Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

Optimality of the Logarithmic Quantizer for Stabilization of Linear Systems: Achieving the Minimum Data Rate

Keyou You, Weizhou Su, Minyue Fu and Lihua Xie*

Abstract— **This paper investigates the optimality of the logarithmic quantizer on stabilization of linear systems. It is shown that a finite-level logarithmic quantizer can asymptotically achieve the well-known minimum average data rate for stabilizing an unstable MIMO linear discrete-time system. A time-sharing protocol is proposed to allocate bits to subsystems, each associated with an unstable eigenvalue of the system, and the stability of the system is established using a single logarithmic quantizer and controller.**

I. INTRODUCTION

Research on quantization has received resurgent interest in recent years due to the emergence of network based control systems. Traditional technique of modeling quantization errors as additive white Gaussian noises began to be challenged in the new environment where only very coarse information is allowed to propagate through the network.

The change of view on quantization can be traced back to the seminal paper [1]. In the paper, the author treated quantization as partial information of the quantized entity, rather than its approximation and showed the significance of the historical values of the quantizer output. After that, methods for studying quantization effects on control and estimation have been developed.

The current research on quantized feedback control can be categorized depending on whether the quantizer is static or dynamic. A static quantizer is a memoryless nonlinear function while a dynamic quantizer takes up memory and is more complicated and potentially more powerful. Following [1], Brockett and Liberzon studied a dynamic finite-level uniform quantizer for stabilization in [2] and pointed out that there exist a dynamic adjustment policy for the quantizer sensitivity and a quantized state feedback to asymptotically stabilize an unstable system. One then raised the fundamental question: how much information needs to be communicated between the quantizer and the controller to stabilize an unstable system? Various authors have addressed this problem under different scenarios, e.g., [3]–[7] and the appealing data rate theorem states that the minimum average data rate R required for the quantizer to achieve stability should satisfy the following inequality:

$$R > \sum_{|\lambda_i(A)|\geq 1} \log_2|\lambda_i(A)| := H, \tag{1}$$

where $\lambda_i(A)$ denotes an eigenvalue of the open-loop matrix A.

Perhaps one of the most interesting static quantizers is the logarithmic quantizer introduced in [8], [9], which is proved to be the coarsest quantizer to quadratically stabilize an unstable SISO linear system. A finite-level logarithmic quantizer is proposed in [10] where a simple dynamic scaling scheme for the quantizer input and output is proposed. Though the study of logarithmic quantizer bears a vast body of literature, e.g., [8], [9], [11]–[14], it is not clear if a logarithmic quantizer can achieve the well-known minimum average data rate required for stabilizing an unstable linear system.

The above motivates the study on the optimality of the logarithmic quantizer which can be formulated in this way: does the logarithmic quantizer require a higher date rate for stabilization? The main result of this paper show that the answer is negative. We confirm the result by showing that the use of a finite-level variable rate logarithmic quantizer for stabilization can asymptotically achieve the minimum average data rate.

Our solution adopts the philosophy that it is more efficient to use more accurate information on some specific states than to use coarser information on all states. That is, it is better to assign different bits to different states rather than to pool total bits among all states. To stabilize an unstable system with bounded disturbance, a static finite-level logarithmic quantizer is explicitly designed and periodically applied on the state that is scaled by a constant scaling factor and a corresponding control law is proposed based on the quantized state. More precisely, the control input is generated by multiplying a constant control gain on the scaled back quantized signal. The striking feature is that the scaling factor remains static in the whole process, rather than dynamically changes as in [10]. For asymptotic stabilization of an unstable system without disturbance, a dynamic quantizer is needed, where the scaling factor evolves with time. However, the quantizer takes the same form as the static one. Meanwhile, a time-sharing protocol for the quantizer is proposed to allocate bits to subsystems, each associated with an unstable eigenvalue of the open-loop matrix, and ensure the feasibility of the proposed control law. Note that an earlier attempt has been made for scalar systems in [15].

*Author for correspondence. This work was supported by the Agency for Science, Technology and Research of Singapore under SERC grant 0521010037 and the National Natural Science Foundation of China under grants NSFC 60828006, NSFC-60834003 and NSFC-60774057.

Keyou You and Lihua Xie are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, 639798. Emails: {youk0001,elhxie}@ntu.edu.sg.

Weizhou Su is with the College of Automation Science and Engineering, South China University of Technology, Guangzhou, China. Email: wzhsu@scut.edu.cn.

Minyue Fu is with the School of Electrical Engineering and Computer Science, University of Newcastle, Callaghan, N.S.W. 2308 Australia. Email: eemf@ee.newcastle.edu.au.

978-1-4244-3872-3/09/$25.00 ©2009 IEEE

The rest of the paper is organized in the following fashion. The problem of interest is formulated in Section II. Section III constitutes the main part of the paper, where the optimality of the logarithmic quantizer is proved by designing a finite-level quantizer and a corresponding control law to stabilize the unstable system. The concluding remarks are drawn in the last section.

II. Problem Formulation

Consider a discrete time-invariant linear system,

$$x_{k+1} = Ax_k + Bu_k + w_k,\ \forall k \in \mathbb{N}, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, and $w_k \in \mathbb{R}^n$ is a bounded additive disturbance satisfying $\|w_k\|_\infty \le d$ with $\|\cdot\|_\infty$ denoting the l^∞ norm for vectors and the induced matrix norm for matrices. (A,B) is a stabilizable pair. To make the problem nontrivial, we further assume that A has at least an unstable eigenvalue.

We are interested in the problem of stabilizing the system using a control law with limited information about the state. In particular, there is a quantizer that observes x_k and generates a discrete-valued sequence $\{b_k, k = 0,1,\dots\}$ at sampling times $\{k = 0,1,\dots\}$. The control law to stabilize the system is designed solely from the quantized value b_k by a pair of quantizer/controller, see Fig. 1, where the encoder and decoder are lumped into the quantizer and controller respectively.

Before proceeding, some definitions are needed.

Definition 1: [9] A quantizer is called *logarithmic* quantizer if it has the form:

$$\begin{aligned} \mathscr{U} \;=\; & \{\pm u^{(i)} : u^{(i)} = \rho^i u^{(0)}, i = \pm 1, \pm 2, \cdots\} \\ & \cup \{\pm u^{(0)}\} \cup \{0\}, \quad 0 < \rho < 1, u^{(0)} > 0. \end{aligned} \tag{3}$$

The associated quantizer $\mathrm{Q}_\infty(\cdot)$ is defined as follows:

$$\mathrm{Q}_\infty(v) = \begin{cases} u^{(i)}, & \text{if } \frac{1}{1+\delta}u^{(i)} < v \le \frac{1}{1-\delta}u^{(i)}, v > 0; \\ 0, & \text{if } v = 0; \\ -\mathrm{Q}_\infty(-v), & \text{if } v < 0 \end{cases} \tag{4}$$

where ρ is called the quantizer *density* and

$$\delta = \frac{1-\rho}{1+\rho}. \tag{5}$$

However, the logarithmic quantizer in (4) has an infinite number of quantization levels and needs an infinite number of bits to represent the quantizer. In the sequel, we shall concentrate on the design of a finite-level logarithmic quantizer that asymptotically achieves the minimum average data rate in (1) and a corresponding control law for stabilizing the system (2). Define a $(2N-1)$-level logarithmic quantizer with quantization density $0 < \rho < 1$ as:

$$\mathrm{Q}_N(x) = \begin{cases} \rho^i(1-\delta), & \text{if } \rho^{i+1} < x \le \rho^i, \\ & \quad 0 \le i \le N-2; \\ 0, & \text{if } 0 \le x \le \rho^{N-1}; \\ \text{undefined}, & \text{if } x > 1; \\ -\mathrm{Q}_N(-x), & \text{if } x < 0. \end{cases} \tag{6}$$

For simplicity, we have chosen $u^{(0)} = \frac{2\rho}{1+\rho}$.

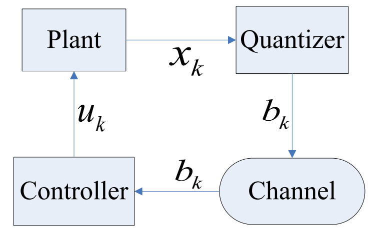

Fig. 1. System structure.

The key problem is to determine the quantizer parameters and the corresponding control law as well as a communication protocol to achieve the stabilization.

III. Main Result

In this section, we show that it is possible to design a finite-level static logarithmic quantizer to asymptotically reach the minimum average data rate to stabilize the unstable system in (2) and the corresponding control law takes a simple form.

The main difficulties involve optimally allocating bits to subsystems, each associated with an unstable eigenvalue of the open-loop matrix A. It is straightforward to show that when A has a stable A-invariant subspace, there is no need to assign bits to the associated stable subsystem since on that subspace, the corresponding state variables will converge to a bounded region (or zero when $w_k = 0$ in (2)). As such, we focus on the most interesting case in which all eigenvalues of A are unstable. Assume that A has distinct unstable eigenvalues $\lambda_1, \dots, \lambda_f$ (if λ_i is complex, its conjugate λ_i^* is excluded from the list) with the corresponding geometric multiplicities $d_1, \dots, d_f$. The real Jordan canonical form of A is given by the following lemma.

Lemma 1: There exists a real nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that

$$SAS^{-1} = J = \mathrm{diag}\{J_1, \dots, J_f\}.$$

The Jordan block J_i associated with the real eigenvalue λ_i takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}_{d_i \times d_i},$$

while the Jordan block J_j associated with the complex eigenvalue $\lambda_j = r_j(\cos\theta + i\sin\theta)$ takes the form

$$J_j = \begin{bmatrix} D_j & I_2 & & \\ & D_j & I_2 & \\ & & \ddots & \\ & & & D_j \end{bmatrix}_{2d_j \times 2d_j},$$

where $D_j = \begin{bmatrix} r_j\cos\theta & r_j\sin\theta \\ -r_j\sin\theta & r_j\cos\theta \end{bmatrix}$ and I_2 is the standard identity matrix with dimension 2×2.

Proof: See [16, pp. 150-153]. ∎

Define the indices $n_i, i = 1, \dots, f$ as follows:

$$n_i = \begin{cases} d_i, & \text{if } \lambda_i \in \mathbb{R}; \\ 2d_i, & \text{otherwise.} \end{cases}$$

The following lemma establishes the relationship between $\|J_i^m\|_\infty$ and $|\lambda_i|, m, n_i$.

Lemma 2: There is a positive ζ such that $\forall i \in [1, \dots, f]$,

$$\|J_i^m\|_\infty \le \zeta \sqrt{n_i} m^{n_i - 1} |\lambda_i|^m := \kappa(m, \lambda_i). \tag{7}$$

Obviously, if $n_i = 1$, we can choose $\zeta = 1$.

Proof: Note that there is a $\zeta > 0$, independent of J_i, n_i, and m, such that [6]

$$\|J_i^m\| \le \zeta m^{n_i - 1} |\lambda_i|^m,$$

where $\|\cdot\|$ is the spectral norm induced from the Euclidean norm. Together with the relationship between $\|\cdot\|$ and $\|\cdot\|_\infty$, see [16, pp.313],

$$\|A\|_\infty \le \sqrt{l}\|A\|, \ \forall\, A \in \mathbb{C}^{l \times l}.$$

The proof is completed. ∎

Since all of the eigenvalues of A are assumed unstable, we have $n_1 + \dots + n_f = n$. Use the nonsingular real matrix S in (1) and define the transformed state $z_k := Sx_k$, the dynamical equation for the transformed systems is written as

$$z_{k+1} = Jz_k + SBu_k + Sw_k. \tag{8}$$

It is trivial that the stability of the above system in (8) is equivalent to that of the original system (2). Due to that (A, B) is a stabilizable pair and all eigenvalues of A are assumed unstable, (A, B) is a controllable pair, implying that (J, SB) is a controllable pair as well. Let the controllability index of (J, SB) be μ which is the least integer such that

$$\text{rank}[SB, JSB, \dots, J^{\mu-1}SB] = n.$$

To utilize the distinct features of the real Jordan form, we partition the transformed state vector into $z_k^{(i)}, i = 1, \dots, f$ respectively corresponding to each Jordan block of A. The dynamical equation of each subsystem can be rewritten as

$$z_{k+1}^{(i)} = J_i z_k^{(i)} + (SBu_k)^{(i)} + (Sw_k)^{(i)}, i \in [1, \dots, f]. \tag{9}$$

The optimality of the logarithmic quantizer is shown in the following theorem, where a finite-level static logarithmic quantizer and a simple control law are explicitly designed. Meanwhile, a time-sharing protocol is proposed to allocate bits to each subsystem in (9).

Theorem 1: Given the stabilizable system (2), stabilization can be achieved using a finite-level logarithmic quantizer if and only if the average data rate exceeds H in (1).

Proof: The necessity part has been well established in [6], [7].

Sufficiency: To better illustrate the quantizer and control law, we use the following way to deliver the proof.

Step 1: Parameter selection for the finite-level logarithmic quantizer in (6).

Let R be a given average data rate greater than H in (1), then there exists an $\alpha > 1$ satisfying

$$R \ge \sum_{i=1}^{f} n_i \log_2(\alpha|\lambda_i|).$$

For any $\varepsilon > 0$, it is feasible to select a pair of sufficiently large τ ($\tau > \mu$) and $N_i, i = 1, \dots, f$ to satisfy

$$\begin{aligned} \log_2\left(1 + \frac{2\log_2 \kappa(\tau, \lambda_i)}{\log_2 \frac{\kappa(\tau,\lambda_i)+\varepsilon+1}{\kappa(\tau,\lambda_i)+\varepsilon-1}}\right) &< \log_2(2N_i - 1) \\ &\le \tau \log_2 \alpha - \mu R + \log_2 |\lambda_i|^\tau, \end{aligned} \tag{10}$$

where $\kappa(\tau, \lambda_i)$ is defined in (7).

Remark 1: The strict inequality in (1) is shown in (10), where we require $\alpha > 1$ to ensure the existence of quantizer parameters τ and $N_i, i \in [1, \dots, f]$. More precisely, we have proved the following equality in the appendix:

$$\lim_{\tau \to \infty} \left(1 + \frac{2\log_2 \kappa(\tau, \lambda_i)}{\log_2 \frac{\kappa(\tau,\lambda_i)+\varepsilon+1}{\kappa(\tau,\lambda_i)+\varepsilon-1}}\right)^{\frac{1}{\tau}} = |\lambda_i|, \tag{11}$$

which implies that for a sufficiently large τ,

$$\log_2\left(1 + \frac{2\log_2 \kappa(\tau, \lambda_i)}{\log_2 \frac{\kappa(\tau,\lambda_i)+\varepsilon+1}{\kappa(\tau,\lambda_i)+\varepsilon-1}}\right) \sim \log_2 |\lambda_i|^\tau.$$

Since $\alpha > 1$, $\tau \log_2 \alpha - \mu R \to \infty$ as $\tau \to \infty$. Thus, it is possible to select an N_i to satisfy (10).

For each state variable of the i-th subsystem in (9), we use the same finite-level logarithmic quantizer with the following parameters: $N = N_i$, $\delta_i = \frac{1}{\kappa(\tau,\lambda_i)+\varepsilon}$ and

$$\rho_i = \frac{\kappa(\tau, \lambda_i) + \varepsilon - 1}{\kappa(\tau, \lambda_i) + \varepsilon + 1}, i \in [1, \dots, f].$$

Step 2: Algorithms on the quantizer and controller.

(1) **Quantizer:**
Divide times $k \in \mathbb{N}$ into cycles $[j\tau, \dots, (j+1)\tau - 1], j \in \mathbb{N}$. At the start of a cycle with time $k = j\tau$, the quantizer receives the state $z_{j\tau}$ from the observation of the system. Choosing a scaling factor $\triangle$ to make the quantizer well-defined on the scaled state, i..e. $\|\frac{z_{j\tau}}{\triangle}\|_\infty \le 1$, the scaled state will be quantized as follows:

$$\sigma_{j\tau}^{(i,h)} := \text{Q}_{N_i}(z_{j\tau}^{(i,h)}/\triangle),$$

$\forall h \in [1, \dots, n_i]$ and $\forall i \in [1, \dots, f]$, where $z_{j\tau}^{(i,h)}$ is the $h-$th component of $z_{j\tau}^{(i)}$ in (9). The quantized signals $\sigma_{j\tau}^{(i,h)}$ will be sent to the controller in a specified sequence.

(2) **Controller:** In view of (10), it can be verified that

$$\begin{aligned} \sum_{i=1}^{f} n_i \left\lceil \frac{\log_2(2N_i - 1)}{R} \right\rceil &< n + \frac{\sum_{i=1}^{f} n_i \log_2(2N_i - 1)}{R} \\ &\le \tau \frac{\sum_{i=1}^{f} n_i \log_2(\alpha|\lambda_i|)}{R} - n(\mu - 1) \\ &\le \tau - (\mu - 1), \end{aligned} \tag{12}$$

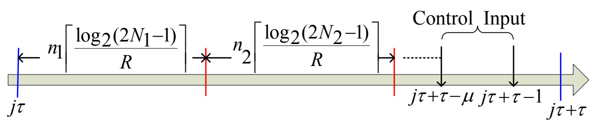

Fig. 2. Time-sharing protocol within one cycle.

where the ceiling function $\lceil\cdot\rceil$ is given by

$$\lceil x\rceil = \min\{n\in\mathbb{Z} : n\geq x\}.$$

With the average data rate R, the transmission of $\sigma_{j\tau}^{(i,h)}$ can be completed within the transmission slots of duration $\lceil\frac{\log_2(2N_i-1)}{R}\rceil$. The total time with respect to the unstable mode λ_i would be $n_i\lceil\frac{\log_2(2N_i-1)}{R}\rceil$. In light of (12) and neglecting the computation time, the quantized signals could reach the controller before the time

$$j\tau+\tau-\mu;$$

See Fig. 2 for the illustration. Assume that the channel is noiseless. The control law within one cycle can be given as

$$\begin{bmatrix} u_{j\tau+\tau-1} \\ \vdots \\ u_{j\tau+\tau-\mu}\end{bmatrix} = -\triangle\mathscr{C}^T(\mathscr{C}\mathscr{C}^T)^{-1}J^\tau Q(\frac{z_{j\tau}}{\triangle}), \qquad (13)$$

where the controllability matrix is defined as

$$\mathscr{C} := [SB, JSB, \ldots, J^{\mu-1}SB].$$

and

$$u_{j\tau+t} = 0, \forall t\in[0,\ldots,\tau-\mu-1]. \qquad (14)$$

The vector quantizer $Q(\cdot)$ is given by

$$Q(x) = \begin{bmatrix} \mathrm{Q}_{N_1}(x^{(1,1)}) \\ \vdots \\ \mathrm{Q}_{N_1}(x^{(1,n_1)}) \\ \vdots \\ \mathrm{Q}_{N_f}(x^{(f,1)}) \\ \vdots \\ \mathrm{Q}_{N_f}(x^{(f,n_f)})\end{bmatrix}, \forall x\in\mathbb{R}^n.$$

Step 3: Proof of the stability.

Inserting the control law in (13) and (14) into the transformed system in (8), we obtain

$$\begin{aligned} z_{(j+1)\tau} &= J^\tau z_{j\tau} + \sum_{t=0}^{\tau-1} J^{\tau-1-t}(SBu_{j\tau+t} + Sw_{j\tau+t}) \\ &= J^\tau z_{j\tau} + \sum_{t=\tau-\mu}^{\tau-1} J^{\tau-1-t}SBu_{j\tau+t} + g_{j\tau} \\ &= J^\tau(z_{j\tau} - \triangle\sigma_{j\tau}) + g_{j\tau}, \end{aligned} \qquad (15)$$

where the uniformly bounded $g_{j\tau}$ with respect to j is given by $g_{j\tau} = \sum_{t=0}^{\tau-1} J^{\tau-1-t}Sw_{j\tau+t}$ and its upper bound is computed as

$$\begin{aligned} \|g_{j\tau}\|_\infty &= \|\sum_{t=0}^{\tau-1} J^{\tau-1-t}Sw_{j\tau+t}\|_\infty \\ &\leq d\|S\|_\infty \sum_{t=0}^{\tau-1}\|J\|_\infty^{\tau-1-t} := D. \end{aligned}$$

The vector $\sigma_{j\tau}$ is compatibly stacked as

$$\sigma_{j\tau} := [\sigma_{j\tau}^{(1,1)}, \ldots, \sigma_{j\tau}^{(1,n_1)}, \ldots, \sigma_{j\tau}^{(f,1)}, \ldots, \sigma_{j\tau}^{(f,n_f)}]^T.$$

We now prove the stability of (15) using arguments of the mathematical induction.

First, the initial state z_0 is assumed to belong to the bounded region

$$\Omega_0 = \{z, \|z\|_\infty \leq \triangle\}.$$

Then, if $\|z_{j\tau}\|_\infty \in \Omega_0$, it is easy to verify that

$$|z_{j\tau}^{(i,h)}| \leq \|z_{j\tau}^{(i)}\|_\infty \leq \|z_{j\tau}\|_\infty \leq \triangle.$$

Thus, it is well-defined for the quantizer to quantize the scaled state $z_{j\tau}^{(i,h)}$ with scaling factor $\triangle$.

Consider subsystem of (15), we have the following results:

$$\begin{aligned} &\|z_{(j+1)\tau}^{(i)}\|_\infty \\ &= \|J_i^\tau[z_{j\tau}^{(i)} - \triangle\sigma_{j\tau}^{(i)}] + g_{j\tau}^{(i)}\|_\infty \\ &\leq \|J_i^\tau\|_\infty\|z_{j\tau}^{(i)} - \triangle\sigma_{j\tau}^{(i)}\|_\infty + D \end{aligned} \qquad (16)$$

$$\leq \begin{cases} \|J_i^\tau\|_\infty\|z_{j\tau}^{(i)}\|_\infty + D, & \text{if } \|z_{j\tau}^{(i)}/\triangle\|_\infty \leq \rho_i^{N_i-1} \\ \|J_i^\tau\|_\infty\delta_i\|z_{j\tau}^{(i)}\|_\infty + D, & \text{if } \rho_i^{N_i-1} < \|z_{j\tau}^{(i)}/\triangle\|_\infty \leq 1 \end{cases}$$

$$\leq \begin{cases} \kappa(\tau,\lambda_i)\rho_i^{N_i-1}\triangle + D, & \text{if } \|z_{j\tau}^{(i)}/\triangle\|_\infty \leq \rho_i^{N_i-1} \\ \kappa(\tau,\lambda_i)\delta_i\triangle + D, & \text{if } \rho_i^{N_i-1} < \|z_{j\tau}^{(i)}/\triangle\|_\infty \leq 1 \end{cases}.$$

Sufficient conditions for the existence of a $\triangle > 0$ to make $\|z_{(j+1)\tau}^{(i)}\|_\infty \leq \triangle$ for $i = 1,\ldots,f$ are

$$\triangle \geq \max\{\frac{D}{1-\kappa(\tau,\lambda_i)\rho_i^{N_i-1}}, \frac{D}{1-\delta_i\kappa(\tau,\lambda_i)}, i = 1,\ldots,f\} \qquad (17)$$

and

$$\begin{cases} \kappa(\tau,\lambda_i)\rho_i^{N_i-1} < 1 \\ \delta_i\kappa(\tau,\lambda_i) < 1 \end{cases}, \forall i\in[1,\ldots,f]. \qquad (18)$$

With the choices of quantizer parameters $\delta_i = \frac{1}{\kappa(\tau,\lambda_i)+\varepsilon}$ and

$$\rho_i = \frac{\kappa(\tau,\lambda_i)+\varepsilon-1}{\kappa(\tau,\lambda_i)+\varepsilon+1},$$

we have $\delta_i\kappa(\tau,\lambda_i) < 1$ and

$$\kappa(\tau,\lambda_i)\rho_i^{N_i-1} = 2^{\log_2\kappa(\tau,\lambda_i)+(N_i-1)\log_2\rho_i} < 1,$$

since by (10), it is easy to verify that

$$N_i > 1 + \frac{\log_2\kappa(\tau,\lambda_i)}{\log_2\rho_i} \Rightarrow \log_2\kappa(\tau,\lambda_i) + (N_i-1)\log_2\rho_i < 0.$$

Therefore, the selection of $\triangle$ to satisfy (17) implies that

$$\|z_{(j+1)\tau}\|_\infty \leq \triangle; \text{ See Fig. (3).}$$

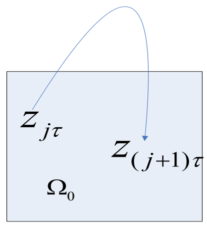

Fig. 3. Trajectory of state.

Hence, $z_{j\tau}$ will be uniformly bounded at all times $j \in \mathbb{N}$. Since $\tau < \infty$, simple calculation shows that $\sup_{k\in\mathbb{N}} \|z_k\|_\infty < \infty$ as well.

Finally, we remove the boundedness assumption of the initial state z_0. To this end, we consider the auxiliary system whose stability property is equivalent to that of the system (15). Choose a scaling factor

$$\gamma > \max\{\zeta\sqrt{n_1}|\lambda_1|, \ldots, \zeta\sqrt{n_f}|\lambda_f|\} \tag{19}$$

and define

$$y_j = \begin{cases} \gamma^{-j\tau} z_{j\tau}, & j < j_0 < \infty, \\ \gamma^{-j_0\tau} z_{j\tau}, & j \geq j_0, \end{cases}$$

where j_0 is to be determined later. At times before j_0, the quantizer keeps quiet and there is no feedback control signal injected to the system. For the introduced state $y_j, j < j_0$, we obtain

$$y_j = \gamma^{-j\tau} z_{j\tau} = (\frac{J}{\gamma})^{j\tau} z_0 + (\frac{J}{\gamma})^{j\tau} \sum_{k=0}^{j\tau-1} J^{-1-k} S w_k. \tag{20}$$

Since $J = \text{diag}\{J_1, \ldots, J_f\}$, using the definition of the induced matrix norm $\|\cdot\|_\infty$ [16] yields

$$\|J\|_\infty = \max_{i\in[1,\ldots,f]} \|J_i\|_\infty.$$

Thus, there is a $\nu \in [1, \ldots, f]$ such that

$$\nu := \arg\max_{i\in[1,\ldots,f]} \|J_i\|_\infty.$$

Based on the assumption $\|w_k\|_\infty \leq d$, $\forall i \in [1, \ldots, f]$, we have

$$\begin{aligned} \|y_j^{(i)}\|_\infty &= \|(\frac{J_i}{\gamma})^{j\tau} z_0^{(i)} + \frac{1}{\gamma^{j\tau}}(\sum_{k=0}^{j\tau-1} J^{j\tau-1-k} S w_k)^{(i)}\|_\infty \\ &\leq \frac{\|J_i\|_\infty^{j\tau}}{\gamma^{j\tau}} \|z_0\|_\infty + \frac{d\|S\|_\infty}{\gamma^{j\tau}} \sum_{k=0}^{j\tau-1} \|J\|_\infty^{j\tau-1-k} \\ &= \frac{\|J_i\|_\infty^{j\tau}}{\gamma^{j\tau}} \|z_0\|_\infty + \frac{d\|S\|_\infty}{\gamma^{j\tau}} \sum_{k=0}^{j\tau-1} \|J_\nu\|_\infty^{j\tau-1-k} \\ &\leq \|z_0\|_\infty (\frac{\zeta\sqrt{n_i}|\lambda_i|}{\gamma})^{j\tau} \\ &\quad + \frac{d\|S\|_\infty}{\gamma^{j\tau}} \sum_{k=0}^{j\tau-1} (\zeta\sqrt{n_\nu}|\lambda_\nu|)^{j\tau-1-k} \\ &\leq \|z_0\|_\infty (\frac{\zeta\sqrt{n_i}|\lambda_i|}{\gamma})^{j\tau} \\ &\quad + \frac{d\|S\|_\infty}{\zeta\sqrt{n_\nu}|\lambda_\nu| - 1} (\frac{\zeta\sqrt{n_\nu}|\lambda_\nu|}{\gamma})^{j\tau} \to 0, \end{aligned} \tag{21}$$

as $j \to \infty$ due to the choice of γ in (19). Consequently, we can find a finite j_0 such that y_{j_0} eventually goes into the bounded region Ω_0. Once y_{j_0} enters Ω_0, the following modified control law will be applied to the system:

$$\begin{bmatrix} u_{j\tau+\tau-1} \\ \vdots \\ u_{j\tau+\tau-\mu} \end{bmatrix} = -\triangle \gamma^{j_0\tau} \mathscr{C}^T (\mathscr{C}\mathscr{C}^T)^{-1} J^\tau Q(\frac{y_j}{\triangle}), \tag{22}$$

and

$$u_{j\tau+t} = 0, \forall t \in [0, \ldots, \tau-\mu-1] \text{ for } j \geq j_0. \tag{23}$$

The rest of proof is the same as the case of bounded initial state $z_0 \in \Omega_0$.

■

The use of the same quantizer in Theorem 1 and a slightly modified control law can asymptotically stabilize the unstable system in (2) with $w_k = 0$.

Corollary 1: When $d = 0$ in (2), asymptotic stabilization can be achieved using a finite-level logarithmic quantizer in (6) if and only if the average data rate exceeds H in (1).

Proof: It is obvious that only the sufficiency part needs to be shown. Define a scaling factor

$$\eta := \max\{\kappa(\tau, \lambda_i)\rho_i^{N_i-1}, \kappa(\tau, \lambda_i)\delta_i, i \in [1, \ldots, f]\}, \tag{24}$$

which is strictly less than one by (18), i.e., $\eta < 1$ and the dynamic equation for $\triangle_j$,

$$\begin{cases} \triangle_{j+1} = \eta \triangle_j; \\ \triangle_0 > 0. \end{cases}$$

The control law within the cycle $[j\tau, j\tau+1, \ldots, (j+1)\tau-1]$ is modified in the following way:

$$\begin{bmatrix} u_{j\tau+\tau-1} \\ \vdots \\ u_{j\tau+\tau-\mu} \end{bmatrix} = -\triangle_j \mathscr{C}^T (\mathscr{C}\mathscr{C}^T)^{-1} J^\tau Q(\frac{z_{j\tau}}{\triangle_j}) \tag{25}$$

and

$$u_{j\tau+t} = 0, \forall t \in [0, \ldots, \tau-\mu-1].$$

Assume that $\|z_0\|_\infty < \triangle_0$, replacing $\triangle$ in (16) with $\triangle_j$ yields

$$\|z_{(j+1)\tau}\|_\infty \leq \eta \triangle_j = \triangle_{j+1}.$$

Thus, $z_{j\tau}$ can be asymptotically driven to zero since

$$\lim_{j\to\infty} \|z_{j\tau}\|_\infty \leq \triangle_0 \lim_{j\to\infty} \eta^j = 0.$$

Due to $\tau < \infty$, it follows that

$$\lim_{k\to\infty} \|z_k\|_\infty = 0.$$

The removal of the boundedness assumption for the initial state is the same as what we have done in Theorem 1. ■

Remark 2: From Corollary 1, it is clear that the convergence rate is determined by $0 < \eta < 1$. The larger the average data rate R, the larger α could be chosen. By (10), it is more likely to select a larger N_i. According to the definition of η in (24), it may result in a smaller η and thus a faster convergence rate.

Remark 3: Although the results are derived for the state feedback case, they can be extended to the output feedback case as well. Consider the system of the form:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + v_k, \end{aligned}$$

where (A,B) and (C,A) are stabilizable and detectable pairs, respectively. The measurement noise is assumed to be uniformly bounded, i.e., $\sup_{k\in\mathbb{N}} \|v_k\|_\infty < \infty$. In this case, we can first design an observer of the form

$$\hat{x}_{k+1} = A\hat{x}_k + K_k(y_k - C\hat{x}_k),\ \hat{x}_0 = x_0$$

to estimate the state, where K_k is the observer gain such that the observer error is stable. Using the triangular inequality

$$\|x_k\|_\infty \le \|\hat{x}_k\|_\infty + \|x_k - \hat{x}_k\|_\infty,$$

and repeating the above proof on the fully observed $\hat{x}_k$ concludes that

$$\sup_{k\in\mathbb{N}} \|\hat{x}_k\|_\infty < \infty.$$

Since $\|x_k - \hat{x}_k\|_\infty$ is uniformly bounded, $\sup_{k\in\mathbb{N}} \|x_k\|_\infty < \infty$ as well.

IV. Conclusion

We addressed the optimality of the logarithmic quantizer in the sense of achieving the minimum average data rate to stabilize an unstable discrete-time linear system. In particular, a pair of the finite-level logarithmic quantizer and controller were constructed to stabilize the system assuming that the state is accessible (or could be estimated with an observer) by the quantizer.

However, our consideration was restricted to a purely deterministic framework, where the boundedness was required for the noise disturbance. One of the future directions is to explore whether the optimality of the logarithmic quantizer continues to work in a stochastic setting by attaching probability distributions to the initial state and noise disturbance.

V. Appendix: Proof of (11)

We give a sketch of its proof.

It is trivial for $|\lambda_i| = 1$. The focus will be on the strictly unstable eigenvalue.

Note that $|\lambda_i| > 1$, then $|\lambda_i|^\tau \to \infty$ and $\kappa(\tau,\lambda_i) \to \infty$ as $\tau \to \infty$. It is well known that $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$, which implies

$$\log_2 \frac{\kappa(\tau,\lambda_i)+\varepsilon+1}{\kappa(\tau,\lambda_i)+\varepsilon-1} \sim 2\kappa^{-1}(\tau,\lambda_i)\log_2 e,$$

if τ is sufficiently large. Substituting the above into (11) yields

$$\begin{aligned} & \left(1+\kappa(\tau,\lambda_i)\ln\kappa(\tau,\lambda_i)\right)^{1/\tau} \\ = & \left(1+\zeta\sqrt{n_i}\tau^{n_i-1}|\lambda_i|^\tau \ln\kappa(\tau,\lambda_i)\right)^{1/\tau} \\ \ge & |\lambda_i|(\zeta\sqrt{n_i})^{1/\tau}(\ln\kappa(\tau,\lambda_i))^{1/\tau}(\tau^{1/\tau})^{n_i-1} \\ = & |\lambda_i| \text{ as } \tau\to\infty, \end{aligned}$$

since τ goes to ∞, $1 \le (\ln\kappa(\tau,\lambda_i))^{1/\tau}$ and

$$\begin{aligned} (\ln\kappa(\tau,\lambda_i))^{1/\tau} &= (\ln(\zeta\sqrt{n_i}) + (n_i-1)\ln\tau + \tau\ln|\lambda_i|)^{1/\tau} \\ &\le (\tau^2)^{1/\tau} \to 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left(1+\kappa(\tau,\lambda_i)\ln\kappa(\tau,\lambda_i)\right)^{1/\tau} \\ \le & \left(2\zeta\sqrt{n_i}\tau^{n_i-1}|\lambda_i|^\tau \ln\kappa(\tau,\lambda_i)\right)^{1/\tau} \\ \le & |\lambda_i|(2\zeta\sqrt{n_i})^{1/\tau}(\tau^{1/\tau})^{n_i+1} \\ = & |\lambda_i| \text{ as } \tau\to\infty. \end{aligned}$$

In view of the above inequities, we get the limit

$$\lim_{\tau\to\infty} \left(1+\kappa(\tau,\lambda_i)\ln\kappa(\tau,\lambda_i)\right)^{1/\tau} = |\lambda_i|.$$

References

[1] D. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Transactions on Automatic Control*, vol. 35, no. 8, pp. 916–924, 1990.

[2] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, 2000.

[3] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints. II. Stabilization with limited information feedback," *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049–1053, 1999.

[4] J. Baillieul, "Feedback coding for information-based control: operating near the data-rate limit," *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 3, 2002.

[5] G. Nair and R. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, no. 4, pp. 585–593, 2003.

[6] ——, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[7] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[8] N. Elia and S. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[9] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1698–1711, 2005.

[10] ——, "Finite-Level Quantization Feedback Control for Linear Systems," *to appear in IEEE Transactions on Automatic Control*.

[11] H. Gao and T. Chen, "A new approach to quantized feedback control systems," *Automatica*, vol. 44, no. 2, pp. 534–542, 2007.

[12] J. Liu and N. Elia, "Quantized feedback stabilization of non-linear affine systems," *International Journal of Control*, vol. 77, no. 3, pp. 239–249, 2004.

[13] H. Hoshina, K. Tsumura, and H. Ishii, "The coarsest logarithmic quantizers for stabilization of linear systems with packet losses," *46th IEEE Conference on Decision and Control*, pp. 2235–2240, 2007.

[14] F. Ceragioli and C. De Persis, "Discontinuous stabilization of nonlinear systems: Quantized and switching controls," *Systems & Control Letters*, vol. 56, no. 7-8, pp. 461–473, 2007.

[15] M. Fu, L. Xie, and W. Su, "Connections between quantized feedback control and quantized estimation," *Int. Conf. Control, Automation, Robotics and Vision, Hanoi*, 2008.

[16] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.