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Affine Formation of Multi-Agent Systems over Directed Graphs

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Abstract— **This paper proposes a simple, distributed control protocol for multi-agent systems with relative sensing capability over a directed network to achieve an affine formation. In contrast to the linear consensus protocol, the control protocol in this paper takes both positive and negative weights that partially encode the target formation shape and is thus able to achieve a formation pattern rather than just consensus. Having possibly negative weights in a graph, the associated Laplacian is called the generalized Laplacian. The connection between the generic rank property of the generalized Laplacian and the connectivity of a directed graph is established, based on which a necessary and sufficient graphical condition is developed to ensure the emergence of an affine formation by the proposed distributed control protocol.**

I. Introduction

This paper proposes a distributed control protocol for multi-agent systems with relative sensing capability over a directed graph to achieve an affine formation. As shown in [1], an affine formation control of distributed agents is of great theoretical interest as an affine formation represents a class of collective patterns that preserve collinearity and ratios of distances (i.e., the agents lying on a line initially still lie on a line and maintain the ratio of distances after transformation). It also provides a new approach for rigid formation control since a network of agents in an affine formation can be reshaped to form a globally rigid formation or a translational formation by controlling only a small number of agents in the network. This is advantageous for better adaptivity to the possibly changing environment. The work of [1] assumes an undirected graph for the interaction of agents in a network. However, the more challenging problem within the directed network setup has not been addressed yet. This paper aims to provide a solution for the affine control problem of multi-agent systems over directed graphs in arbitrarily dimensional space.

In recent years, a large body of work has been looking at local interaction rules using relative states between neighboring agents. Suppose each agent updates its state according to a weighted sum of its neighbors's relative states. A collective pattern (namely, consensus) occurs provided that the graph (either undirected or directed) modeling the interaction topology has certain connectivity properties [2], [3]. A notion of *rooted graph*, meaning that there is a node in the graph called root so that every other node is reachable (having a path) from the root, is able to unify the necessary and sufficient connectivity properties for consensus in undirected and directed graphs. If multiple nodes in a network play the role of leaders while the others update their states according to their neighbor's relative states, then the resulting collective pattern is either a straight line ([2], [4]) or a configuration within a convex hull ([5], [6]).

The works above assume that the weights to generate the local interaction rules are all positive and real. Only few papers address collective behaviors using negative real weights. [7] shows that the use of negative weights may lead to faster convergence for distributed averaging consensus, while [8] shows that the use of negative weights can lead to a consensus value, which is the same for all agents except for the sign. Moreover, the works by [9], [10] consider negative weights as the inhibitory mechanism to desynchronize the interacting agents in different clusters. Nevertheless, these works all concentrate on consensus for all agents in the network or agents in the same cluster while the network is divided into several clusters. More recently, a novel idea using complex weights in local interaction rules is proposed in [11]–[13], which achieves a collective pattern called *similar formation* in the plane. In other words, the achieved pattern is similar to a target configuration subject to rotations, translations and dilations. A necessary and sufficient graphical condition is given in [12], [13] for realizability of a similar formation. It states that a graph (either undirected or directed) modeling the interaction topology requires to be 2-rooted, i.e., there exists a subset of two nodes, called *roots*, from which every other node in the graph has two disjoint paths.

This paper considers a similar local interaction law to update each agent's state using a weighted sum of its neighbor's relative states. However, in contrast to the linear consensus control law, the local interaction law in this paper takes possibly negative weights that partially encode the target formation shape. That is, the weights are designed to meet certain algebraic constraints related to a given target configuration and then a collective pattern, called an affine formation, emerges, which is similar to the target configuration subject to rotations, translations, dilations and shears. Also, in contrast to the complex-weight based local interaction law in [11]–[13], which limits to the 2-dimensional space for each agent's state, the local interaction law in this paper can deal with pattern formation in arbitrarily dimensional space, but the emergent pattern does not preserve angles between lines like the ones in [11]–[13].

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The main contribution of this paper is twofold. First, the adoption of possibly negative weights in a simple local interaction law makes it possible to achieve an affine formation in arbitrarily dimensional space. Secondly, a necessary and sufficient graphical condition is presented to verify from the structural viewpoint whether such a simple local interaction law exists to stabilize the agents to an affine formation. For distributed agents over directed networks, it is shown that a $(d+1)$-rooted graph is necessary and sufficient for affine formation control in the d-dimensional space. This complements the results in [1] where it shows a globally rigid graph is necessary and sufficient for affine formation control of distributed agents over undirected networks.

The organization of this paper is as follows. Section II introduces some basic notation and preliminary results in directed graphs and then formulates the affine formation control problem. Section III develops a necessary and sufficient condition for affine formation control and provides an approach for the design of local control law for a given target configuration. Section IV presents simulations to validate our theoretic results. Section V concludes the paper.

Notation: $\mathbb{R}$ represents the set of real numbers. $\mathbf{1}_n$ stands for the n-dimensional vector of ones and I_d denotes the $d \times d$ identity matrix. span$\{p_1, \ldots, p_n\}$ represents the linear span of vectors $p_1, \ldots, p_n$. diag$(a_1, \ldots, a_n)$ denotes the diagonal (or block diagonal) matrix with its diagonal (or block diagonal) entries being $a_1, \ldots, a_n$.

II. Preliminaries and Problem Formulation

In this section, we first introduce some basic notation and preliminary results in directed graphs and then formulate the affine formation control problems we study.

A. Basic notation and preliminary results for directed graphs

A *directed graph* is a set of n nodes $\mathcal{V}$ and m edges $\mathcal{E}$, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A *configuration in* $\mathbb{R}^d$ (or simply called a configuration in this paper) of a set of n nodes is defined by their coordinates in the Euclidean space $\mathbb{R}^d$, denoted as $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$, where each $p_i \in \mathbb{R}^d$ for $1 \le i \le n$. A configuration p is *generic* if the coordinates $p_1, \ldots, p_n$ do not satisfy any nontrivial algebraic equation with integer coefficients ([14]). Next, several new definitions are introduced and a preliminary result is presented.

Definition 2.1: For a directed graph $\mathcal{G}$, a node v is said to be *k-reachable* from a non-singleton set $\mathcal{U}$ of nodes if there exists a path from a node in $\mathcal{U}$ to v after removing any $k-1$ nodes except node v (i.e., there are k disjoint paths from $\mathcal{U}$ to v).

Definition 2.2: A directed graph is k*-rooted* if there exists a subset of k nodes called roots, from which every other node is k-reachable.

Definition 2.3: For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a spanning k-tree of $\mathcal{G}$ rooted at $\mathcal{R} = \{r_1, r_2, \cdots, r_k\} \subset \mathcal{V}$ is a spanning subgraph $\mathcal{T} = (\mathcal{V}, \bar{\mathcal{E}})$ such that

(1) every node $r \in \mathcal{R}$ has no in-neighbor;
(2) every node $v \notin \mathcal{R}$ has k in-neighbors;
(3) every node $v \notin \mathcal{R}$ is k-reachable from $\mathcal{R}$.

Lemma 2.1: A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is k-rooted if and only if $\mathcal{G}$ has a spanning k-tree.

Proof. (Sufficiency) If $\mathcal{G}$ has a spanning k-tree, then by the definition of k-rooted graph, it is certain that $\mathcal{G}$ is k-rooted.

(Necessity) By the definition of k-rooted graph, we know that there exists a subset of k nodes, called roots, such that every other node is k-reachable from them. Denote by $\mathcal{R} = \{r_1, r_2, \cdots, r_k\}$ the set composed of k roots.

First we remove all incoming edges to nodes in $\mathcal{R}$. By doing so, every node $v \notin \mathcal{R}$ is still k-reachable from $\mathcal{R}$. Second, we remove extra incoming edges for node $v \notin \mathcal{R}$ such that there remain k incoming edges for node v and is still k-reachable from $\mathcal{R}$. It is obvious that the removal of edges on node v does not affect the k-reachability from $\mathcal{R}$ to other nodes that do not have paths from $\mathcal{R}$ via node v. Moreover, for those nodes that have paths from $\mathcal{R}$ via node v, there must be another disjoint path not containing v connecting from $\mathcal{R}$ to u due to the k-reachability property. So the removal of the extra incoming edges on v also does not affect their k-reachability. Therefore, by Definition 2.3, a spanning k-tree is constructed. ∎

B. Problem formulation

We consider a group of n agents, whose states are denoted by $z_1, \cdots, z_n \in \mathbb{R}^d$, (for example, mobile robots or unmanned aerial vehicles with their states being the positions in the plane or in the 3-dimensional space). Suppose each agent is governed by a single-integrator dynamics as follows.

$$\dot{z}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $u_i \in \mathbb{R}^d$ represents the control input of each agent. Define the aggregate state $z = [z_1^\intercal, \cdots, z_n^\intercal]^\intercal$, as a column vector in $\mathbb{R}^{nd}$.

Moreover, suppose that each agent i is able to access the relative states $(z_j - z_i)$ of its neighbors $j \in \mathcal{N}_i$ where $\mathcal{N}_i$ denotes the set of agent i's neighbors, i.e., $(j, i) \in \mathcal{E}$ in the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that models the information flow structure among the n agents. We use

$$z_{ij} = z_i - z_j, \quad j \in \mathcal{N}_i$$

to denote the relative state available to agent i.

Consider a target configuration $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$ in $\mathbb{R}^{nd}$, where each $p_i \in \mathbb{R}^d$ for $1 \le i \le n$. We denote the *affine image* of p as

$$\mathcal{A}(p) := \left\{ q = [q_1^\intercal, \ldots, q_n^\intercal]^\intercal \;\middle|\; \begin{array}{l} q_i = Ap_i + a,\ A \in \mathbb{R}^{d\times d} \\ a \in \mathbb{R}^d,\ \text{and } i = 1, \ldots, n \end{array} \right\}$$

or equivalently,

$$\mathcal{A}(p) := \left\{ q = (I_n \otimes A)p + \mathbf{1}_n \otimes a \,\middle|\, A \in \mathbb{R}^{d\times d},\ a \in \mathbb{R}^d \right\}.$$

Notice that a real matrix A can be factorized by singular value decomposition as $A = U\Sigma V$ where U and V are unitary matrices, and Σ is a $d \times d$ diagonal matrix. It means that a configuration in $\mathcal{A}(p)$ is attained via an affine motion from p, namely, a translation a, followed by a rotation V, a scaling along different axis by Σ, and then another rotation U. So indeed, $\mathcal{A}(p)$ represents a class of collective patterns for the n agents in the state space. We say that the n agents form an *affine formation* if the state z belongs to $\mathcal{A}(p)$.

In this paper, we study the *affine formation control problems* with the objective of steering the state of a group of agents to the affine image of a target configuration and uncovering a necessary and sufficient graphical condition in order to achieve an affine formation using the following local interaction law

$$u_i = -\sum_{j\in\mathcal{N}_i} a_{ij}z_{ij},\ i = 1,\cdots,n \tag{2}$$

where a_{ij} is a non-zero scalar weight (possibly negative) on the edge (j,i) of $\mathcal{G}$. In other words, we aim to find the graphical conditions for $\mathcal{G}$ and weights a_{ij}'s such that the trajectories of the closed-loop system satisfy

$$\lim_{t\to\infty} z(t) = z^* \quad \text{ where } z^* \in \mathcal{A}(p).$$

In the d-dimensional space, if there are only $d+1$ or fewer agents, the problem is trivial. Therefore, in this paper we make the following assumption.

A1. The total number n of agents is greater than $d+1$.

Remark 2.1: If the matrix A in the definition of $\mathcal{A}(p)$ is an unitary matrix, the affine image $\mathcal{A}(p)$ is called a *rotation/translation image*, denoted as $\mathcal{R}(p)$. In other words, if $z \in \mathcal{R}(p)$, the agents form a *rigid formation* congruent to p, meaning that their configuration is congruent to the target configuration. Moreover, if the matrix A in the definition of $\mathcal{A}(p)$ is an identity matrix, the affine image $\mathcal{A}(p)$ is called a *translation image*, denoted as $\mathcal{T}(p)$. That is, if $z \in \mathcal{T}(p)$, the agents form a rigid formation with the same orientation as the target configuration. In [1], it has been shown that if a group of agents are in an affine formation, then by adding a few extra constraints for a small number of agents, the whole team achieves a globally rigid formation. Therefore, controlling a group of agents to achieve an affine formation not only is of its own interest, but also provides a new approach for rigid formation control.

III. Affine formation over directed networks

In this section we develop a necessary and sufficient graphical condition for achieving an affine formation and then provide a procedure for the design of local control laws.

A. Necessary and sufficient condition for realizability of affine formation

As the first step towards the goal of achieving an affine formation, we explore a necessary and sufficient condition under which the equilibrium set of the closed-loop system under the local interaction law (2) is exactly the affine image $\mathcal{A}(p)$ of the target configuration p.

Under the local interaction law (2), the closed-loop system is of the following form.

$$\dot{z} = -(L\otimes I_d)z \tag{3}$$

where $L\in\mathbb{R}^{n\times n}$ is the matrix whose (i,j)th off-diagonal element is the weight a_{ij} on edge (j,i) if it is an edge of $\mathcal{G}$ and 0 otherwise, and whose diagonal entry is the negative row sum of off-diagonal entries in the same row. In this paper, we call L the *generalized Laplacian matrix* associated with $\mathcal{G}$ as its off-diagonal entries may contain both positive and negative entries, which is different from the classic Laplacian matrix used in consensus study.

An affine formation is said to be *realizable* if for a graph $\mathcal{G}$ and a target configuration p there exists a generalized Laplacian L associated with $\mathcal{G}$ such that the equilibrium set of system (3) equals to $\mathcal{A}(p)$. Next we present our main result.

Theorem 3.1: Consider a generic configuration $p = [p_1^T,\dots,p_n^T]^T$ with every $p_i\in\mathbb{R}^d$. An affine formation of p is realizable if and only if $\mathcal{G}$ is $(d+1)$-rooted.

The proof requires two auxiliary results.

Lemma 3.1: Consider $p = [p_1^\intercal,\dots,p_n^\intercal]^\intercal$ with every $p_i\in\mathbb{R}^d$. If $\text{span}\{p_1,\dots,p_n\} = \mathbb{R}^d$, then $\mathcal{A}(p)$ is a linear subspace of dimension d^2+d.

Lemma 3.2: For a generic generalized Laplacian L of a directed graph $\mathcal{G}$, if $\mathcal{G}$ is k-rooted with the root set $\mathcal{R} = \{r_1,\cdots,r_k\}$, then

(a) all principal minors of $L_{\mathcal{R}}$ are distinct from zero, where $L_{\mathcal{R}}$ is the sub-matrix of L with the rows and columns corresponding to nodes in $\mathcal{R}$ crossed out;

(b) $\det(M)\neq 0$ where M is the sub-matrix of L by deleting the k rows corresponding to the k roots and any k columns.

The proofs of the above two lemmas are omitted due to space limitations but they can be found in [15].

Proof of Theorem 3.1. (**Necessity**) If an affine formation is realizable, then by definition, there exists a generalized Laplacian matrix L associated with $\mathcal{G}$ such that the equilibrium set of system (3) equals to $\mathcal{A}(p)$. Thus, it can be inferred that $(L\otimes I_d)p = 0$ due to $p\in\mathcal{A}(p)$. Moreover, since the dimension of $\mathcal{A}(p)$ is d^2+d as shown in Lemma 3.1, it then follows that $\text{rank}(L) = n-d-1$. Thus, there exist $d+1$ rows of L, which can be transformed to zero vectors by elementary row operations. Denote by $\mathcal{R}$ the set of nodes corresponding to the indices of these $d+1$ rows.

Now suppose by contradiction that $\mathcal{G}$ is not $(d+1)$-rooted. Then there exists a node $i\notin\mathcal{R}$ such that after deleting d nodes, without loss of generality say $\{1,2,\dots,d\}$, i is not reachable from $\mathcal{R}$. Let $\mathcal{U}$ be the set of nodes not in $\mathcal{R}$ (including node i) such that all nodes in $\mathcal{U}$ are not reachable from $\mathcal{R}$ after removing $\{1,2,\dots,d\}$. Let

$$\bar{\mathcal{U}} = \mathcal{V} - \mathcal{U} - \{1,2,\dots,d\}.$$

Then it is clear that there is no edge from any node in $\bar{\mathcal{U}}$ to any node in $\mathcal{U}$. So by relabeling the nodes in $\mathcal{U}$ and $\bar{\mathcal{U}}$ in a consecutive manner respectively, the matrix L transforms to the following form by a permutation matrix P, i.e.,

$$PLP^\intercal = L' := \left[\begin{array}{c|c|c} L_{11} & L_{12} & L_{13} \\ \hline L_{21} & L_{22} & 0 \\ \hline L_{31} & L_{32} & L_{33} \end{array}\right].$$

where the rows and columns in L_{11} correspond to nodes $1,2,\dots,d$, the rows and columns in L_{22} correspond to the nodes in $\mathcal{U}$, and the rows and columns in L_{33} correspond to the nodes in $\bar{\mathcal{U}}$. Thus, $(L \otimes I_d)p = 0$ is equivalent to

$$(L' \otimes I_d)(P \otimes I_d)p = 0,$$

from which we have

$$\left(\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix} \otimes I_d\right)(P \otimes I_d)p = 0.$$

Moreover, we have $\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix}\mathbf{1} = 0$. Therefore, $\begin{bmatrix} L_{21} & L_{22} & 0 \end{bmatrix}$ is not of full row rank, which together with the fact that $d+1$ rows corresponding nodes not in $\mathcal{U}$ can be transformed to zero vectors by elementary row operations, imply rank$(L) \leq n-d-2$. It contradicts to the above conclusion rank$(L) = n-d-1$. Therefore, $\mathcal{G}$ is $(d+1)$-rooted.

(**Sufficiency**) If $\mathcal{G}$ is $(d+1)$-rooted, we show in the following that there exists a generalized Laplacian matrix L satisfying $(L \otimes I_d)p = 0$ and rank$(L) = n-d-1$.

Since $\mathcal{G}$ is $(d+1)$-rooted, by Lemma 2.3 we know that $\mathcal{G}$ has a spanning $(d+1)$-tree. Denote by $\mathcal{T}$ the spanning $(d+1)$-tree. Let T be a generic generalized Laplacian matrix associated with $\mathcal{T}$. It is known that the rows of T corresponding to the $d+1$ roots are all zero vectors. Moreover, by Lemma 3.2, rank$(T) \geq n-d-1$. So the kernel of T is a $(d+1)$-dimensional subspace, for which one basis is $\mathbf{1}_n$. Denote $\eta_0 = \mathbf{1}_n$ and the other linearly independent basis as $\eta_1, \cdots, \eta_d$.

Next we show that for the tall matrix $[\eta_0, \eta_1, \ldots, \eta_d]$, by removing any $n-d-1$ rows, the remaining square matrix is of full rank. To see this, suppose by contradiction that it is not. That is, by row switching, $[\eta_0, \eta_1, \ldots, \eta_d]$ transforms to the form $[M \;\; N]^T$ where $M \in \mathbb{R}^{(d+1)\times(d+1)}$ is not of full rank. In other words, there is a nonzero vector ξ such that $M\xi = 0$. By corresponding column switching for T followed by removing the rows in T corresponding to the $d+1$ roots, denote the resulting transformed sub-matrix as $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$ where T_1 is $(n-d-1)$-by-$(d+1)$ and T_2 is $(n-d-1)$-by-$(n-d-1)$. Then we have

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} = 0. \tag{4}$$

Plugging $M\xi = 0$ into eq. (4) leads to $T_2 N\xi = 0$. On the other hand, from Lemma 3.2, we know that for a generic generalized Laplacian T, T_2 is of full rank. Therefore, $N\xi = 0$, which together with $M\xi = 0$ imply $\eta_0, \eta_1, \ldots, \eta_d$ are not linearly independent, a contradiction.

Since $\mathcal{T}$ is a $(d+1)$-tree, each non-root node has exactly $(d+1)$ neighbors, which implies the corresponding row of T has at most $(d+2)$ nonzero entries. Denote by μ the sub-vector of a row of T corresponding to a non-root node, which aggregates only the $(d+2)$ nonzero entries. Since we just showed that for the tall matrix $[\eta_0, \eta_1, \ldots, \eta_d]$, by removing any $n-d-1$ rows the remaining $(d+1)$-by-$(d+1)$ square matrix is of full rank, it means μ lies in a one-dimensional subspace. Therefore, for generic p and for a generalized Laplacian T' associated with tree $\mathcal{T}$ and satisfying $(T' \otimes I_d)p = 0$, the corresponding μ' of T' also comes from a subspace of at least one dimension. Therefore, T' has the same zero/nonzero pattern as T and

$$\text{rank}(T') = \text{rank}(T) = n-d-1$$

in a generic sense.

For a $(d+1)$-rooted graph $\mathcal{G}$ and a generalized Laplacian matrix L satisfying $(L \otimes I_d)p = 0$, T' can be considered as a Laplacian of $\mathcal{G}$ for a special choice of weights with edges not in $\mathcal{T}$ being 0. Thus, by using the fact that either a polynomial is zero or it is not zero almost everywhere, it follows that rank$(L) = n-d-1$, too.

On the other hand, since $(L \otimes I_d)p = 0$, it turns out that for any $A \in \mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$,

$$\begin{aligned}&(L \otimes I_d)[(I_n \otimes A)p + \mathbf{1}_n \otimes a] = (L \otimes A)p \\ &= (I_n \otimes A)(L \otimes I_d)p = 0,\end{aligned}$$

which means the affine image $\mathcal{A}(p)$ is a subset of the equilibrium set. Moreover, from Lemma 3.1 we know that $\mathcal{A}(p)$ is a linear subspace of dimension $(d+1)d$, which equals to the dimension of null space of $L \otimes I_d$. Therefore, it is certain that the equilibrium subspace of system (3) equals to $\mathcal{A}(p)$, that means affine formation is realizable. ■

B. *Stabilizability of affine formation*

The preceding subsection shows that in order to make an affine formation an equilibrium of the overall system under the local interaction law (2), the graph $\mathcal{G}$ needs to be $(d+1)$-rooted. The result in this section will show that a $(d+1)$-rooted graph is also sufficient to ensure the existence of a generalized Laplacian matrix L making the system globally exponentially stable.

An affine formation is said to be *stabilizable* if for a graph $\mathcal{G}$ and a target configuration p there exists a generalized Laplacian matrix L associated with $\mathcal{G}$ such that the overall closed-loop system (3) is globally exponentially stable with respect to $\mathcal{A}(p)$. The theorem below provides a necessary and sufficient condition for stabilizability of an affine formation.

Theorem 3.2: Consider a generic configuration $p = [p_1^T, \ldots, p_n^T]^T$ with every $p_i \in \mathbb{R}^d$. An affine formation of p is stabilizable if and only if $\mathcal{G}$ is $(d+1)$-rooted.

The proof requires the following result related to the multiplicative inverse eigenvalue problem over the real field.

Lemma 3.3 ([16]): Let A be an $n \times n$ real matrix with all of its leading principal minors being nonzero. Then there is an $n \times n$ diagonal matrix D such that all the roots of DA are positive and simple.

Proof of Theorem 3.2. (**Necessity**) If an affine formation is stabilizable, then it can be inferred that affine formation must be realizable. So by Theorem 3.1, $\mathcal{G}$ is $(d+1)$-rooted.

(**Sufficiency**) If $\mathcal{G}$ is $(d+1)$-rooted, then by Lemma 3.2 it follows that for a generic generalized Laplacian matrix L, there is a permutation operation (multiplying with a permutation matrix P, equivalent to relabeling the nodes) such that PLP^T has the form

$$L' = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

with $B_1 \in \mathbb{R}^{(n-d-1)\times(n-d-1)}$ and B_2, B_3 and B_4 of appropriate dimension, and all principal minors of B_1 are nonzero. The property also holds for L satisfying $(L \otimes I_d)p = 0$ for a generic p, which can be shown by the same argument as to show $\text{rank}(L) = n - d - 1$ in Theorem 3.1. Thus, by Lemma 3.3, there exists a diagonal matrix $D_1 = \text{diag}(d_1, \cdots, d_{n-d-1})$ such that all the eigenvalues of $D_1 B_1$ are in the right half plane.

Denote $D_2 = \text{diag}(d_{n-d}, \cdots, d_n)$ and $D = \text{diag}(D_1, D_2)$. Moreover, define $M(D_1, D_2) = DL'$. Then it is clear that

$$M(D_1, 0) = \begin{bmatrix} D_1 B_1 & D_1 B_2 \\ 0 & 0 \end{bmatrix}.$$

So the eigenvalues of $M(D_1, 0)$ consist of $d + 1$ zero eigenvalues and the eigenvalues of $D_1 B_1$. Then by the continuity property of eigenvalues, for sufficiently small entries in D_2, $M(D_1, D_2)$ also has eigenvalues in the right half plane except the fixed $d + 1$ zero eigenvalues. This means, $P^\intercal DPL$ can be used to replace the original L and stabilize the overall system to an affine formation, where $P^\intercal DPL$ is another generalized Laplacian matrix associated with $\mathcal{G}$ because $P^\intercal DP$ is a diagonal matrix just scaling each row of L. Thus, the conclusion follows. ■

C. Design of local interaction protocol

The design of local interaction law (2) requires to find proper weights a_{ij}'s so that $L\mathbf{1}_n = 0$, $(L \otimes I_d)p = 0$, $\text{rank}(L) = n - d - 1$ and the nonzero eigenvalues of L all have positive real parts (i.e., system (3) is globally exponentially stable with respect to $\mathcal{A}(p)$ for a generic target configuration p). Theorem 3.2 shows that such L exists if $\mathcal{G}$ is $(d + 1)$-rooted. From the proof of Theorem 3.2, finding proper weights a_{ij} for this purpose can be decomposed into two steps.

First, find weights a'_{ij} so that $L'\mathbf{1}_n = 0$, $(L' \otimes I_d)p = 0$, and $\text{rank}(L') = n - d - 1$, where L' is the generalized Laplacian associated to $\mathcal{G}$ with weights a'_{ij}. This step can be done locally. For a given target configuration $p = [p_1^\intercal, \ldots, p_n^\intercal]^\intercal$, each agent i is supposed to know p_i and p_j for $j \in \mathcal{N}_i$ as otherwise the problem is not feasible. Then agent i computes a'_{ij}'s according to the following formula

$$\sum_{j \in \mathcal{N}_i} a'_{ij}(p_j - p_i) = 0. \quad (5)$$

Since $\mathcal{G}$ is $(d+1)$-rooted, by the definition of a rooted graph, it is known that each node has at least $(d + 1)$ neighbors, which means eq. (5) must have a solution and the solution space is of at least one dimension. Eq. (5) is a linear equation and can be solved easily. For each agent i, picking any one solution to eq. (5) gives a choice of a'_{ij} for $j \in \mathcal{N}_i$, which certainly ensures $L'\mathbf{1}_n = 0$ and $(L' \otimes I_d)p = 0$. Moreover, from the proof of Theorem 3.1, we know that for arbitrarily picking a'_{ij}'s as described above, $\text{rank}(L') = n - d - 1$ holds almost surely since $\mathcal{G}$ is $(d + 1)$-rooted.

Secondly, design a diagonal matrix D such that the nonzero eigenvalues of DL' all have positive real parts. The existence of D is assured by Theorem 3.2. Usually, this step requires centralized computation. However, an observation from simulations shows that by picking 1 or -1 for d_i (the ith diagonal entry of D) to make $d_i L'(i, i)$ positive (where $L'(i, i)$ is the ith diagonal entry of L'), the nonzero eigenvalues of DL' may all become to have positive real parts in most cases. In this way, this step can be done locally as well.

IV. Simulations

In this section, we present two simulation results to demonstrate the correctness of our results: One is in the plane with 9 agents and the other is in the three dimensional space with 12 agents.

For the simulation in the plane we consider a target configuration p in Fig. 1, which is a 3-by-3 grid. Suppose the agents interact with each other over a directed graph $\mathcal{G}$ in Fig. 1.

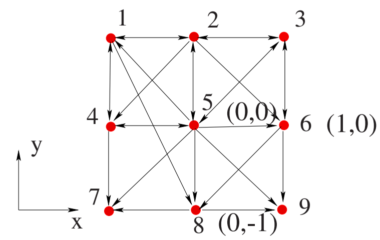

Fig. 1. A target configuration p in $\mathbb{R}^2$ an a directed graph $\mathcal{G}$, according to which the agents interact locally.

It can be checked that $\mathcal{G}$ is 3-rooted. Thus by Theorem 3.2, a local interaction law exists to achieve an affine formation of p in $\mathbb{R}^2$. In other words, proper weights a_{ij}'s can be found for this purpose. By the design procedure described in Subsection III-C, proper weights a_{ij}'s can be found, e.g.,

$a_{12} = 1$, $a_{13} = 1$, $a_{14} = -1$, $a_{21} = 1$, $a_{23} = 1$, $a_{32} = 1$, $a_{35} = -1$, $a_{36} = 1$, $a_{41} = 1$, $a_{42} = -1$, $a_{45} = 1$, $a_{52} = -1$, $a_{53} = 1$, $a_{54} = 1$, $a_{62} = -1$, $a_{63} = 1$, $a_{65} = 1$, $a_{74} = 1$, $a_{75} = -1$, $a_{78} = 1$, $a_{81} = -1$, $a_{85} = 3$, $a_{86} = -1$, $a_{95} = -2$, $a_{96} = 2$, $a_{98} = 2$.

One simulation result is presented in Fig. 2 with randomly selected initial states. It can be seen that a group of 9 agents achieves an affine formation of p, which preserves the collinearity and the ratio of distances, but not angles. The final configuration plotted in Fig. 2 is an affine map of p via rotations, translations, dilations, and shears.

The second example demonstrates the application of our result in the three dimensional space. A target configuration p in 3D is given in Fig. 3 and suppose the agents in 3D interact locally over a directed graph $\mathcal{G}$ in Fig. 3.

The graph $\mathcal{G}$ is 4-rooted as required in 3D by Theorem 3.2. By the same design procedure as for the 2D case, proper weights a_{ij}'s can be found so that the local interaction law (2) with these weights leads the 12 agents to an affine formation of p in 3D asymptotically for arbitrarily initial states. The simulation result is presented in Fig. 4 with randomly initial states. It can be seen that the system asymptotically converge to an equilibrium in the affine image. The

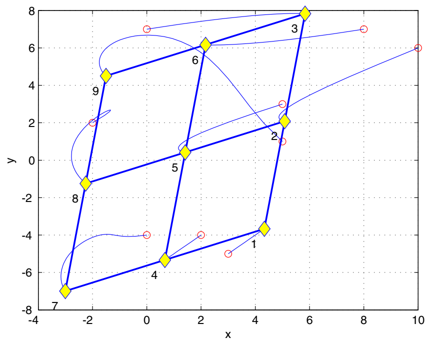

Fig. 2. Achieved affine formation in $\mathbb{R}^2$ and the trajectories of all agents starting from randomly selected initial states.

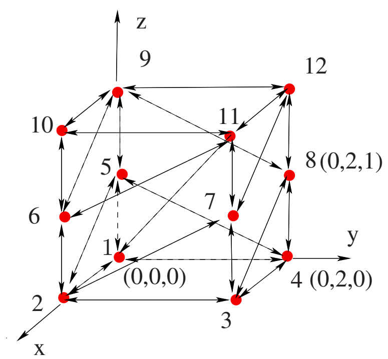

Fig. 3. A target configuration p in $\mathbb{R}^3$ and a directed graph $\mathcal{G}$, according to which the agents interact locally in 3D. (The doted edges in the figure are those edges that cannot be seen from the front view)

final configuration plotted in Fig. 4 is an affine map of p with rotations, translations, dilations, and shears.

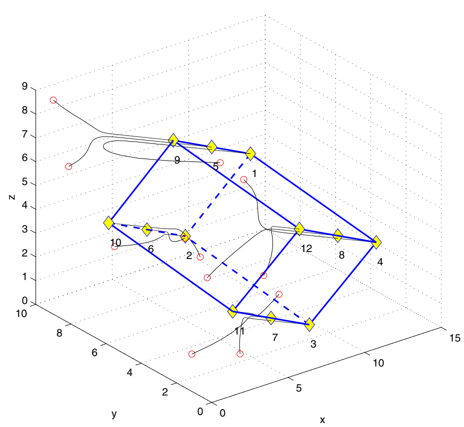

Fig. 4. Achieved affine formation in $\mathbb{R}^3$ and the trajectories of all agents starting from one initial state.

V. Conclusions

The paper solves a multi-agent control problem of achieving an affine formation in arbitrarily dimensional space. Each agent updates its own state according to a weighted sum of its neighbor's relative states, similar to the linear consensus protocol, but the weights here can be either positive or negative. The graph modeling the interaction topology is directed, which is more general. The connection of the generic rank property of generalized Laplacian with both positive and negative weights and the connectivity of a directed graph is established, with which a necessary and sufficient condition is obtained for a multi-agent system to ensure the emergence of collective formation pattern by local interaction with both positive and negative weights. It is shown that a $(d+1)$-rooted graph is necessary and sufficient for both realizability and stabilizability of an affine formation in the d-dimensional space. Moreover, how to design proper weights of the interaction protocol for a given target configuration is also developed. However, a main challenge still remains on computing the interaction weights in a fully distributed manner.

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