

# A Linear Control Approach to Distributed Multi-Agent Formations in $d$ -Dimensional Space

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**Abstract**—The paper presents a linear approach for formation control of multiple autonomous agents in  $d$ -dimensional space. The generalized notion of graph Laplacian, associated to a graph with possibly negative weights on the edges, is introduced aiming to solve the formation problem of controlling a network of point agents to form a given pattern (including rotation, translation, and scaling). By assuming the number of agents is larger than  $d + 1$ , we derive that a linear distributed control law exists for this purpose if and only if certain algebraic conditions hold (or equivalently, the graph is globally rigid). Next, it is shown that the generalized graph Laplacian used to stabilize the formations can be obtained by solving a convex optimization problem. Further results are also provided to reveal the conditions under which the attained formations are congruent to or just translations of the desired one.

## I. INTRODUCTION

Multi-agent systems represent a class of systems composed of many autonomous agents, interacting locally to achieve desirable collective behaviors, among which forming a group pattern is a typical one. Such behaviors have been observed a lot in nature such as bird flocking, fish schooling etc. In addition to these amazing phenomena, there are also a bunch of potential applications in engineering. For example, use unmanned aerial vehicles (UAVs) to form a team pattern for surveillance or localization, use autonomous underwater vehicles (AUVs) for ocean data retrieval or exploration, and use smaller satellites in formation flying to view research targets from multiple angles or at multiple times.

In coordination control of multi-agent systems, the concept of graph Laplacian, perfectly reflecting the topological interconnection and the usage of local relative information, naturally leads to distributed linear control strategies in consensus [1]–[3], row straightening [4], [5], and containment control [6], [7]. The Laplacian based control laws are modified in [8] and [9] to accomplish circular and rectilinear formation motions in the plane. The ideas are then generalized in [10]–[13] by formally introducing complex Laplacians, based on which the formation control problem in the plane is solved using linear distributed control laws for both undirected and directed graphs. However, the extension to distributed formations in three or higher dimensions is still missing.

On the other hand, there have also been other techniques for formation characterization and formation control in the

plane [14]–[17] or in three dimensional space [18], [19]. The mainstream is to use graph rigidity [20] and gradient-descent nonlinear control laws [21] to achieve rigid formations. However, global stability analysis is always a critical issue using the gradient-descent nonlinear control laws. So far, there are only a few known results ensuring global or almost global stability of spatial formations in the plane by assuming particular topologies [22], [23] or simple formations [24]. Deriving distributed local control laws ensuring global convergence towards desired formation shapes in three or higher dimensional space is still challenging.

The paper aims to present a new approach for formation control with global convergence properties ensured, by calling for a generalized graph Laplacian. That is, the weights on the edges of the graph can be negative numbers, based on which linear distributed formation control strategies are developed. The approach overcomes the two technical challenges: attaining formations in high dimensional spaces and ensuring global asymptotic stability. Under the proposed linear control strategies, the achievable formations are outcomes of a desired configuration via affine motions if and only if certain rank condition of the graph Laplacian holds, or equivalently the graph is globally rigid. The design of the generalized Laplacian for formation control is transformed to solve a convex optimization problem, which can be solved efficiently and globally. Moreover, we reveal that in order to achieve a formation subject to rigid motions or sole translations, only a small number of agents is required to have additional control while the rest can remain to use the same Laplacian based linear control. Thus, it offers a new approach for rigid formation control by controlling only a portion of agents to meet certain distance or relative position constraints. Two simulation results (one in the plane and the other in the three dimensional space) are presented to demonstrate the correctness of the analytical results.

## II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, we first introduce several notions from graph theory and then formulate the problem we study.

### A. Notations

The following notations are used throughout the paper.

$\mathbb{R}$  — The set of real numbers.

$\mathbf{1}_n$  — The  $n$ -dimensional vector of ones.

$I_d$  — The  $d \times d$  identity matrix.

$\text{span}\{p_1, \dots, p_n\}$  — The linear span of vectors  $p_1, \dots, p_n$ .

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## B. Graph and framework

A *graph* is a set of  $n$  nodes  $\mathcal{V}$  and  $m$  edges  $\mathcal{E}$ , denoted as  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . A graph is *connected* if, for all pairs of nodes  $i$  and  $j$ , there exists a path of nodes starting from  $i$  and ending at  $j$ . A graph is *k-connected* if deleting any subset of  $k - 1$  nodes and edges incident on those nodes results in a connected graph. A graph is called a *complete graph* if every pair of distinct nodes is connected by an edge.

A *configuration in  $\mathbb{R}^d$*  (or simply called a configuration in this paper) of a set of  $n$  nodes  $\mathcal{V}$  is defined by their coordinates in the Euclidean space  $\mathbb{R}^d$ , denoted as  $p = [p_1^\top, \dots, p_n^\top]^\top$ , where each  $p_i \in \mathbb{R}^d$  for  $1 \leq i \leq n$ . A *framework in  $\mathbb{R}^d$*  (or simply called a framework in this paper) is a graph  $\mathcal{G}$  equipped with a configuration  $p$  in  $\mathbb{R}^d$ , denoted as  $\mathcal{F} = (\mathcal{G}, p)$ .

We say that two frameworks  $(\mathcal{G}, p)$  and  $(\mathcal{G}, q)$  with  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  are *equivalent*, and we write  $(\mathcal{G}, p) \cong (\mathcal{G}, q)$ , if  $\|p_i - p_j\| = \|q_i - q_j\|, \forall (i, j) \in \mathcal{E}$ . We say that two frameworks  $(\mathcal{G}, p)$  and  $(\mathcal{G}, q)$  are *congruent*, and we write  $(\mathcal{G}, p) \equiv (\mathcal{G}, q)$  (or simply  $p$  and  $q$  are congruent,  $p \equiv q$ ), if  $\|p_i - p_j\| = \|q_i - q_j\|, \forall i, j \in \mathcal{V}$ . A framework  $(\mathcal{G}, p)$  is called *globally rigid* if

$$(\mathcal{G}, p) \cong (\mathcal{G}, q), \forall q \in \mathbb{R}^{nd} \Leftrightarrow (\mathcal{G}, p) \equiv (\mathcal{G}, q).$$

A configuration  $p$  is *generic* if the coordinates  $\{p_1, \dots, p_n\}$  do not satisfy any nontrivial algebraic equation with rational coefficients ([25]). By abuse of notion, we also say a graph  $\mathcal{G}$  is *globally rigid* if for any generic configuration  $p$  the framework  $(\mathcal{G}, p)$  is globally rigid. In [26], it has been proven that if a graph  $\mathcal{G}$  is globally rigid in  $\mathbb{R}^d$ , then it is  $(d + 1)$ -connected, which also implies, every node has at least  $d + 1$  neighbors. However, the reverse is not true.

## C. Problem formulation

We consider a group of  $n$  agents in  $d$ -dimensional space (for example, mobile robots in the plane, unmanned aerial vehicles in the 3-dimensional space). The positions of the  $n$  agents are denoted by  $z_1, \dots, z_n \in \mathbb{R}^d$ . We consider that each agent  $i$  has a point kinematic model given by the single integrator

$$\dot{z}_i = u_i, \quad (1)$$

where  $u_i \in \mathbb{R}^d$  is the velocity control input. Define the state  $z = [z_1^\top, \dots, z_n^\top]^\top$ , as a column vector in  $\mathbb{R}^{nd}$ .

Suppose each agent  $i$  has an onboard sensor allowing it to measure the relative positions of some of the other agents, namely,  $z_j - z_i$  when agent  $j$  lies in the sensing field of agent  $i$ . Assume that the agents can sense mutually. We then use an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  to model the sensing relationship with each  $i \in \mathcal{V}$  representing an agent and an edge  $(i, j) \in \mathcal{E}$  meaning that the two agents  $i$  and  $j$  can be mutually sensed. Denote  $\mathcal{N}_i$  the set of neighbors of agent  $i$ . Thus, we use  $z_{ij} = z_i - z_j, j \in \mathcal{N}_i$  to denote the measurement available to agent  $i$ .

Consider a target configuration  $p = [p_1^\top, \dots, p_n^\top]^\top$  in  $\mathbb{R}^d$ , where each  $p_i \in \mathbb{R}^d$  for  $1 \leq i \leq n$ . It has been proved in [27] that a generic framework  $(\mathcal{G}, p)$  in  $\mathbb{R}^d$  with  $d + 1$  or

fewer nodes is globally rigid if and only if  $\mathcal{G}$  is a complete graph (i.e., a simplex). Therefore in the paper we assume that our graph has  $d + 2$  or more nodes, i.e.,  $n \geq d + 2$ . This also implies that a generic framework does not lie in a proper affine subspace of  $\mathbb{R}^d$ . Moreover, we assume that  $p_i \neq p_j$  for  $i \neq j$ , indicating that there is no overlap for any two points in  $\mathbb{R}^d$ .

We are interested in the problem whether there exists a distributed and local control law using only the available measurement  $z_{ij}, j \in \mathcal{N}_i$ , for each agent such that all the agents converge to the *affine image* of  $p$ , i.e.,

$$\mathcal{A}(p) := \left\{ q = [q_1^\top, \dots, q_n^\top]^\top \left| \begin{array}{l} q_i = Ap_i + a, A \in \mathbb{R}^{d \times d} \\ a \in \mathbb{R}^d, \text{ and } i = 1, \dots, n \end{array} \right. \right\}$$

or equivalently,

$$\mathcal{A}(p) := \{ q = (I_n \otimes A)p + \mathbf{1}_n \otimes a \mid A \in \mathbb{R}^{d \times d}, a \in \mathbb{R}^d \}.$$

Notice that a real matrix  $A$  can be factorized by singular value decomposition as  $A = U\Sigma V$  where  $U$  and  $V$  are unitary matrices, and  $\Sigma$  is a  $d \times d$  diagonal matrix. It means that a configuration in  $\mathcal{A}(p)$  is attained via an affine motion from  $p$ , namely, a translation  $a$ , followed by a rotation  $V$ , a scaling along different axis by  $\Sigma$ , and then another rotation  $U$ .

If the matrix  $A$  in the definition of  $\mathcal{A}(p)$  is an unitary matrix, the affine image  $\mathcal{A}(p)$  is called a *rotation/translation image*, denoted as  $\mathcal{R}(p)$ . In other words, if  $z \in \mathcal{R}(p)$ , the agents achieve a rigid formation, meaning that their configuration is congruent to the target configuration.

Moreover, if the matrix  $A$  in the definition of  $\mathcal{A}(p)$  is an identity matrix, the affine image  $\mathcal{A}(p)$  is called a *translation image*, denoted as  $\mathcal{T}(p)$ . That is, if  $z \in \mathcal{T}(p)$ , the agents achieve a rigid formation with the same orientation as the target configuration.

From the relationships among the affine image, rotation/translation image, and translation image of a target configuration discussed above, it is clear that the rotation/translation image and translation image lie in the affine image. In other words, if a few extra constraints can be reached by a portion of agents, then the whole team can achieve a rigid formation subject to rotation/translation or translation only. Therefore, controlling a group of agents to an affine image of a target configuration not only is of its own interest, but also serves as a starting point and provides a new approach for rigid formation control.

More precisely, the control objective is formulated as follows.

*Control objective:* For the  $n$  agent dynamics (1) and a desired framework  $(\mathcal{G}, p)$ , we aim to design a linear control

$$u_i = - \sum_{j \in \mathcal{N}_i} k_{ij} z_{ij}, \quad i = 1, \dots, n \quad (2)$$

with possibly negative  $k_{ij}$ 's, also called the weights on the edges  $(i, j)$ 's, such that the trajectories of the closed-loop system satisfy

$$\lim_{t \rightarrow \infty} z(t) = z^*$$

where  $z^* \in \mathcal{A}(p)$  is the configuration  $p$  with translation, rotation, and scaling.

Moreover, we aim to reveal the conditions under which  $z^* \in \mathcal{R}(p)$  is the configuration  $p$  with translation and rotation, and the conditions under which  $z^* \in \mathcal{T}(p)$  is the configuration  $p$  with sole translation.

### III. DISTRIBUTED MULTI-AGENT FORMATIONS

In this section we will explore necessary and sufficient conditions (both algebraic and topological) for the correspondence of the affine image of a generic configuration and the equilibrium set of a closed-loop multi-agent system. The design problem for an asymptotic stabilizing controller will also be investigated. Further results will be presented to reveal the conditions under which the attained formations are the desired configuration with translations and rotations, and the conditions under which the attained formations are the desired configuration with sole translations.

#### A. Necessary and sufficient conditions for $\mathcal{A}(p)$ to be the equilibrium subspace

Considering the distributed local control (2), we obtain the overall closed system

$$\dot{z} = -(H \otimes I_d)z \quad (3)$$

where  $H \in \mathbb{R}^{n \times n}$  is the generalized Laplacian matrix corresponding to the graph with possible negative weights  $k_{ij}$ 's attributed on its edges. It should be pointed out that the weights  $k_{ij}$  here might be negative, so it is different from the consensus control protocol.

First, we present a necessary and sufficient condition for the equilibrium set of the generalized Laplacian based control system to be the affine image of a desired configuration.

**Theorem 3.1:** Consider a generic configuration  $p = [p_1^T, \dots, p_n^T]^T$  in  $\mathbb{R}^d$ . The equilibrium set of system (3) is the affine image  $\mathcal{A}(p)$  if and only if  $\text{rank}(H) = n - d - 1$  and  $(H \otimes I_d)p = 0$ .

**Proof:** (Sufficiency) Firstly, since  $\text{rank}(H) = n - d - 1$ , it is then known that the null space of  $H \otimes I_d$  has dimension  $(d+1)d$ . Secondly, from  $(H \otimes I_d)p = 0$ , it turns out that for any  $A \in \mathbb{R}^{d \times d}$  and  $a \in \mathbb{R}^d$ ,

$$\begin{aligned} (H \otimes I_d)[(I_n \otimes A)p + \mathbf{1}_n \otimes a] &= (H \otimes A)p \\ &= (I_n \otimes A)(H \otimes I_d)p \\ &= 0, \end{aligned}$$

which means the affine image  $\mathcal{A}(p)$  belongs to the equilibrium set. Since the null space of  $H \otimes I_d$  has the same dimension  $(d+1)d$  as  $\mathcal{A}(p)$ , it is certain that the equilibrium set of system (3) is exactly the affine image  $\mathcal{A}(p)$ .

(Necessity) If the equilibrium set of system (3) is exactly the affine image  $\mathcal{A}(p)$ , then we take  $p \in \mathcal{A}(p)$  and can obtain that  $(H \otimes I_d)p = 0$ . Moreover, since it can be verified that  $\mathcal{A}(p)$  is a linear subspace of dimension  $d^2 + d$ , it then follows that  $\text{rank}(H) = n - d - 1$ . ■

Second, we present a necessary and sufficient topological condition for the equilibrium set of the generalized Laplacian

based control system to be the affine image of a desired configuration.

In [28], a symmetric matrix  $H$  satisfying  $H\mathbf{1}_n = 0$  and  $(H \otimes I_d)p = 0$  is called the *stress matrix*. Next we introduce a result that a generic framework is globally rigid if and only if it has a stress matrix with kernel of dimension  $d+1$ . The sufficiency is shown in [28]. The necessity is conjectured in [28] and proven in [25].

**Theorem 3.2** ([25], [28]): Suppose a graph  $\mathcal{G}$  has  $n$  nodes with  $n \geq d+2$  and  $p = [p_1^T, \dots, p_n^T]^T$  is a generic configuration in  $\mathbb{R}^d$ . Then the framework  $(\mathcal{G}, p)$  is globally rigid if and only if there exists a stress matrix  $H$  whose rank is  $n - d - 1$ .

With the help of Theorem 3.2, we are now ready to present a topological condition to ensure that the equilibrium set of system (3) is the affine image  $\mathcal{A}(p)$ .

**Theorem 3.3:** Suppose a graph  $\mathcal{G}$  has  $n$  nodes with  $n \geq d+2$  and  $p = [p_1^T, \dots, p_n^T]^T$  is a generic configuration in  $\mathbb{R}^d$ . Then there exists an  $H$  such that the equilibrium set of system (3) is the affine image  $\mathcal{A}(p)$  if and only if the graph  $\mathcal{G}$  is globally rigid.

**Proof:** From Theorem 3.2 we know that if the graph  $\mathcal{G}$  is globally rigid, then there exists a stress matrix  $H$  (satisfying  $(H \otimes I_d)p = 0$ ) whose rank is  $n - d - 1$ . Thus applying Theorem 3.1 we could get that if we use the stress matrix in system (3) then its equilibrium set is the affine image  $\mathcal{A}(p)$ . The necessity is also straightforward. ■

#### B. Stability and stabilization analysis

We come to design  $k_{ij}$ 's for a given globally rigid graph  $\mathcal{G}$  such that the affine image  $\mathcal{A}(p)$  is exactly the equilibrium set of system (3), and moreover the equilibrium set is asymptotically stable.

Denote  $q_1$  the  $n$ -dimensional vector by aggregating the first components of  $p_1, \dots, p_n$ . Similarly, we denote  $q_2, \dots, q_d$  the corresponding aggregate vectors. Then we let  $Q$  be an  $(n-d-1) \times n$  matrix with orthonormal rows that are each orthogonal to  $\mathbf{1}_n, q_1, \dots, q_d$ ; that is

$$Q\mathbf{1}_n = 0, Qq_1 = 0, \dots, Qq_d = 0, QQ^T = I_{n-d-1}.$$

Then we are ready to present a stability criteria that can be used in the control design.

**Theorem 3.4:** The system (3) is asymptotically stable with respect to its equilibrium set  $\mathcal{A}(p)$ , i.e.,  $\lim_{t \rightarrow \infty} z(t) = z^*$  for  $z^* \in \mathcal{A}(p)$ , if and only if  $\lambda_{\min}(QHQ^T) > 0$  where  $\lambda_{\min}(\cdot)$  represents the smallest eigenvalue of a symmetric matrix.

**Proof:** Since  $\mathcal{A}(p)$  is the equilibrium set of system (3), it can be verified that

$$H\mathbf{1}_n = 0, Hq_1 = 0, \dots, Hq_d = 0.$$

Therefore,  $QHQ^T$  inherits all eigenvalues of  $H$  except the ones at zero. Thus the conclusion follows. ■

Based on Theorem 3.4, we reformulate the control design problem as an optimization problem in the following.

We suppose the graph  $\mathcal{G}$  has  $m$  edges, with labels  $1, \dots, m$ . We arbitrarily assign an orientation for each edge.

The choice of orientation does not change the analysis. The incidence matrix  $B \in \mathbb{R}^{n \times m}$  is defined as

$$B_{il} = \begin{cases} 1 & \text{if edge } l \text{ starts from node } i, \\ -1 & \text{if edge } l \text{ ends at node } i, \\ 0 & \text{otherwise.} \end{cases}$$

Since we consider symmetric weights, each edge  $l$  of the graph is associated with a single weight  $w_l = k_{ij} = k_{ji}$ , where edge  $l$  is incident to nodes  $i$  and  $j$ . We let  $w \in \mathbb{R}^m$  denote the vector of weights on the edges. Using this notation, the matrix  $H$  can be written as

$$H(w) = B \text{diag}(w) B^T$$

where  $\text{diag}(w)$  stands for the  $m \times m$  diagonal matrix with the  $l$ th diagonal entry being  $w_l$ . Thus, the control design problem turns out to be the design problem of the weight vector  $w$  subject to certain equality or inequality constraints, to meet the stability requirement of the multi-agent formation. That is,

$$\begin{aligned} & \underset{w}{\text{maximize}} && \lambda \\ & \text{subject to} && QH(w)Q^T \succ \lambda I_{n-d-1}, \\ & && H(w)q_i = 0, \quad i = 1, \dots, d \end{aligned}$$

where  $A \succ B$  refers that  $A - B$  is positive definite. The optimization above is a semi-definite programming that can be efficiently and globally solved by using polynomial-time interior point methods.

### C. Extra conditions to make rigid formations

In this subsection, we will show that if the agents are in the affine image  $\mathcal{A}(p)$  and if additionally an agent and its neighbors are able to maintain the desired distances or the desired relative positions, then the whole group is in a formation subject to rotations together with translations, or translations only.

**Theorem 3.5:** Suppose  $\mathcal{G}$  is globally rigid and  $p = [p_1^T, \dots, p_n^T]^T$  is a generic configuration in  $\mathbb{R}^d$ . For any equilibrium  $z^* = [z_1^{*T}, \dots, z_n^{*T}]^T$  of system (3), if there exists an agent  $k$  such that

$$\|z_k^* - z_j^*\| = \|p_k - p_j\| \quad \text{for all } j \in \mathcal{N}_k,$$

then the equilibrium  $z^*$  is a rotation/translation image of  $p$ , i.e.,  $z^* \in \mathcal{R}(p)$ .

**Proof:** Note from Theorem 3.1 that if  $z^*$  is an equilibrium point of system (3), then  $z^*$  is in the affine image  $\mathcal{A}(p)$ . In other words, there exist  $A \in \mathbb{R}^{d \times d}$  and  $a \in \mathbb{R}^d$  such that  $z_i^* = Ap_i + a$  for all  $i$ . Moreover, recall from the condition that for all  $j \in \mathcal{N}_k$ ,  $\|z_k^* - z_j^*\| = \|p_k - p_j\|$ . Thus, we have for all  $j \in \mathcal{N}_k$ ,

$$\|p_k - p_j\|^2 = \|z_k^* - z_j^*\|^2 = (p_k - p_j)^T A^T A (p_k - p_j)$$

and

$$(p_k - p_j)^T (I - A^T A) (p_k - p_j) = 0.$$

Since  $\mathcal{G}$  is globally rigid, it then follows that  $\mathcal{G}$  is  $(d+1)$ -connected [26], which implies, every node has at least  $d+1$  neighbors. Moreover since  $p = [p_1^T, \dots, p_n^T]^T$  is a generic

configuration in  $\mathbb{R}^d$ , then it is certain that the linear span of  $(p_k - p_j)$ ,  $j \in \mathcal{N}_k$  equals to  $\mathbb{R}^d$ . Thus,

$$(p_k - p_j)^T (I - A^T A) (p_k - p_j) = 0$$

for all  $j \in \mathcal{N}_k$  implies

$$I - A^T A = 0.$$

As a result, for any  $i$  and  $j$ , we have

$$\|z_i^* - z_j^*\|^2 = (p_i - p_j)^T A^T A (p_i - p_j) = \|p_i - p_j\|^2,$$

meaning that the distance between any pair of nodes is preserved, i.e., the equilibrium  $z^*$  is congruent to  $p$ . ■

**Theorem 3.6:** Suppose  $\mathcal{G}$  is globally rigid and  $p = [p_1^T, \dots, p_n^T]^T$  is a generic configuration in  $\mathbb{R}^d$ . For any equilibrium  $z^* = [z_1^{*T}, \dots, z_n^{*T}]^T$  of system (3), if there exists an agent  $k$  such that

$$z_k^* - z_j^* = p_k - p_j \quad \text{for all } j \in \mathcal{N}_k,$$

then the equilibrium  $z^*$  is a translation image of  $p$ , i.e.,  $z^* \in \mathcal{T}(p)$ .

**Proof:** Similar to the proof of Theorem 3.5, for any equilibrium point  $z^*$  of system (3), we have  $A \in \mathbb{R}^{d \times d}$  and  $a \in \mathbb{R}^d$  such that  $z_i^* = Ap_i + a$  for all  $i$ . If in addition,

$$z_k^* - z_j^* = p_k - p_j \quad \text{for all } j \in \mathcal{N}_k,$$

then we have for all  $j \in \mathcal{N}_k$ ,

$$p_k - p_j = z_k^* - z_j^* = A(p_k - p_j)$$

and

$$(I - A)(p_k - p_j) = 0.$$

For the same reason as given in the proof of Theorem 3.5, it is known that the linear span of  $(p_k - p_j)$ ,  $j \in \mathcal{N}_k$ , equals to  $\mathbb{R}^d$ . Thus,

$$(I - A)(p_k - p_j) = 0 \quad \text{for all } j \in \mathcal{N}_k$$

implies

$$A = I.$$

As a result, for any  $i$  and  $j$ , we have

$$z_i^* - z_j^* = p_i - p_j$$

meaning that the formation corresponding to the equilibrium point  $z^*$  is a translation of the configuration  $p$ . ■

**Remark 3.1:** From the proof of Theorem 3.5, we can see that as long as a number of  $d$  neighbors (not necessary all neighbors) of a node  $k$  are able to preserve the edge lengths,  $I - A^T A = 0$  holds, which means the equilibrium is a realization of globally rigid formation  $p$  subject to only rotations and translations. Therefore, if we could additionally control  $d+1$  agents (which play the role of leaders in the network) to attain the desired distances, then the whole group of agents with the proposed linear distributed control law can achieve a globally rigid formation.

**Remark 3.2:** Similarly, in addition to make the agents converge to the affine image  $\mathcal{A}(p)$ , if we could control  $d+1$  agents to attain the desired relative positions in a

common reference frame, then the whole group of agents with the proposed distributed control law can achieve a globally rigid formation subject to only translations. This can also be interpreted from the dimension of the affine image  $\mathcal{A}(p)$  and the number of constraints. In a generic sense, the preservation of relative positions for  $d$  edges in  $\mathbb{R}^d$  results in  $d^2$  linearly independent constraints, which reduces the  $(d^2 + d)$ -dimensional equilibrium subspace  $\mathcal{A}(p)$  to a subspace of dimension  $d$ , corresponding to the translation motions. However, the distance constraints in Theorem 3.5 are nonlinear with respect to  $z^*$ . Though there are totally  $d$  independent constraints at least, it reduces the  $(d^2 + d)$ -dimensional equilibrium subspace  $\mathcal{A}(p)$  to a smaller dimensional manifold corresponding to the rigid motions (translations and rotations).

#### IV. SIMULATIONS

In this section, we present two simulation results of five agents to demonstrate the correctness of our results: one is in the plane and the other is in the three dimensional space.

For the simulation in the plane, the desired framework  $(\mathcal{G}, p)$  is given in Fig. 1, for which it can be checked that the

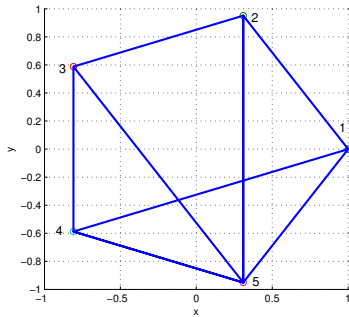


Fig. 1. A desired framework  $(\mathcal{G}, p)$  in the plane.

graph is globally rigid. Thus, according to Theorem 3.3, there exists a generalized Laplacian matrix with possibly negative weights such that the closed-loop system has its equilibrium set equal to the affine image of  $p$ . Moreover, by solving the convex optimization problem in the preceding section, we obtain that the following weights

$$\begin{aligned} k_{12} = k_{21} &= 2.5792, & k_{14} = k_{41} &= -2.5792, \\ k_{15} = k_{51} &= 4.1732, & k_{23} = k_{32} &= 1.5940, \\ k_{25} = k_{52} &= -1.5940, & k_{34} = k_{43} &= 2.5792, \\ k_{35} = k_{53} &= -1.5940, & k_{45} = k_{54} &= 4.1732 \end{aligned}$$

attributed on the edges of  $\mathcal{G}$  result in a generalized Laplacian matrix  $H$ , with its spectrum  $\sigma(H) = \{0, 0, 0, 5.7672, 12.8958\}$ . Therefore, the closed-loop system (3) under the proposed distributed control law is asymptotically stable with respect to its equilibrium set. A simulation result is presented in Fig. 2 for a randomly generated initial state.

Next, we consider distributed formations in the three dimensional space, whose desired framework is given in Fig. 3. The graph  $\mathcal{G}$  is a complete graph and thus it is globally

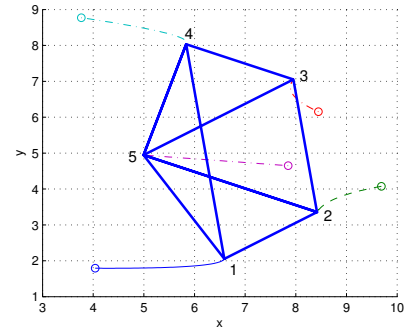


Fig. 2. An achieved formation in the plane. Small circles represent the initial states of the agents.

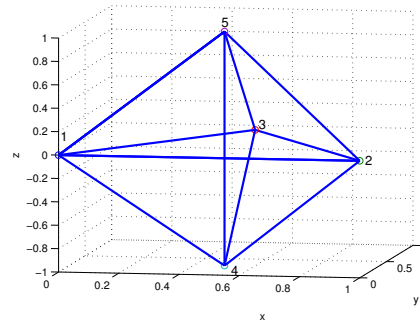


Fig. 3. A desired framework  $(\mathcal{G}, p)$  in  $\mathbb{R}^3$ .

rigid in  $\mathbb{R}^3$ . By the same procedure, we obtain the following weights

$$\begin{aligned} k_{12} = k_{21} &= -1.2491, & k_{13} = k_{31} &= -1.2491, \\ k_{14} = k_{41} &= 1.8736, & k_{15} = k_{51} &= 1.8736, \\ k_{23} = k_{32} &= -1.2491, & k_{24} = k_{42} &= 1.8736, \\ k_{25} = k_{52} &= 1.8736, & k_{34} = k_{43} &= 1.8736, \\ k_{35} = k_{53} &= 1.8736, & k_{45} = k_{54} &= -2.8104, \end{aligned}$$

which can make the equilibrium set be the affine image  $\mathcal{A}(p)$  and ensure asymptotic stability of the closed-loop system (3). The spectrum of  $H$  using the above weights is  $\sigma(H) = \{0, 0, 0, 0, 9.3682\}$ . The Laplacian matrix  $L$  has only one positive eigenvalue, coinciding with Theorem 3.1. Two simulation results are presented in Fig. 4 and Fig. 5 with different initial states. It is shown that both asymptotically converge to an equilibrium in the affine image though the final configurations are different depending on their initial states.

#### V. CONCLUSIONS

The paper proposes a novel approach to study distributed formations based on the generalized graph Laplacian with possibly negative weights at its entries. Linear distributed control laws are developed for the problem of formation control using only local measurements, yet ensuring globally asymptotic stability. The design of the generalized Laplacian in solving the formation control problem is transformed to a convex optimization problem. Necessary and sufficient (alge-

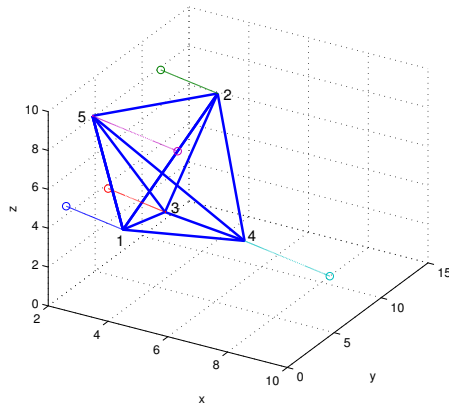


Fig. 4. An achieved formation in  $\mathbb{R}^3 - I$ .

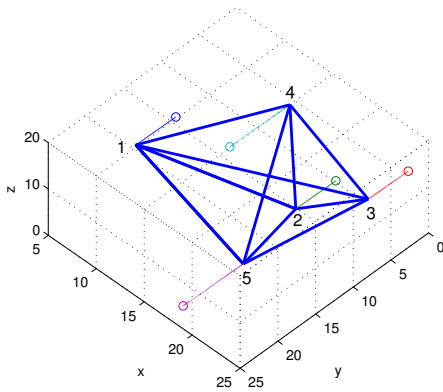


Fig. 5. An achieved formation in  $\mathbb{R}^3 - II$ .

braic and topological) conditions are explored to characterize the equilibrium set for the system with the Laplacian based control. Moreover, it is shown that with a small number of agents playing the role of leaders to additionally control their distances or relative positions, the whole group is able to achieve globally rigid formations subject to rotations and translations, or sole translations. However, it is left for future study on how to design effective control laws for leaders for these purposes.

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