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ROBUST STABILIZATION OF LINEAR UNCERTAIN SYSTEMS VIA QUANTIZED FEEDBACK

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Abstract—This paper studies the problem of robust stabilization for linear uncertain systems via logarithmic quantized feedback. Our work is based on a new method for the analysis of quantized feedback. More specifically, we characterize the quantization error using a simple sector bound. It is shown in our previous work that this method yields the same result on the coarsest quantization density as in the work of Elia and Mitter, when the system does not involve uncertainties. The advantage of this new method is that it is applicable to multi-input-multi-output systems and to performance control problems. In this paper, we apply this method to robust stabilization of linear uncertain systems. We give conditions under which there exists a quadratic stabilizing controller for a given quantization density. Both state feedback and output feedback are considered. For output feedback, we consider two cases: 1) quantization occurs at the control input; and 2) quantization occurs at the measured output.

I. INTRODUCTION

Control using quantized feedback can be traced back to the work of Kalman [1] in 1956 which studied the limit cycle behavior of a system with a finite-alphabet quantizer in the control loop. Since then, a lot of research has been done on understanding and mitigation of quantization effects; see, e.g., [2], [3], [4].

Recently, there is a surge of interest on quantized feedback control, with the aim to understand the required quantization density or information rate for control purposes. Noticeable works include [7], [8], [9], [10], [11]. The most pertinent reference to this paper is the work by Elia and Mitter [11]. In [11], the problem of quadratic stabilization of discrete-time single-input-single-output (SISO) linear time-invariant (LTI) systems using quantized feedback is studied. The quantizer is assumed to be static and time-invariant (i.e. memoryless and with fixed quantization levels). It is proved in [11] that for a quadratically stabilizable system, the quantizer needs to be *logarithmic* (i.e., the quantization levels are linear in logarithmic scale). Further, the coarsest quantization density is given explicitly in terms of the system's unstable poles. The work of [11] is also generalized to some extent to guaranteed performance control [12], stabilization of two-input systems [13], and multi-input systems [14].

In Fu and Xie [15], the work of [11] is generalized to general multi-input-multi-output (MIMO) systems and to control problems requiring performances. This is done using the so-called *sector bound method*, which is based on using a simple sector bound to model the quantization error. For a SISO system with quantized state feedback (which is the

most fundamental problem), the sector bound method gives an identical result as in [11]. But the main advantage of the sector bound method is that it is easy to understand and easy to generalize to more complicated quantized feedback control scenarios such as those mentioned above.

In this paper, we study the problem of robust stabilization for linear uncertain systems via logarithmic quantized feedback. Our work is based on the sector bound method. We give conditions under which there exists a quadratic stabilizing controller for a given quantization density. Both state feedback and output feedback are considered. For output feedback, we consider two cases: 1) quantization occurs at the control input; and 2) quantization occurs at the measured output. We also give an example to illustrate our results and to demonstrate how the required quantization density increases as the level of uncertainties increases.

II. QUANTIZED STATE FEEDBACK

The uncertain system to be considered is in the following form:

$$x(k + 1) = (A + \Delta A)x(k) + (B + \Delta B)u(k) \qquad (1)$$

where $x(k) \in \mathbf{R}^n$ is the state, $u(k) \in \mathbf{R}$ is the (single) control input, ΔA and ΔB represent the uncertainties in the system, and they satisfy the following:

$$[\Delta A\ \Delta B] = HF(k)[E_1\ E_2], \quad \|F(k)\| \leq 1 \qquad (2)$$

for some matrices H, E_1 and E_2, where $F(k) \in \mathbf{R}^{n_1 \times n_2}$ represents a norm-bounded uncertainty. This description of uncertainty is commonly used in the robust control literature; see, e.g., [17].

Quantized state feedback requires the control input in the following form:

$$u(k) = f(v(k)) \qquad (3)$$

$$v(k) = Kx(k) \qquad (4)$$

In the above, $K \in \mathbf{R}^{1 \times n}$ is the feedback gain, and $f(\cdot)$ is a quantizer which is assumed to be symmetric, i.e., $f(-v) = -f(v)$. Note that the quantizer is static and time-invariant.

The set of quantized levels is denoted by

$$\mathcal{U} = \{\pm u_i, i = 0, \pm 1, \pm 2, \cdots\} \cup \{0\} \qquad (5)$$

Denote by $\#g[\epsilon]$ the number of quantization levels in the interval $[\epsilon, 1/\epsilon]$. The density of the quantizer $f(\cdot)$ is defined as follows:

$$\eta_f = \limsup_{\epsilon \to 0} \frac{\#g[\epsilon]}{-\ln \epsilon} \qquad (6)$$

Consider a logarithmic quantizer as in [11]:

$$\mathcal{U} = \{\pm u^{(i)} : u^{(i)} = \rho^i u^{(0)}, i = \pm 1, \pm 2, \cdots\} \\ \cup \{\pm u^{(0)}\} \cup \{0\}, \quad 0 < \rho < 1, u^{(0)} > 0 \qquad (7)$$

It is easy to compute that the quantization density for (7) is given by

$$n_f = \frac{2}{\ln 1/\rho} \qquad (8)$$

In particular, n_f is a monotonically increasing function of ρ. For this reason, we will call ρ, instead of n_f, the quantization density in the sequel.

For the quadratic stabilization problem, a quadratic Lyapunov function $V(x) = x^T P x, P = P^T > 0$, is used to assess the stability of the feedback system. That is, the quantizer must satisfy

$$\nabla V(x) = V((A + \Delta A)x + (B + \Delta B)f(Kx)) - V(x) \\ < 0, \forall x \neq 0 \qquad (9)$$

for all admissible uncertainties. The coarsest quantizer is the one which minimizes η_f subject to (9).

The density of the quantizer depends on $V(x)$ (or P) and K. This raises the key question: What is the coarsest density among all possible P and K? In [11], this problem is studied for systems without uncertainties, and the answer is given for a specially chosen K:

$$K = K_{GD} = -\frac{B^T P A}{B^T P B} \qquad (10)$$

More specifically, for the K as above, the coarsest density is given by

$$\rho = \frac{1 - \delta}{1 + \delta} \qquad (11)$$

with

$$\delta^{-1} = \prod_i |\lambda_i^u| \qquad (12)$$

where λ_i^u are the unstable eigenvalues of A. But it is shown in [15] that the result on ρ (or δ) remains the same even when K is allowed to be a free variable.

When the system is subject to uncertainties, the approach in [11] seems to be difficult to generalize. It turns out the coarsest quantization density is in general difficult to characterize. We therefore aim to searching for an upper bound of it which guarantees quadratic stabilizability. That is, we consider the following problem: Given a (logarithmic) quantization density $\rho > 0$, determine (possibly sufficient) conditions under which there exist a quadratically stabilizing quantized state feedback controller with a given quantization density. Once an algorithm is found for this problem, the required quantization density can be easily searched by repeatedly applying the algorithm.

To solve the above problem, we resort to the sector bound method used in [15]. This method uses the following simple observation: For a given quantization density $\rho > 0$, the quantization error is bounded by

$$f(v) - v = \Delta v, \quad |\Delta| \leq \delta \qquad (13)$$

where δ is related to ρ by (11). When there are no uncertainties, it is shown in [15] that the quantized state feedback controller (3)-(4) is quadratically stabilizing if and only if the (unquantized) state feedback controller (4) is quadratically stabilizing in the presence of the sector bound uncertainty (13). That is, the quantized feedback stabilization problem is equivalent to well-known quadratic stabilization problem with a sector-bounded uncertainty. This is a key observation which allows [15] to generalize the work of [11] to stabilization problem for MIMO systems and performance control problems.

Now let us return to the uncertain system (1). Given a quantization density $\rho > 0$, we apply the sector bound (13). It turns out that this leads to the following auxiliary system:

$$x(k+1) = Ax(k) + Bv(k) + \begin{bmatrix} B & \tau^{-1}H \end{bmatrix} w(k) \\ \zeta(k) = \begin{bmatrix} \delta v(k) \\ \tau (E_1 x(k) + E_2 v(k) + [E_2\ 0]w(k)) \end{bmatrix} \qquad (14)$$

where $v(k)$ is the control input, $\zeta(k)$ is the controlled output, and $\tau > 0$ is a scaling parameter.

Theorem 1: The system (1) is quadratically stabilizable for a given a quantization density $\rho > 0$ if there exists a scaling parameter $\tau > 0$ and a state feedback controller

$$v(k) = Kx(k) \qquad (15)$$

for the auxiliary system (14) such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1. Further, the control gain K and Lyapunov matrix P for the auxiliary system will also work for the uncertain system (1).

Proof. Combining (1) and (3) gives

$$x(k+1) = (A + \Delta A)x + (B + \Delta B)(1 + \Delta)v \qquad (16)$$

Defining

$$\bar{\Delta} = \Delta/\delta \\ \xi = \Delta v = \bar{\Delta}\delta v \\ \eta = E_1 x + E_2 v + E_2 \xi \\ w = [\xi^T \ \tau \eta^T] \qquad (17)$$

The equation (16) becomes

$$x(k+1) = Ax + Bv + B\xi + H\eta \\ = Ax + Bv + \begin{bmatrix} B & \tau^{-1}H \end{bmatrix} w \qquad (18)$$

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and

$$w = \text{diag } \{\bar{\Delta}, F\}\zeta \qquad (19)$$

It is clear that

$$w^T w \le \zeta^T \zeta \qquad (20)$$

Note that (18)-(20) resembles the auxiliary system (14). It is well-known that the system (18), (19) and (15) is quadratically stabilizable if the H_∞ norm for the transfer function of (14) from w to ζ is less than or equal to 1 for some $\tau > 0$. Hence, our result is proved. $\quad\square$

If the system (1) does not involve any uncertainty, Theorem 1 reduces the following result (see [15]):

Corollary 1: Suppose the system (1) does not involve any uncertainty (i.e., $H = 0, E_1 = 0, E_2 = 0$). Then, it is quadratically stabilizable for a given a quantization density $\rho > 0$ *if and only if* there exists a state feedback controller (15) for the following auxiliary system

$$\begin{aligned} x(k+1) &= Ax(k) + Bv(k) + Bw(k) \\ \zeta(k) &= \delta v(k) \end{aligned} \qquad (21)$$

such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1. Further, the control gain K and Lyapunov matrix P for the auxiliary system will also work for the system (1).

III. QUANTIZED OUTPUT FEEDBACK

The system of concern has the following form:

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B + \Delta B)u(k) \\ y(k) &= (C + \Delta C)x(k) + (D + \Delta D)u(k) \end{aligned} \qquad (22)$$

where x and u are as before, and $y(k) \in \mathbf{R}^r$ is the measured output.

When quantized output feedback is used for control, there are at least two basic cases for the location where quantization takes places:

Case 1. The control input is quantized. In this case, the construction of a pre-quantized control signal is done at the output end where the measured output perfectly available. The control signal is then quantized and transmitted to the input side. We will call this case the *output feedback with quantized input.*

Case 2. The measured output is quantized. In this case, the construction of the control signal is done at the input end using quantized output signal. No more quantization happens to the control input signal. We will call this case the *output feedback with quantized output.*

Obviously, it is possible to have more complicated scenarios. For example, quantization may happen to both measured output and control input. But we are only concerned with Cases 1 and 2 for simplicity reasons. Also, the ideas for these cases can be easily generalized.

Output feedback with quantized input

In this case, we take the controller to be of the form below:

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c \bar{y}(k) + B_1 u(k) \\ v(k) &= C_c x_c(k) + D_c \bar{y}(k) + D_1 u(k) \\ u(k) &= f(v(k)) \end{aligned} \qquad (23)$$

where

$$\bar{y}(k) = y(k) - Du(k) = (C + \Delta C)x_c(k) + \Delta Du(k) \quad (24)$$

$x_c(k)$ is the state of the controller with its dimension and matrices A_c, B_c, C_c, D_c, B_1 and D_1 to be designed. Note that using $\bar{y}(k)$ instead of $y(k)$ does not alter the available feedback information.

We first consider the special case when no uncertainties exist in the system. This case has been studied in [15], and the result is that output feedback with quantized input is equivalent to quantized state feedback for quadratic stabilization, provided the system is detectable. That is, if state feedback can quadratically stabilize the system for a given quantization density, so can the output feedback. The corresponding output feedback controller is an observer-based one, taking the following form:

$$\begin{aligned} x_c(k+1) &= Ax_c(k) + L(\bar{y}(k) - Cx_c(k)) + Bu(k) \\ v(k) &= Kx_c(k) \\ u(k) &= f(v(k)) \end{aligned} \qquad (25)$$

where L is the observer gain and K is the state feedback gain. Note that $L = B_c$, $B_1 = B$ and $D_1 = 0$ if we compare (25) with (23).

Now let us return to the uncertain system (22). Motivated by the above, we will also choose

$$B_1 = B; \quad D_1 = 0 \qquad (26)$$

Next, we define an auxiliary system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bv(k) + \begin{bmatrix} B & \tau^{-1}H_1 \end{bmatrix} w(k) \\ \bar{y}(k) &= Cx(k) + [0 \quad \tau^{-1}H_2]w(k) \\ \zeta &= \begin{bmatrix} \delta v(k) \\ \tau(E_1 x(k) + E_2 v(k) + [E_2 \quad 0]w(k) \end{bmatrix} \end{aligned} \qquad (27)$$

where δ is computed from a given quantization density $\rho > 0$, and $\tau > 0$ is a scaling parameter.

Theorem 2: Consider the uncertain system (22) and a given quantization density $\rho > 0$. Suppose there exists an output feedback controller without quantization (i.e., (25) with $u = v$) for the auxiliary system (27) such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1. Then, the system (22) is quadratically stabilizable via the same controller with quantized control input and quantization density ρ.

Proof. The proof is similar to that of Theorem 1. Using the definitions in (17), the system (22) is converted into the auxiliary system (27) with (19)-(20). Hence, the relationship between the quadratic stabilizability of (22) and H_∞ control of (27) follows. □

When no uncertainty is involved in the system, Theorem 2 reduces to the following [15]:

Corollary 2: Suppose the system (22) does not involve any uncertainty (i.e., $H_1 = 0, H_2 = 0, E_1 = 0, E_2 = 0$). Then, the following two problems are equivalent:

- The system (22) is quadratically stabilizable via output feedback with quantized input and quantization density $\rho > 0$;
- There exists an output feedback controller without quantization for the following auxiliary system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bv(k) + Bw(k) \\
\bar{y}(k) &= Cx(k) \\
\zeta &= \delta v(k)
\end{aligned}
\tag{28}
$$

such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1.

Output feedback with quantized output

In this case, the feedback information is

$$
v(k) = f(y(k))
\tag{29}
$$

and the controller has the following form:

$$
\begin{aligned}
x_c(k+1) &= A_c x_c(k) + B_c v(k) \\
u(k) &= C_c x_c(k) + D_c v(k)
\end{aligned}
\tag{30}
$$

Note that the terms B_1 and D_1 are not needed because the mapping from v to u is linear.

The corresponding auxiliary system is now given by

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + [0 \ \ H_1]w(k) \\
v(k) &= Cx(k) + Du(k) + [\tau^{-1} \ \ H_2]w(k) \\
\zeta(k) &= \begin{bmatrix} \tau\delta(Cx(k) + Du(k) + [0 \ \ H_2]w(k)) \\ E_1 x(k) + E_2 u(k) \end{bmatrix}
\end{aligned}
\tag{31}
$$

Theorem 3: Consider the uncertain system (22) and a given quantization density $\rho > 0$. Suppose there exists an output feedback controller (30) for the auxiliary system (31) such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1. Then, the system (22) is quadratically stabilizable via the same controller with quantized output and quantization density ρ.

Proof. The proof is similar to that of Theorem 2. □

When system uncertainties disappear, again we have the following special result [15]:

Corollary 3: Suppose there is no uncertainty in (22). Then the following are equivalent:

- The system (22) is quadratically stabilizable via output feedback with quantized output and quantization density $\rho > 0$.
- There exists an unquantized output feedback controller for the following auxiliary system

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
v(k) &= Cx(k) + Du(k) + \tau^{-1}w(k) \\
\zeta(k) &= \delta(Cx(k) + Du(k))
\end{aligned}
\tag{32}
$$

such that the H_∞ norm of the transfer function from w to ζ is less than or equal to 1.

IV. An Example

In this section, we give an example to show the effects of three quantization schemes as studied before. The system to be considered is given by (22) with

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}; \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ C = [-3 \ \ 1]; \ D = 0
$$

$$
H_1 = \epsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ H_2 = \epsilon; \ E_1 = [1 \ \ 0]; \ E_2 = 1
\tag{33}
$$

In the above, the parameter $\epsilon > 0$ represents the size of uncertainties.

When $\epsilon = 0$, the uncertainties vanish and the transfer function of the system becomes $G(z) = C(zI - A)^{-1}B = (z - 3)/z(z - 2)$. This example is analyzed in [15]. When quantized state feedback is used, the coarsest quantization density is computed to be $\rho = 1/3$. The same quantization density is reached when output feedback with quantized control input is used. For output feedback with quantized output, the coarsest quantization density turns out to be $\rho = 0.8182$. That is, the latter scheme requires a much denser quantizer.

When $\epsilon > 0$, Theorems 1-3 are applied and the coarsest quantization densities are searched by solving the H_∞ control problems associated with the auxiliary systems for various τ and ρ. The results are plotted in Figure 1.

It is clear that increasing ϵ will increase the required quantization density. Also, output feedback with quantized output requires a denser quantizer compared with output feedback with quantized input. Finally, although the output feedback with quantized input requires the same quantization density as the quantized state feedback when there is no uncertainty, the former requires a denser quantizer when ϵ increases. This is because the existence of uncertainty makes it difficult to recover the state information from the output measurement.
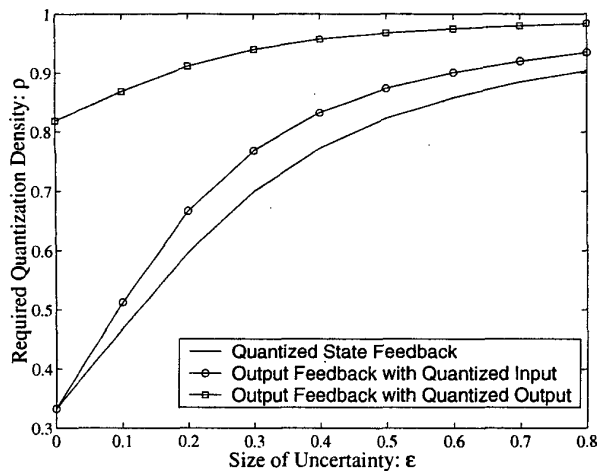
Fig. 1. Required Quantization Density vs. Size of Uncertainty

V. CONCLUSIONS

We have studied three robust stabilization problems associated with logarithmic quantized feedback, namely, quantized state feedback, output feedback with quantized input, and output feedback with quantized output. In each of these cases, we have shown the connection between quadratic stabilizability for a given quantization density and H_∞ control for a corresponding auxiliary system. This allows us to use the standard H_∞ design tools to deal with quantized feedback control for uncertain systems.

In the output feedback control case, we have noted an interesting phenomenon that the quadratic stabilizability depends on where the quantization occurs. In particular, a coarser quantization density can be achieved in general when quantization occurs at the control input rather than at the measured output. Intuitively, this is because the measured information is better preserved in the former case.

Although the sector bound method gives only sufficient conditions for quantized feedback stabilization, we make two points: 1) The results become non-conservative when uncertainties are not present, as shown in Corollaries 1-3. 2) The technical difficulties for quantized feedback stabilization of the uncertain systems as in Theorems 1-3 are essentially the same as quadratic stabilization of systems with two blocks of uncertainties (one from F and one from Δ). This problem has been studied for a long time in the robust control literature, and there is no non-conservative solution to it.

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