochastic network model. The method applies to general discrete-time LTI systems and adopts the finite state Markov channel model.**

I. Introduction

Characterizing the behavior of a Kalman filter when measurements are intermittently available has attracted a great interest in the recent years. This is partly due to the development of communications technologies, which today permit distributed control and monitoring in a broad range of applications. When measurements sent through a communication channel are subject to random losses, the estimation accuracy of a Kalman filter deteriorates. In [1], the authors established the mathematical foundations for the basic problem and pointed out that the covariance of the estimation error does not reach a steady state. Since then, several authors have studied different aspects of the problem, using different assumptions on network models and protocols.

When a Kalman filter is subject to intermittent observations (KFIO), the error covariance (EC) matrix becomes of random nature. The study of the stochastic properties of the EC is a central issue for performance and stabitity analysis of KFIO. Several authors have studied different properties of the EC. In [2], [3] the authors present methods to derive bounds on the asymptotic probability distribution of the error covariance matrix. Although previous works were concerned with finding these bounds, it was not until [4], [5] that the existence of a unique and invariant APDEC was shown.

One of the first problems proposed in the area is whether the asymptotic expected value of the EC can be bounded by a constant matrix or not [1]. Since then, several authors presented important contributions that permit assessing the stability of KFIO for particular classes of systems and communication channels [1], [5]–[21]. In particular, [1] described the measurement dropout using a Bernouli process, i.e., the dropout process is described by a sequence of independent and identically distributed (i.i.d.) binary random variables. Under this assumption, the authors derived a condition that is necessary and sufficient for the stability of KFIO assciated to systems whose observalion matrix $\mathbf{C}$ is invertible. The assumption was later relaxed to only requiring that the part of $\mathbf{C}$ associated with the observable subspace has full column ranksystems whose observalion matrix $\mathbf{C}$ has full column rank [10]. The set of systems for which the necessary conditions for stability are also sufficient was further extended in [11], where the authors studied the case where the unstable eigenvalues of $\mathbf{A}$ have different magnitudes.

In an attempt to account for some communication channel phenomena, such as fading and congestion, [6] introduced the Gilbert-Elliott model [22], [23] in the context of KFIO. In order to do that, [6] introduced the notion of peak covariance for the estimation error. More recently, the equivalence between the two notions of stability has been studied in [5], [7]. For scalar plants, [6] was able to provide a necessary and sufficient for stability of KFIO. In [8], a new sufficient condition for the stability of the peak covariance was established. peak covariance matches the necessary one presented in [12]. In [7], a necessary and sufficient condition for the stability of the peak covariance for second order systems was derived, while for higher order systems, only a necessary condition was presented. In [9] the authors derive a necessary and sufficient condition for the stability of the KFIO for non-degenerate systems. A further extension on the class of systems for which the stalility conditions are known is by [9], where the authors showed that the condition still holds for a larger class of systems, called non-degenerate. Finally, in [24], the present authors presented a necessary and sufficient condition for systems whose dynamics matrix $\mathbf{A}$ can be diagonalised.

In many applications, particularly when the network conditions change slowly in comparison with the sampling time, the use of a higher order Markov process (also known as finite state Markov channel (FSMC) [25]) produces a more accurate description of the packet dropouts. In the context of KFIO, this network model has been studied in [26], where the existence of a stationary APDEC was investigated, and in [24] where the authors provided the stability conditions for diagonalisable systems.

In this paper we present a method for assessing the stability of a KFIO for general discrete-time LTI systems and semi-Markov network models. This communication model includes the i.i.d. and Gilberto Elliott models studied previously as special cases. By doing so, we extend the class of systems for which the stability conditions are known by relaxing the assumption that the matrix $\mathbf{A}$ is diagonalisable. That is, the result presented in this paper applies for general discrete-time finite-dimensional LTI systems.

This problem has remained open for over a decade, and as it may be expected the proofs are lenghty and technically involving. Due to the limited space in this paper we supress

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the proofs of our results. A journal paper with the complete proofs is currently under preparation.

II. Problem Formulation

Consider the discrete-time LTI system

$$\begin{cases} \mathbf{x}_{t+1} &= \mathbf{A}\mathbf{x}_t + \mathbf{w}_t \\ \mathbf{y}_t &= \mathbf{C}\mathbf{x}_t + \mathbf{v}_t \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^n$ is the vector of states, $\mathbf{y} \in \mathbb{R}^p$ is the vector of measurements, $\mathbf{w} \sim N(\mathbf{0}, \mathbf{Q})$ with $\mathbf{Q} \geq 0$ is the process noise, $\mathbf{v} \sim N(\mathbf{0}, \mathbf{R})$ with $\mathbf{R} \geq 0$ is the measurement noise, $\mathbf{A} \in \mathbb{C}^{n\times n}$ is the state matrix and $\mathbf{C} \in \mathbb{C}^{p\times n}$ is the measurement matrix. The initial state is $\mathbf{x}_0 \sim N(\mathbf{0}, \mathbf{P}_0)$, with $\mathbf{P}_0 \geq 0$. The measurements are sent to an estimator through a network subject to random packet losses, but without delays. We assume that an error correcting scheme is used such that if an error is introduced during transmission, it can be detected. If the transmission error cannot be corrected, then the corresponding measurement is discarded. Let g_t be a binary random variable describing the arrival of a valid measurement at time t. We denote $g_t = 1$ when $\mathbf{y}_t$ is available for the estimator and $g_t = 0$ otherwise.

We run a Kalman filter to obtain an estimate $\hat{\mathbf{x}}_t$ of the state $\mathbf{x}_t$. The update equation of the EC matrix $\mathbf{P}_t$ (i.e., the covariance of the error $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ given the measurements received up to time $t-1$) can be written as follows [1]:

$$\mathbf{P}_t = \begin{cases} \Phi_0(\mathbf{P}_{t-1}) & , \ g_{t-1} = 0, \\ \Phi_1(\mathbf{P}_{t-1}) & , \ g_{t-1} = 1, \end{cases} \tag{2}$$

with

$$\Phi_0(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{Q},$$
$$\Phi_1(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{Q} - \mathbf{A}\mathbf{X}\mathbf{C}^*\left(\mathbf{C}\mathbf{X}\mathbf{C}^* + \mathbf{R}\right)^{-1}\mathbf{C}\mathbf{X}\mathbf{A}^*.$$

Define $\mathbb{B} \triangleq \{0, 1\}$ and let Γ_t be the binary sequence indicating whether the measurements $y_\tau, \tau = 0, \cdots, t-1$ are available, i.e.,

$$\Gamma_t \triangleq (g_0, \cdots, g_{t-1}). \tag{3}$$

For a given matrix $0 \leq \mathbf{X} \in \mathbb{R}^{n\times n}$ and sequence $S \in \mathbb{B}^T$, we define the map $\mathbf{\Psi} : \mathbb{R}^{n\times n} \times \mathbb{B}^T \to \mathbb{R}^{n\times n}$, by

$$\mathbf{\Psi}(\mathbf{X}, S) = \Phi_{S(T)} \circ \Phi_{S(T-1)} \circ \ldots \Phi_{S(1)}(\mathbf{X}). \tag{4}$$

Notice that the EC at time t only depends on the initial EC $\mathbf{P}_0$ and the sequence of available measurements up to time $t-1$, i.e.,

$$\mathbf{P}_t = \mathbf{\Psi}(\mathbf{P}_0, \Gamma_t) = \Phi_{g_{t-1}} \circ \Phi_{g_{t-2}} \circ \ldots \Phi_{g_0}(\mathbf{P}_0). \tag{5}$$

Notice that above, g_t and Γ_t are random quantities, while S is deterministic.

In this paper we derive a necessary condition and a sufficient condition, with a trivial gap between them, for the boundedness of the asymptotic value of the norm of the expected error covariance (AEEC). The definition of this quantity is given below.

Definition 1: For a given initial EC $\mathbf{P}_0 \geq 0$, the AEEC norm is defined as

$$G(\mathbf{P}_0) \triangleq \limsup_{t\to\infty} G_t(\mathbf{P}_0), \tag{6}$$

where

$$G_t(\mathbf{P}_0) \triangleq \|\mathbb{E}(\mathbf{P}_t)\| = \left\| \sum_{S\in\mathbb{B}^t} \mathbb{P}(\Gamma_t = S)\mathbf{\Psi}(\mathbf{P}_0, S) \right\|. \tag{7}$$

The results are obtained under the assumptions that the measurements are dropped according to the FSMC model. We formally introduce this assumption below.

Definition 2: Let g_t, $t \in \mathbb{Z}$, be a stationary random process. Its Markov order ν is defined as the smallest non-negative integer such that, for all $\mu \geq 1$, the following holds

$$\mathbb{P}(g_t = 1 | g_{t-\nu-\mu}, \cdots, g_{t-1}) = \mathbb{P}(g_t = 1 | g_{t-\nu}, \cdots, g_{t-1}). \tag{8}$$

We say that the communication channel follows the FSMC model if the measurement drop process g_t is a stationary random process with Markov order ν.

We use the following assumption.

Assumption 1: The packet dropout process g_t is stationary, and its Markov order ν is finite. Also, $0 < \mathbb{P}(g_t = 1 | g_{t-\nu}, \cdots, g_{t-1}) < 1$, for any $g_{t-\nu}, \cdots, g_{t-1}$.

Remark 1: Notice that the i.i.d. network model is a special case of the FSMC model with $\nu = 0$. It is fully characterized by the parameter $\mathbb{P}(g_t = 1) \triangleq \lambda$ [1]. Similarly, a Gilbert-Elliott model is obtained using the FSMC model with $\nu = 1$ and is fully characterized by two parameters: the recovery rate $q = \mathbb{P}(g_t = 1 | g_{t-1} = 0)$ and the failure rate $p = \mathbb{P}(g_t = 0 | g_{t-1} = 1)$ [6].

Remark 2: The general FSMC model has been widely used to model wireless channels in a variety of applications (see [25] for a survey of principles and applications). The problem of channel modeling, i.e., how to obtain a model of the channel based on its statistics, has been studied by several authors. For instance, [27] and [28] presented methods to obtain a model given a set of observations from a channel.

III. Main Result

The results uses a particular division of the system (1) into subsystems, or finite multiplicative order (FMO) blocks. We assume that the system is in its upper Jordan cannonical form, i.e., the matrix $\mathbf{A}$ is Jordan. Notice this is without loss of generality since the similarity transformation used to obtain the Jordan form is always well-defined and applying it will not change whether the AEEC of the KFIO is bounded or not.

Before presenting the main result, the definition of FMO block is introduced.

Definition 3: [FMO block] Consider the following partition of $\mathbf{A}$

$$\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \ \cdots, \ \mathbf{A}_K), \tag{9}$$

where the sub-matrices $\mathbf{A}_k$ are chosen such that, for any k, the diagonal entries of $\mathbf{A}_k$ have a common FMO up to a constant (i.e., there exists $N_k \in \mathbb{N}$ such that all the entries in the main diagonal of $\mathbf{A}_k^{N_k}$ are equal to $\alpha_k^{N_k}$, with $\alpha_k \in \mathbb{C}$), and for any k and l with $k \neq l$, the diagonal entries of the matrix $\text{diag}(\mathbf{A}_k, \mathbf{A}_l)$ do not have FMO up to any constant. Also, consider the partition

$$\mathbf{C} = [\mathbf{C}_1 \ \cdots \ \mathbf{C}_K], \tag{10}$$

such that each $\mathbf{C}_k$ has the same number of columns as $\mathbf{A}_k$. Then the pair $(\mathbf{A}_k, \mathbf{C}_k)$ is called an FMO block of the system (1).

Notice that if $(\mathbf{A}_k, \mathbf{C}_k)$ is an FMO block, then each sub-matrix $\mathbf{A}_k$ can be written as

$$\begin{aligned}\mathbf{A}_k &= \alpha_k \tilde{\mathbf{A}}_k \\ \tilde{\mathbf{A}}_k &= \text{diag}(\exp(i2\pi\theta_{k,1}), \ \cdots, \exp(i2\pi\theta_{k,K_k})) + \mathbf{J}_k,\end{aligned}$$

where $\mathbf{J}_k$ is strictly upper triangular, i.e., its non-zero entries lie above its main diagonal. Also, $\alpha_k \in \mathbb{C}$ and $\theta_{k,j} \in \mathbb{Q}$ for $j = 1, \cdots, K_k$. Notice that for any k and l with $k \neq l$, α_k/α_l is not a root of unity, i.e., $(\alpha_k/\alpha_l)^m \neq 1$ for all $m \in \mathbb{N}$. For convenience, in the rest of this chapter it will be assumed that the sub-matrices $\mathbf{A}_k$ are ordered such that $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_K|$.

Consider a sampling time interval $[0, t-1]$ and its associated measurement arrival sequence Γ_t. Let t_i, $i = 1, \cdots, s$, be all the time instants such that $\Gamma_t(t_i + 1) = 1$. Then, the available measurements can be written in a vector $\mathbf{z}_t$ as follows

$$\mathbf{z}_t = \begin{bmatrix}\mathbf{y}'_{t_1} & \mathbf{y}'_{t_2} & \cdots & \mathbf{y}'_{t_s}\end{bmatrix}' = \mathbf{O}(\Gamma_t)\mathbf{x}_0 + \mathbf{f}(\Gamma_t) \tag{11}$$

where

$$\mathbf{O}(\Gamma_t) \triangleq \begin{bmatrix}\mathbf{O}_1(\Gamma_t) & \mathbf{O}_2(\Gamma_t) & \ldots & \mathbf{O}_K(\Gamma_t)\end{bmatrix}, \tag{12}$$

with

$$\mathbf{O}_k(\Gamma_t) \triangleq \left[\left(\mathbf{C}_k\mathbf{A}_k^{t_1}\right)' \quad \left(\mathbf{C}_k\mathbf{A}_k^{t_2}\right)' \quad \cdots \quad \left(\mathbf{C}_k\mathbf{A}_k^{t_s}\right)'\right]'. \tag{13}$$

Also, $\mathbf{f}(\Gamma_t) = [\mathbf{f}'_1, \cdots, \mathbf{f}'_s]'$, with $\mathbf{f}_i = \sum_{j=0}^{t_i-1} \mathbf{C}\mathbf{A}^{t_i-1-j}\mathbf{w}_j + \mathbf{v}_{t_i}$, for $i = 1, \cdots, s$. For $k = 1, \cdots, K$, let $\mathcal{N}_k^t$ denote the subset of sequences S in $\mathbb{B}^t$ such that $\mathbf{O}_k(S)$ does not have FCR, i.e.,

$$\mathcal{N}_k^t \triangleq \{S \in \mathbb{B}^t : \mathbf{O}_k(S) \text{ does not have FCR}\}. \tag{14}$$

To simplify the notation, in the rest of the paper we will use $\mathbb{P}(\mathcal{N}_k^t)$ to denote $\mathbb{P}(\Gamma_t \in \mathcal{N}_k^t)$. The main result is presented in terms of the quantity $\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t}$. We present in Lemmas 2 and 3 a method compute this quantity.

We now state the main result of the paper.

Theorem 1: Consider the system (1) satisfying Assumption 1. If

$$|\alpha_k|^2 \limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} < 1, \text{ for all } k \in \{1, \cdots, K\}, \tag{15}$$

then the AEEC norm $G(\mathbf{P}_0)$ is finite for any $\mathbf{P}_0 \geq 0$, and if

$$|\alpha_k|^2 \limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} > 1, \text{ for some } k \in \{1, \cdots, K\}, \tag{16}$$

then $G(\mathbf{P}_0)$ is infinite for any $\mathbf{P}_0 \geq 0$.

Notice that the result is valid for any initial condition $\mathbf{P}_0 \geq 0$, so we will omit the argument of $G(\cdot)$ in the rest of the paper. Also, there is a trivial gap between (15) and (16), i.e., we do not state whether G is finite or not when $|\alpha_k|^2 \limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} = 1$. This gap is common in the literature, see e.g., [1], [9].

Remark 3: Notice that the result in Theorem 1 permits carrying out the stability analysis in each FMO block independently. This allows splitting the problem into smaller sub-problems that are easier to analyze. This is a key property of our decomposition of the system into FMO blocks, and showing this property is a central issue for proving our result.

Contrary to most results available in the literature, where the conditions for the AEEC to be finite are cast in terms of the parameters of the network model, we state our result in terms of the quantity $\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t}$. This permits stating the result for the general FSMC network model. We now address the issue of how to evaluate this quantity. The following 2 definitions are used to this end.

Definition 4 (Reduced FMO block): Let L_k be the number of Jordan blocks of $\mathbf{A}_k$. Consider the following partition of $\mathbf{A}_k$:

$$\mathbf{A}_k = \text{diag}\left(\mathbf{A}_{k,1}, \cdots, \mathbf{A}_{k,L_k}\right),$$

where $\mathbf{A}_{k,l}$ is a Jordan block with size $J_{k,l}$ and eigenvalue $\alpha_{k,l}$. Also, consider the consistent partition of $\mathbf{C}_k$, i.e., $\mathbf{A}_{k,l}$ and $\mathbf{C}_{k,l}$ have the same number of columns. Let $[\mathbf{C}_{k,l}]_1$ be the first column of $\mathbf{C}_{k,l}$. Define

$$\hat{\mathbf{A}}_k = \text{diag}\left(\alpha_{k,l}, \cdots, \alpha_{k,L_k}\right) \tag{17}$$

$$\hat{\mathbf{C}}_k = \left[[\mathbf{C}_{k,1}]_1 \ \cdots \ [\mathbf{C}_{k,L_k}]_1\right]. \tag{18}$$

We call the pair $(\hat{\mathbf{A}}_k, \hat{\mathbf{C}}_k)$ the *reduced FMO block* of $(\mathbf{A}_k, \mathbf{C}_k)$.

Definition 5: An FMO block $(\mathbf{A}_k, \mathbf{C}_k)$ is said to be degenerate if in its reduced FMO block $(\hat{\mathbf{A}}_k, \hat{\mathbf{C}}_k)$, the matrix $\hat{\mathbf{C}}_k$ does not have FCR, and non-degenerate otherwise. Also, the system (1) is said to be degenerate if at least one of its FMO blocks is degenerate, and non-degenerate otherwise.

This definition differs from the one presented in [9] in two points. In [9], the authors split the matrix $\mathbf{A}$ in blocks with eigenvalues of the same magnitude, while in our approach each block is also required to have FMO, hence the set of systems that are non-degenerate according to Definition 5 is slightly larger that the one in [9, Definition 5]. The second difference is that we include systems whose dynamics matrix $\mathbf{A}$ is non-diagonalizable in our classification.

Notice that according to both definitions, observable systems whose matrix $\mathbf{A}$ have all its eigenvalues with different magnitudes are non-degenerate.

The computation of $\limsup_{t\to\infty} \mathbb{P}(\mathcal{N}_k^t)^{1/t}$ is addressed below.

The next lemma tells us that the reduced FMO blocks can be used to compute $\limsup_{t\to\infty} \mathbb{P}(\mathcal{N}_k^t)^{1/t}$.

Lemma 1: We have

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} = \limsup_{t\to\infty} \mathbb{P}\left(\hat{\mathcal{N}_k^t}\right)^{1/t}. \tag{19}$$

Degenerate and non-degenerate FMO blocks are treated separately. The results are first derived for general FSMC networks and then used to state the conditions for the i.i.d. and the Gilbert-Elliott packet loss models.

For non-degenerate FMO blocks, we show in Lemma 2 that $\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t}$ is simply the recovery rate of the network, i.e., the probability to receive a measurement after a long sequence of lost ones. This result, together with Theorem 1, extends the class of communication channels for which the stability conditions of the KFIO are known. It also recovers most known results in the literature. For the case when the FMO block is degenerate, we cannot give a closed-form expression for this quantity. However, we give a method to evaluate it in Lemma 3.

A. Non-Degenerate FMO blocks

Lemma 2: Let $(\mathbf{A}_k, \mathbf{C}_k)$ be a non-degenerate FMO block, and suppose that Assumption 1 holds. Then,

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} = \mathbb{P}\left(g_t = 0 | g_{t-\nu} = 0, \cdots, g_{t-1} = 0\right). \tag{20}$$

Combining Theorem 1 and Lemma 2 we obtain the following corollary.

Corollary 1: For an i.i.d. network with $\mathbb{P}(g_t = 1) = \lambda$, the AEEC norm G is finite, if

$$|\alpha_1|^2(1-\lambda) < 1, \tag{21}$$

and G is infinite if

$$|\alpha_1|^2(1-\lambda) > 1. \tag{22}$$

Also, for a Gilbert-Elliott network model with recovery rate $\mathbb{P}(g_t = 1|g_{t-1} = 0) = q$, if

$$|\alpha_1|^2(1-q) < 1, \tag{23}$$

then G is finite. If

$$|\alpha_1|^2(1-q) > 1, \tag{24}$$

then G is infinite.

B. Degenerate FMO blocks

When the matrix $\mathbf{C}_k$ does not have FCR for some k, then more than one measurement in the sequence Γ_t must be available in order for $\mathbf{O}_k(\Gamma_t)$ to have FCR. Moreover, the time when the measurement is received is important to determine how many measurements are needed. We show this with the following example:

Example 1: Consider the FMO block $(\mathbf{A}, \mathbf{C})$, with

$$\mathbf{A} = \alpha \text{diag}(1, -1) \qquad \mathbf{C} = [1\ 1]. \tag{25}$$

Notice that if all the available measurements are from even time instants, i.e., $\Gamma_t = (1, 0, 1, 0, 1, 0, \cdots)$, then the matrix $\mathbf{O}(\Gamma_t)$ does not have FCR:

$$\mathbf{O}\left((1,0,1,0,1,0)\right) = \begin{bmatrix} 1 & \alpha^2 & \alpha^4 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}'.$$

More generally, since there exists a positive integer N such that $\mathbf{A}^N = \alpha^N \mathbf{I}$, if the measurement y_t is available, the measurements y_{t+jN}, for $j \in \mathbb{N}$, will not increase the rank of $\mathbf{O}$.

It is clear from the previous example that the probability to observe a sequence of measurements Γ_t such that $\mathbf{O}(\Gamma_t)$ has FCR depends on the structure of the system under analysis as well as on the parameters of the communication channel. Lemma 3 below states a numerical method to evaluate $\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}^t\right)^{1/t}$ for general degenerate systems using the FSMC channel model. Furthermore, we show in Corollary 2 that this method results in a closed-form expression when the packet loss model is i.i.d..

In order to compute the probability to observe a sequence Γ_t such that $\mathbf{O}_k(\Gamma_t)$ does not have FCR, we first group the sequences that share some important properties together. Then, we build the probability transition matrix that describes the probability that a sequence change groups when concatenated with a new sequence.

We start with some necessary definitions. Fix $k \in \{1, \cdots, K\}$. For convenience of notation we omit the subindex k and assume w.l.g. that $\mathbf{A}$ is a reduced FMO block, i.e., $\mathbf{A} = \hat{\mathbf{A}}_1$. Let $N \in \mathbb{N}$ be the smallest positive integer greater than or equal to ν (the order of the packet drop model) such that $\mathbf{A}^N = \alpha^N \mathbf{I}$. Since, for all $n, l \in \mathbb{N}_0$, $\mathbf{C}\mathbf{A}^n$ and $\mathbf{C}\mathbf{A}^{n+lN}$ are linearly dependent, we can restrict our characterization to sequences of length N. To do so, for each $t \in \mathbb{N}$, we define the map $\psi : \mathbb{B}^t \to \mathbb{B}^N$ such that, for all $n = 1, \cdots, N$, the n-th element of $\psi(S)$ is given by

$$\psi(S)(n) \triangleq \begin{cases} 1, & S(n + lN) = 1 \text{ for some } l \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

i.e., ψ maps a sequence $S \in \mathbb{B}^t$ of arbitrary length t, to a sequence $\psi(S) \in \mathbb{B}^N$ of length N, such that $\text{rank}\left(\mathbf{O}\left(S\right)\right) = \text{rank}\left(\mathbf{O}\left(\psi\left(S\right)\right)\right)$. Let $F_i \in \mathbb{B}^N$, $i = 1, \cdots, I$, be the set of all the sequences of length N such that $\mathbf{O}(F_i)$ does not have FCR, i.e., $\mathcal{N}^N = \{F_1, \cdots, F_I\}$. Let the elements F_i be enumerated such that, for all $t \in \mathbb{N}$ and $S \in \mathbb{B}^t$, if $\psi\left((F_i, S)\right) = F_j$, then $j \geq i$. Hence, we have $F_1 = (0, \cdots, 0)$. Then, for each $i = 1, \cdots, I$, let

$$\mathcal{L}_i^n \triangleq \{S \in \mathbb{B}^{nN} : \psi(S) = F_i\}, \tag{27}$$

i.e., $\mathcal{L}_i^n$ is the set of sequences S of length nN such that its associated sequence $\psi(S)$ equals F_i. Notice that $\mathcal{N}^{nN} = \mathcal{L}_1^n \cup \cdots \cup \mathcal{L}_I^n$.

Define the map $\delta : \mathbb{B}^t \to \mathbb{B}^\nu$ by

$$\delta\left(S\right) \triangleq \left(S\left(t - \nu + 1\right), \cdots, S\left(t\right)\right), \tag{28}$$

i.e., $\delta(S)$ is a sequence of length ν whose elements equal the last ν elements of S. Notice that, in view of Assumption 1, for two sequences $S_a, S_b \in \mathbb{B}^t$ with $t \geq \nu$, if $\delta(S_a) = \delta(S_b)$, then for $g_t \in \{0, 1\}$, we have $\mathbb{P}\left(g_t | \Gamma_t = S_a\right) = \mathbb{P}\left(g_t | \Gamma_t = S_b\right)$.

For each $i = 1, \cdots, I$, let $H_{i,j} \in \mathbb{B}^\nu$, $j = 1, \cdots, J_i$, denote the sequences such that, if $S \in \mathbb{B}^{nN}$, for some $n \in \mathbb{N}$, and $\psi(S) = F_i$, then $\delta(S) \in \{H_{i,1}, \cdots, H_{i,J_i}\}$. That is, $\{H_{i,1}, \cdots, H_{i,J_i}\}$ is the set of all possible tails $\delta(S)$ of a sequence S whose length is a multiple of N and $\psi(S) = F_i$. For $i = 1, \cdots, I$ and $j = 1, \cdots, J_i$, define the set

$$\mathcal{L}_{i,j}^n \triangleq \{S \in \mathcal{L}_i^n : \delta(S) = H_{i,j}\}. \quad (29)$$

For every $i = 1, \cdots, I$, we now build the $J_i \times J_i$ probability transition matrix $\mathbf{D}_i$, whose (r,c)-th entry equals the probability that $\Gamma_{3N} \in \mathcal{L}_{i,r}^3$ given that $\Gamma_{2N} \in \mathcal{L}_{i,c}^2$, i.e.,

$$[\mathbf{D}_i]_{r,c} \triangleq \mathbb{P}\left(\Gamma_{3N} \in \mathcal{L}_{i,r}^3 | \Gamma_{2N} \in \mathcal{L}_{i,c}^2\right). \quad (30)$$

Notice that $\mathcal{L}_{i,j}^2$ is not an empty set, since $(F_i, (0)^{N-\nu}, H_{i,j}) \in \mathcal{L}_{i,j}^2$. Let

$$p_i = \max \operatorname{eig}(\mathbf{D}_i). \quad (31)$$

Recall that p_i is a quantity associated to the FMO block k. In general, for $k = 1, \cdots, K$ define

$$\sigma_k \triangleq \left(\max_{1 \le i \le I} p_i\right)^{1/N}. \quad (32)$$

We can now state our result for numerically evaluating $\limsup_{t\to\infty} \mathbb{P}(\mathcal{N}_k^t)^{1/t}$.

Lemma 3: Let σ_k be as defined in (32). We have

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}_k^t\right)^{1/t} = \sigma_k. \quad (33)$$

Example 2: Consider the FMO block in Example 1 subject to random measurement losses according to the Gilbert-Elliott model with recovery rate q and failure rate p. We have that $N = 2$ and $\mathcal{N}^N = \{F_1, F_2, F_3\}$, with $F_1 = (0,0)$, $F_2 = (0,1)$, and $F_3 = (1,0)$. We have $\mathbf{D}_1 = \mathbb{P}\left(\Gamma_{3N} = (0)^6 | \Gamma_{2N} = (0)^4\right) = (1-q)^2$. For $i = 2$, we have $H_{2,1} = (0)$ and $H_{2,2} = (1)$. Then,

$$\begin{aligned}
[\mathbf{D}_2]_{1,1} &= \mathbb{P}(\Gamma_{3N} = (0,1,0,0,0,0) | \Gamma_{2N} = (0,1,0,0)) \\
[\mathbf{D}_2]_{1,2} &= \mathbb{P}(\Gamma_{3N} = (0,1,0,1,0,0) | \Gamma_{2N} = (0,1,0,1)) \\
[\mathbf{D}_2]_{2,1} &= \mathbb{P}(\Gamma_{3N} = (0,1,0,0,0,1) | \Gamma_{2N} = (0,1,0,0)) \\
[\mathbf{D}_2]_{2,2} &= \mathbb{P}(\Gamma_{3N} = (0,1,0,1,0,1) | \Gamma_{2N} = (0,1,0,1)) \\
\mathbf{D}_2 &= \begin{bmatrix} (1-q)^2 & p(1-q) \\ q(1-q) & pq \end{bmatrix},
\end{aligned}$$

and $\max \operatorname{eig}(\mathbf{D}_2) = (1-q)^2 + pq$. For $i = 3$, we have $H_{3,1} = (0)$. Then,

$$\begin{aligned}
\mathbf{D}_3 &= \mathbb{P}(\Gamma_{3N} = (1,0,0,0,x,0) | \Gamma_{2N} = (1,0,0,0)) \\
&= (1-q)^2 + q(1-q) \\
&= 1 - q,
\end{aligned}$$

where x can be either 0 or 1. Hence,

$$\begin{aligned}
\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}^t\right)^{1/t} &= (\max\{(1-q), (1-q)^2 + pq\})^{1/2} \\
&= \begin{cases} (1-q)^{1/2}, & p \le 1-q \\ \left((1-q)^2 + pq\right)^{1/2}, & p > 1-q. \end{cases}
\end{aligned}$$

When the packet loss model is i.i.d., the following corollary can be used to evaluate $\limsup_{t\to\infty} \mathbb{P}(\mathcal{N}_k^t)^{1/t}$.

Corollary 2: To simplify the notation, suppose that $\mathbf{A} = \mathbf{A}_1$, so we can omit the subindex k. Consider an i.i.d. network model with packet receiving probability λ. Let $F_i \in \mathbb{B}^N, i = 1, \cdots, I$ such that $\bigcup_{i=1}^{I} F_i = \mathcal{N}^N$. Let ζ_i be the number of zeros in the sequence $F_i \in \mathcal{N}^N$. Then,

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}^t\right)^{1/t} = \max_i (1-\lambda)^{\zeta_i/N}. \quad (34)$$

Example 3: Consider the system in Example 1 with an i.i.d. network model with packet receiving probability λ. We have that $N = 2$, $\zeta_1 = 2$, $\zeta_2 = \zeta_3 = 1$, $\mathbf{D}_1 = (1-\lambda)^2$, $\mathbf{D}_2 = \mathbf{D}_3 = (1-\lambda)$, $\max p_i = (1-\lambda)$, and

$$\limsup_{t\to\infty} \mathbb{P}\left(\mathcal{N}^t\right)^{1/t} = (1-\lambda)^{1/2}. \quad (35)$$

Using Theorem 1, the critical probability to receive a measurement λ_c can be obtained by solving $|\alpha|^2(1-\lambda_c)^{1/2} = 1$, hence, $\lambda_c = 1 - |\alpha|^{-4}$. That is, if $\lambda < \lambda_c$, then $G = \infty$ and if $\lambda > \lambda_c$, then the AEEC is bounded.
The critical value obtained in Example 3 matches the result reported in [20, Theorem 4].

C. Resulting method for assessing stability

We now present a procedure that summarizes how the main results of the paper can be used to determine if the KFIO corresponding to a given system and communication channel is stable.

1) *Obtain the FSMC model of the communication channel.* The FSMC model is determined by the probabilities

$$\mathbb{P}(g_\nu = 1 | \Gamma_\nu = S) \text{ for all } S \in \mathbb{B}^\nu, \quad (36)$$

where ν is the order of the FSMC model. There are several methods to obtain these probabilities from a set of channel observations, see e.g., [27] and [28].

2) *Partition the system into FMO blocks* $(\mathbf{A}_k, \mathbf{C}_k)$*,* $k = 1, \cdots, K$*, according to* (9)-(10). Recall that each FMO block has an associated scalar α_k which equals the magnitude of all the eigenvalues of $\mathbf{A}_k$.

3) *Determine whether the system is degenerate or non-degenerate using to Definition 5.*

   a) If the system is non-degenerate: the associated KFIO is stable if

   $$|\alpha_1|^2 \mathbb{P}(g_t = 0 | g_{t-\nu} = 0, \cdots, g_{t-1} = 0) < 1 \quad (37)$$

   and unstable if

   $$|\alpha_1|^2 \mathbb{P}(g_t = 0 | g_{t-\nu} = 0, \cdots, g_{t-1} = 0) > 1. \quad (38)$$

   b) If the system is degenerate:

      i) *For each* $k = 1, \cdots K$*, compute* σ_k*. This in turn requires:*

         A) *Enumerate the sequences* $F_i \in \mathbb{B}^N, i = 1, \cdots, I$*, such that* $\mathcal{N}_k^N = \{F_1, \cdots, F_I\}$*.*

         B) *For each* $i = 1, \cdots, I$*, enumerate all the possible tails* $H_{i,j} \in \mathbb{B}^\nu$ $j = 1, \cdots, J_i$*, such that, for any* $n \in \mathbb{N}$ *and* $S \in \mathbb{B}^{nN}$*, if* $\psi(S) = F_i$*, then* $\delta(S) = H_{i,j}$*, for some* j*.*

         C) *For* $1 \le r, c \le J_i$*, compute each entry* $[\mathbf{D}_i]_{r,c}$ *using* (30) *and* (36).

D) *Compute* p_i *using* (31).

E) *Compute* σ_k *using* (32).

ii) The associated KFIO is stable if

$$|\alpha_k|^2\sigma_k < 1, \text{ for all } k \in \{1, \cdots, K\}, \quad (39)$$

and unstable if

$$|\alpha_k|^2\sigma_k > 1, \text{ for some } k \in \{1, \cdots, K\}. \quad (40)$$

IV. Conclusion

We have derived a necessary condition and a sufficient condition, having only a trivial gap, for the boundedness of the expected value of the estimation error covariance of a Kalman filter subject to random measurements losses. The results were obtained using the general finite state Markov channel (FSMC) packet loss model. The existing literature usually adopts either i.i.d. or Gilbert-Elliott packet loss model and assumes non-degenerate systems or special cases of degenerate systems. In these cases, our conditions retrieve the known results for the boundedness of the asymptotic expected error covariance. When the more general FSMC packet loss model and non-degenerate systems are adopted, we extend the known results by providing a closed-form expression to determine the critical parameter that determines the boundedness of the asymptotic expected error covariance. Finally, when degenerate systems and an FSMC packet loss model are considered, we provide a numerical method to assess whether the asymptotic expected error covariance is bounded or not.

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