53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Consensus Based Bisection Approach for Economic Power Dispatch

Hao Xing, Yuting Mou, Minyue Fu, and Zhiyun Lin

Abstract— **Economic dispatch problem (EDP) is an important optimization problem in power systems, aiming at minimizing the aggregate cost of a group of power generators, which cooperatively generate a given amount of power within their individual capacity constraints. In this paper, we present an average consensus based bisection approach for the EDP with quadratic cost functions, which is fully distributed and especially desirable in a smart grid scenario. Under the connected topology condition, we show that the proposed iterative solution converges to the globally optimal solution of EDP, without the need for a central decision maker or a leader node. Finally, numerical simulation based on the IEEE 14-bus system is given to show the performance of the approach.**

I. Introduction

Economic dispatch problem (EDP) has attracted broad attention in the electric power industry for optimal operation and planning of energy resources, which is usually formulated as an optimization problem [1]. The classic EDP mainly concerns the economic dispatch of fossil-fired power generation systems to achieve a minimum operational cost within their capacity limits. Many centralized solutions have been proposed, e.g., the conventional Lagrangian Relaxation approach and the first order gradient method [1], direct search method [2], parallel micro genetic algorithm (PMGA) for ramp-rate constrained EDP [3].

Distributed algorithms for control, estimation and optimization have been intensively investigated for large-scale systems [4]. A smart grid with distributed renewable power generation is a typical such large-scale system [5]. Inspired by some natural phenomena, such as bird flocking and fish schooling, multi-agent systems (MAS) problems, including average consensus [6], finite-time consensus [7], biological cell coupling [8], leader-following and formation [9], have been heavily investigated [10], where the agents (or nodes) can collectively achieve a common global goal without global information and a central controller. Compared with centralized algorithms, distributed algorithms have many advantages, including enhanced robustness, reduction in communication between agents, and uniform power consumption for each agent.

A lot of distributed algorithms for solving EDP in a smart grid scenario have been proposed so far. In [11] and [12], the authors propose a consensus based decentralized algorithm, where a master node aware of the total power demand is required. In [13], the authors present a ratio consensus based decentralized algorithm to find the optimal incremental cost, under the assumption that each node (i.e., generator) knows the parameters of all the nodes. In [14], an algorithm based on a consensus + innovation framework is proposed to find the optimal incremental cost. In [15], the authors propose a consensus based approach, which can be treated as a distributed implementation of standard Lambda-Iteration method, without requiring other nodes' parameters.

In this paper a *distributed bisection algorithm* based on average consensus is proposed. Compared with the existing work, our algorithm has the following features. Firstly, the algorithm proposed in our paper requires no prior knowledge of the systems, while in [15], several global parameters need to be known in order to design an appropriate learning gain for the convergence purpose. And in [13], each node needs to know every other nodes parameters, which implies that the computation and communication package size will explode as the network size grows. Secondly, in this paper, we assume that a bus in the power grid with pure load, pure generation, or both only knows its local demand, which is often the case with some equipped devices on the bus. However, in [15], every bus with a generator is required to be aware of the total demand that may not be known by the equipped measurement devices. In particular, since the power demand is spatially distributed over the entire system, i.e., most loads are located at the buses not having generators, it is unrealistic for a generation bus to access the power demand of other buses, in a distributed fashion. Thirdly, no master or leader node aware of the total power demand is needed in our algorithm, whereas such a node is required in [11], [12] and [13]. Furthermore, in our algorithm, none of the nodes knows the total demand, yet the demand and supply balance is achieved by the algorithm.

The rest of the paper is organized as follows. In Section II, preliminaries on graph theory and average consensus are introduced, and the problem formulation and a centralized solution are given. In Section III, we present our distributed algorithms for the EDP and give the stopping criteria for practical use. Illustrative numerical simulation based on the IEEE 14-bus system is given in Section IV. We conclude our paper in Section V.

H. Xing and Y. Mou are with the State Key Laboratory of Industrial Control Technology and the Department of Control Science and Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027 P. R. China e-mail: haoxing@zju.edu.cn, ytmou@zju.edu.cn

M. Fu is with the School of Electrical Engineering and Computer Science, University of Newcastle, NSW 2308 Australia, and also with the State Key Laboratory of Industrial Control Technology and the Department of Control Science and Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027 P. R. China e-mail: minyue.fu@newcastle.edu.au

Z. Lin is with the State Key Laboratory of Industrial Control Technology and the College of Electrical Engineering, Zhejiang University, 38 Zheda Road, Hangzhou, 310027 P. R. China e-mail: linz@zju.edu.cn

This work was supported by Zhejiang Provincial Natural Science Foundation of China LR13F030002.

978-1-4673-6088-3/14/$31.00 ©2014 IEEE

II. Preliminaries and Problem Formulation

In this section, we first present some basic concepts and results in graph theory and average consensus. Then we formulate the EDP as a quadratic optimization problem with both equality and inequality constraints, and introduce a centralized solution.

A. Graph Theory

An undirected graph $G = (V, E)$ consists of a non-empty finite set of nodes $V = \{1, 2, \ldots, n\}$ and a finite set of unordered paths $E \subseteq V \times V$. For node $i \in V$, its neighbor set is denoted by $N_i = \{j \in V - \{i\} : (j, i) \in E\}$, i.e., node i can bidirectionally communicate with its neighbors. The degree of node i is the cardinality of N_i, denoted by $d_i = |N_i|$. For all $i \in V$ and $j \in N_i$, $(i, j) \in E$ implies $(j, i) \in E$. An undirected graph is connected if there is a path from any node to any other node. It's reasonable to assume that each node can communicate with itself, i.e., $\forall i \in V,\ (i, i) \in E$. The diameter of a connected undirected graph G is defined as the length of the longest among the shortest paths connecting any two nodes in G. We also say that a non-negative matrix $Q \in \mathbb{R}^{n \times n}$ is associated with graph G, where $[Q]_{ij} > 0$ if and only if $(j, i) \in E$.

B. Average Consensus

We consider a MAS consisting of n autonomous agents (nodes), labeled 1 through n. Each agent has the ability of computation and local communication, where the communication network can be represented by an undirected graph $G = (V, E)$, as described in section II-A. The state of agent i is denoted by $\pi_i \in \mathbb{R}$, and the aggregate state is denoted by $\pi = [\pi_1, \pi_2, ..., \pi_n]^T \in \mathbb{R}^n$.

Let us consider the following linear iteration, with the iteration index denoted by $t = 0, 1, 2, ...$ and initial value $\pi(0)$:

$$\pi(t + 1) = Q\pi(t), \tag{1}$$

where matrix $Q \in \mathbb{R}^{n \times n}$ is non-negative and associated with graph G. Iteration (1) can be implemented in a distributed form, i.e.,

$$\pi_i(t + 1) = q_{ii}\pi_i(t) + \sum_{j \in N_i} q_{ij}\pi_j(t),\ \forall i = 1, \ldots, n, \tag{2}$$

where q_{ij} is the entry in the ith row and the jth column of Q.

The iterative algorithm (1) is said to solve the *consensus problem asymptotically* if for any initial state $\pi(0)$, there exists $\pi^* \in \mathbb{R}$, such that $\lim_{t\to\infty} \pi_i(t) = \pi^*$, $\forall i = 1, 2, ..., n$. Moreover, if $\pi^* = \left(\sum_{j=1}^{n} \pi_j(0)\right)/n$, the iterative algorithm (1) is said to solve the *average consensus problem asymptotically*. We now give a well-known theorem on average consensus with regard to connected graphs, and give a local determination of Q.

Theorem 1: [16] Given a connected graph G with self-loops and a associated non-negative matrix Q, if Q is (row) stochastic (i.e., $q_{ij} \geqslant 0$, $\sum_{j=1}^{n} q_{ij} = 1, \forall i$), then Q solves consensus problem, and

$$\lim_{t\to\infty} Q^t = \underline{1}\eta^T,$$

where $\underline{1} = [1, 1, \ldots, 1]^T$, and $\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T$ is the left eigenvector for the eigenvalue 1 of Q, with the properties $\eta_i > 0$ for all i and $\underline{1}^T\eta = 1$. In addition, if Q is doubly stochastic (i.e., both Q and Q^T are stochastic), then the iterative algorithm (1) solves the average consensus problem, i.e., $\eta = \underline{1}/n$.

On an undirected graph G, to solve the average consensus problem, a doubly stochastic matrix Q associated with graph G can be determined locally by:

$$q_{ij} = \begin{cases} \dfrac{1}{\max(d_i, d_j) + 1} & \text{if } j \in N_i, \\ 1 - \displaystyle\sum_{j \in N_i} q_{ij} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

C. Problem Formulation

Now we formulate the EDP into a quadratic optimization problem. We only consider active power in this paper and ignore power transmission loss and transmission capacity constraints, which is valid for many power networks. Each generator is associated with a local variable $x_i \geqslant 0$, i.e., the (active) power generated by generator i, and a quadratic cost function $C_i(x_i)$ given by:

$$C_i(x_i) = a_i x_i^2 + b_i x_i + c_i, \tag{4}$$

where $a_i > 0$, $b_i \geqslant 0$ and $c_i \geqslant 0$ are cost parameters. For simplicity of expression in the following sections, we use an equivalent function by changing a constant term:

$$C_i(x_i) = \frac{(x_i - \alpha_i)^2}{2\beta_i} + \gamma_i \tag{5}$$

with constants $\beta_i > 0$, $\alpha_i \leqslant 0$, and any γ_i.

Denote by m the total number of buses in the grid and the number of buses with power generators by n. In general, each bus can be associated with a generator only, loads only, or both. Since not all the buses are attached to power generators, we have $m > n$. Denoting by P_j the power demand (load) of bus j, the aggregate power demand $P^\star$ is given by

$$P^\star = \sum_{j=1}^{m} P_j,$$

where $P_j = 0$ if bus j is a pure generation bus.

Denoting by $\underline{x}_i$ and $\bar{x}_i$ the lower and upper bounds of x_i, we have

$$0 \leqslant \underline{x}_i \leqslant x_i \leqslant \bar{x}_i.$$

It is obvious that the EDP is feasible (i.e., the optimal solution above holds) if and only if

$$\sum_{i=1}^{n} \underline{x}_i \leqslant P^\star \leqslant \sum_{i=1}^{n} \bar{x}_i. \tag{6}$$

In the framework assumed above, the EDP can be formulated as follows :

$$\min \quad \sum_{i=1}^{n} C_i(x_i), \tag{7}$$

$$\text{s.t.} \quad \underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \ \forall i = 1, 2, \ldots, n, \tag{8}$$

$$\sum_{i=1}^{n} x_i = P^\star. \tag{9}$$

Throughout the paper, we assume that communication networks are imposed on the power grid so that each bus corresponds to a node in the communication network. Here we set up two communication networks, denoted by $G_m = (V_m, E_m)$ and $G_n = (V_n, E_n)$, respectively. The node set V_m consists of all the m buses in the grid, while V_n consists of all the n generation buses, i.e., $V_m = \{1, 2, ..., n, n+1, ..., m\}$ and $V_n = \{1, 2, ..., n\}$. The edge sets of G_m and G_n are denoted by E_m and E_n, respectively. To make our distributed algorithm meaningful, we also assume that G_m and G_n are sparse graphs in the sense that

$$\max_{1 \leqslant i \leqslant m} d_{m,i} \ll m, \quad \max_{1 \leqslant i \leqslant n} d_{n,i} \ll n.$$

D. Centralized Solution to EDP

Since the EDP has been formulated as a quadratic optimization problem, its centralized solution can be easily achieved by the Lagrange dual method [17].

Denote the incremental cost of generator i by

$$f_i(x_i) = \frac{dC_i(x_i)}{dx_i} = \frac{x_i - \alpha_i}{\beta_i}, \ \forall i \in V_n,$$

which is strictly increasing with respect to x_i since $\beta_i > 0$.

The Lagrange dual problem is given by

$$\max \sum_{i=1}^{n} C_i^\star(\lambda) + \lambda P^\star, \tag{10}$$

where

$$C_i^\star(\lambda) = \begin{cases} C_i(\underline{x}_i) - \lambda \underline{x}_i, & \lambda < f_i(\underline{x}_i), \\ -\lambda(\alpha_i + \dfrac{\lambda \beta_i}{2}), & f_i(\underline{x}_i) \leqslant \lambda < f_i(\bar{x}_i), \\ C_i(\bar{x}_i) - \lambda \bar{x}_i, & f_i(\bar{x}_i) \leqslant \lambda, \end{cases} \tag{11}$$

and $\lambda \in \mathbb{R}$ is the Lagrange multiplier.

From the above, we have

$$g_i(\lambda) = \frac{dC_i^\star(\lambda)}{d\lambda} = \begin{cases} -\underline{x}_i, & \lambda < f_i(\underline{x}_i), \\ -\alpha_i - \lambda \beta_i, & f_i(\underline{x}_i) \leqslant \lambda < f_i(\bar{x}_i), \\ -\bar{x}_i, & f_i(\bar{x}_i) \leqslant \lambda. \end{cases} \tag{12}$$

If the primal solution is feasible, the Lagrange dual problem (10) has a unique optimal solution λ^*, satisfying

$$P^\star + \sum_{i=1}^{n} g_i(\lambda^*) = 0.$$

Accordingly, the primal EDP has a unique optimal solution given by $x_i^* = -g_i(\lambda^*), i = 1, 2, \ldots, n$, i,e.,

$$x_i^* = \begin{cases} \underline{x}_i, & \lambda^* < f_i(\underline{x}_i), \\ \alpha_i + \lambda^* \beta_i, & f_i(\underline{x}_i) \leqslant \lambda^* < f_i(\bar{x}_i), \\ \bar{x}_i, & f_i(\bar{x}_i) \leqslant \lambda^*. \end{cases} \tag{13}$$

In the special case where the inequality constraints (8) are void, i.e., if we consider the following unconstrained problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} C_i(x_i), \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = P^\star. \end{aligned} \tag{14}$$

That is, we take $\underline{x}_i = 0$ and $\bar{x}_i = \infty$ in the above.

Defining

$$h_i(\lambda) = -\alpha_i - \lambda \beta_i, \ \forall i \in V_n, \tag{15}$$

it is easily verified that the optimal Lagrange multiplier λ^* reduces to

$$\lambda^* = \frac{P^\star - \sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} \beta_i}, \tag{16}$$

and the optimal solution x^* of problem (14) is given by

$$x_i^* = -h_i(\lambda^*) = \alpha_i + \lambda^* \beta_i, \ \forall i \in V_n. \tag{17}$$

III. Main Results

In this section, we present a distributed bisection method to obtain the optimal Lagrange multiplier λ^* for the problem (7)-(9). This is done based on the average consensus algorithm (1) or (2), with no need for a central decision maker or a leader node. We first propose a distributed algorithm for gathering the aggregate power demand. We then give a distributed solution to the EDP. Finally, we give analysis on convergence and stopping criteria for practical use.

A. Distributed Algorithm for Aggregate Demand

The first step of solving the EDP is to collect the aggregate power demand $P^\star = \sum_{j=1}^{m} P_j$. However, it is a difficult task to compute $P^\star$ directly in a fully distributed fashion (relatively negative results is given in [18]). Instead, our algorithm is to make every node i (generation bus) in V_n share a common value $y^* = P^\star / n$. As we will show in the next subsection, it turns out that such common value y^* will be sufficient to solve the EDP.

In graph $G_m = (V_m, E_m)$, using (3), an associated doubly stochastic matrix $Q \in \mathbb{R}^{m \times m}$ is given by:

$$[Q]_{ij} = q_{ij} = \begin{cases} \dfrac{1}{\max(d_{m,i}, d_{m,j}) + 1}, & \text{if } j \in N_{m,i}, \\ 1 - \sum\limits_{j \in N_{m,i}} q_{ij}, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where the subscript m indicates that the parameters are defined with regard to graph G_m.

For every node $i \in V_m$, we establish two variables $p_i(t)$ and $s_i(t)$, respectively initialized by $p_i(0) = P_i$, and

$$s_i(0) = \begin{cases} 1, & i = 1, 2, ..., n, \\ 0, & i = n+1, n+2, ...m. \end{cases}$$

We then run the following average consensus algorithms simultaneously until convergence:

$$p_i(t+1) = q_{ii}p_i(t) + \sum_{j\in N_{m,i}} q_{ij}p_j(t), \tag{19}$$

$$s_i(t+1) = q_{ii}s_i(t) + \sum_{j\in N_{m,i}} q_{ij}s_j(t). \tag{20}$$

Defining $p^* = \lim_{t\to\infty} p_i(t)$ and $s^* = \lim_{t\to\infty} s_i(t)$, we have:

$$p^* = P^\star/m, \ s^* = n/m.$$

For every node $i \in V_n$, we have:

$$y^* = \frac{p^*}{s^*} = P^\star/n. \tag{21}$$

B. Distributed Bisection Algorithm for EDP

We now propose a distributed bisection algorithm for the EDP, which involves graph G_n only. From (19)-(21), we assume that each node $i \in V_n$ gets $y^\star = P^\star/n$.

Let $k \geqslant 0$ denote the bisection step index for the bisection method. We let each node establish two variables $\lambda^-(k)$ and $\lambda^+(k)$, representing the lower and upper bounds of the Lagrange multiplier. Their initial values are given such that $\lambda^-(0)$ is sufficiently small and $\lambda^+(0)$ is sufficiently large, or alternatively by

$$\lambda^-(0) = \min_{i\in V_n} f_i(\underline{x}_i),$$

$$\lambda^+(0) = \max_{i\in V_n} f_i(\bar{x}_i),$$

which can be achieved by the *minimum/maximum consensus* algorithms, which are fully distributed for connected network [6].

Define a variable $\lambda(k)$, which acts as the approximation of the Lagrange multiplier, as

$$\lambda(k) = (\lambda^+(k) + \lambda^-(k))/2. \tag{22}$$

Each node $i \in V_n$ then takes

$$x_i(k) = -g_i(\lambda(k)). \tag{23}$$

In graph $G_n = (V_n, E_n)$, an associated doubly stochastic matrix $R \in \mathbb{R}^{n\times n}$ is given by:

$$[R]_{ij} = r_{ij} = \begin{cases} \dfrac{1}{\max(d_{n,i}, d_{n,j}) + 1} & \text{if } j \in N_{n,i}, \\ 1 - \displaystyle\sum_{j\in N_{n,i}} r_{ij} & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \tag{24}$$

where the subscript n indicates that the parameters are defined with regard to graph G_n.

Each node establishes a local variable $z_i(t)$ initialized by $z_i(0) = x_i(k)$, and runs the following average consensus algorithm until convergence:

$$z_i(t+1) = r_{ii}z_i(t) + \sum_{j\in N_{n,i}} r_{ij}z_j(t). \tag{25}$$

Defining $z^* = \lim_{t\to\infty} z_i(t)$, we have

$$z^* = \left(\sum_{i=1}^{n} x_i(k)\right)/n. \tag{26}$$

Every node updates $\lambda^+(k+1)$ and $\lambda^-(k+1)$ by comparing y^* and z^* as follows:

$$\begin{cases} \lambda^+(k+1) = \lambda(k), \ \lambda^-(k+1) = \lambda^-(k) & \text{for } z^* > y^*, \\ \lambda^+(k+1) = \lambda^+(k), \ \lambda^-(k+1) = \lambda(k) & \text{for } z^* \leqslant y^*. \end{cases} \tag{27}$$

It is clear from (22) and (27) that $\lambda^\star = \lim_{k\to\infty} \lambda(k)$ exists and that each node obtains its local optimal solution from (23), i.e.,

$$x_i^* = -g_i(\lambda^*), \ \forall i \in V_n.$$

For clarity, we summarize the distributed bisection method in Algorithm 1. The convergence property of Algorithm 1 is formally stated below.

Algorithm 1 Distributed Bisection Method for EDP

Input: P_i: local power demand, $\forall i \in V_m$;
Output: x_i^*: power assignment, $\forall i \in V_n$;
Each node gets y^* from (19)-(21);
Initialization $\lambda^-(0)$ and $\lambda^+(0)$;
for $k = 0, 1, 2, \ldots$ **do**
  Each node computes $\lambda(k) = \frac{1}{2}(\lambda^-(k) + \lambda^+(k))$;
  Each node computes $x_i(k) = -g_i(\lambda(k))$;
  Each node runs (25) to get z^*;
  Each node computes $\lambda^+(k+1)$ and $\lambda^-(k+1)$ according to (27);
end for

Theorem 2: Under the assumption that the EDP problem (7)-(9) is feasible, Algorithm 1 converges to the unique optimal solution as $k \to \infty$.

Proof: For all $i \in V_n$, the function $g_i(\lambda)$ is monotonically decreasing with respect to λ. Therefore $-g_i(\lambda)$ is monotonically increasing. Especially, $-g_i(\lambda)$ is strictly increasing with respect to λ for

$$f_i(\underline{x}_i) \leqslant \lambda \leqslant f_i(\bar{x}_i).$$

Define

$$\bar{\lambda} = \max_{i\in V_n} f_i(\bar{x}_i), \ \underline{\lambda} = \min_{i\in V_n} f_i(\underline{x}_i).$$

Since the problem is feasible, the optimal Lagrange multiplier must satisfy

$$\underline{\lambda} \leqslant \lambda \leqslant \bar{\lambda}.$$

Therefore, $-\sum_{i=1}^n g_i(\lambda)$ is strictly increasing with respect to $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. Thus, for every node i, $\left(\sum_{j=1}^n x_j(k)\right)/n$ is strictly increasing with respect to $\lambda(k) \in [\underline{\lambda}, \bar{\lambda}]$. Therefore, Algorithm 1 converges. Moreover, since the optimal solution is unique, Algorithm 1 converges to the unique one. ■

Remark 1: The bisection algorithm above does not need a central information collector to compute $P^\star$, i.e., the total power demand $P^\star$ is not needed explicitly for solving the EDP. The equality constraint (9) is gradually satisfied through the bisection iterations.

Remark 2: The algorithm above is fully distributed due to the following properties. Information exchange between nodes occurs only when running average consensus. All the computations are performed locally. Also, each node only requires the knowledge of local parameters α_i and β_i, without the need to know other nodes' parameters.

Remark 3: Besides getting around the difficulty of directly obtaining $P^\star$ in a fully distributed fashion, another benefit of procedures (19)-(21) is that it spares the nodes representing buses with pure loads the burden of communication and computation after y^* is computed using (19)-(21) because the rest part of Algorithm 1 only involves the generation buses, i.e., the bisection steps are performed in G_n only.

C. Convergence and Stopping Criteria

Now we give analysis on convergence and stopping criteria for practical use. Two stopping criteria are needed, one for the average consensus iterations at each bisection step, and the other for the bisection iterations.

As shown in Algorithm 1, the average consensus iterations (25) are performed in each bisection step. Iteration (1) solves the average consensus asymptotically, but it is impossible to run it for infinite time. In most scenarios, when implementing the average consensus protocol (1), a stopping criterion may be setting a fixed number of iterations t^* such that

$$\frac{||\pi(t)-\pi^*\underline{1}||_2}{||\pi(0)-\pi^*\underline{1}||_2} < \epsilon, \forall t \geqslant t^*, \tag{28}$$

i.e., for a specific threshold $\epsilon > 0$, which is small enough. Accordingly there exists t^* such that when $t = t^*$, consensus is reached approximately in a practical sense. However, such stopping criterion is not suitable for consensus iteration (25) in Algorithm 1. In fact, at each bisection step, we only need to run iteration (25) till every node reaches agreement on the direction in which they shall bisect their incremental cost intervals.

Now we initialize $z_i(0) = x_i(k) - y^*$, and then run iteration (25). One can easily verify that (27) is equivalent to

$$\begin{cases} \lambda^+(k+1) = \lambda(k),\ \lambda^-(k+1) = \lambda^-(k) & \text{for } z^* > 0, \\ \lambda^+(k+1) = \lambda^+(k),\ \lambda^-(k+1) = \lambda(k) & \text{for } z^* \leqslant 0. \end{cases} \tag{29}$$

Defining $\rho_i(t) = \operatorname{sgn}(z_i(t))$, we say that iteration (25) reaches *sign consensus*, if $\rho_i(t) = \rho_j(t), \forall i, j$. It's easily verified that for iteration (25), sign consensus can always be reached in finite time, i.e., there exists $t^\dagger \geqslant 0$, such that for all $t > t^\dagger$, $\rho_i(t) = \rho_j(t), \forall i, j$, unless $z^* = 0$.

Remark 4: The reason why the commonly used stopping criterion for average consensus, i,e., setting some fixed number of iteration steps, is unsuitable for our bisection algorithm is that when average consensus is assumed to be reached for $t > t^*$, it does not imply that sign consensus is reached. This could happen when $z^* \approx 0$ and the iteration stops at t^* with some $z_i(t^*) < 0$ and some $z_i(t^*) > 0$. Consequently, different decisions on bisection are made.

Now we give a distributed method to judge whether sign consensus is reached or not, which is based on maximum/minimum consensus. We assume each node knows the diameter of G_n, denoted by D_n. At bisection step $k+1$, each node i establishes three variables, $\underline{\rho}_i$, $\bar{\rho}_i$, and $\rho_i^\dagger$, where

$$\rho_i^\dagger(t) = \begin{cases} 1 & z_i(t) > 0, \\ 0 & z_i(t) \leqslant 0. \end{cases} \tag{30}$$

At $t = 1$, we set $\underline{\rho}_i(1) = \rho_i^\dagger(1)$, $\bar{\rho}_i(1) = \rho_i^\dagger(1)$. Then for $t \geqslant 1$, at step $t+1$, we update $\underline{\rho}_i(t+1)$ and $\bar{\rho}_i(t+1)$ by:

$$\underline{\rho}_i(t+1) = \begin{cases} \min\limits_{j \in \{N_i, i\}} \underline{\rho}_i(t), & \text{if } \rho_i^\dagger(t+1) = \rho_i^\dagger(t), \\ \rho_i^\dagger(t+1), & \text{otherwise,} \end{cases} \tag{31}$$

$$\bar{\rho}_i(t+1) = \begin{cases} \max\limits_{j \in \{N_i, i\}} \bar{\rho}_i(t), & \text{if } \rho_i^\dagger(t+1) = \rho_i^\dagger(t), \\ \rho_i^\dagger(t+1), & \text{otherwise.} \end{cases} \tag{32}$$

The stopping criterion for iteration (25) is that for any node i, if there is some $t^\dagger \geqslant 1$ such that

$$\underline{\rho}_i(t) = \bar{\rho}_i(t),\ t = t^\dagger, t^\dagger+1, ..., t^\dagger + D_n, \forall i,$$

then sign consensus is reached. Besides, $\underline{\rho}_i(t) = \bar{\rho}_i(t) = 1$ implies that $\lambda^+(k+1) = \lambda(k)$ and $\lambda^-(k+1) = \lambda^-(k)$, while $\underline{\rho}_i(t) = \bar{\rho}_i(t) = 0$ implies $\lambda^+(k+1) = \lambda^+(k)$ and $\lambda^-(k+1) = \lambda(k)$.

Remark 5: A special situation is that sign consensus is already reached without any step of iteration (25), e.g., for all i, $z_i(0) > 0$. Since there are no central decision makers, we still need D_n steps of iteration. However, it is often the case that $D_n \ll t^*$, thus another benefit of using sign consensus is the reduction of iteration steps.

As for bisection, a stopping criterion can be established by either setting a fixed number of bisections K or using

$$|\lambda(k) - \lambda^*| \leqslant \varepsilon/2, \tag{33}$$

for some sufficiently small $\varepsilon > 0$. Since λ^* is not available when solving EDP, an alternative can be

$$|\lambda^+(k) - \lambda^-(k)| \leqslant \varepsilon, \tag{34}$$

which can be easily achieved in a distributed fashion, provided sign consensus is always reached in each bisection step.

IV. SIMULATION

In this section, we show the performance of the distributed bisection algorithm using numerical experiment based on the IEEE 14-bus system [19]. The generator parameters are adopted from [14], as shown in Table I. We set $\underline{x}_i = 10MW$ for all i. We take $\varepsilon = 0.005$ for the stopping criterion.

In our simulation, generators buses are $\{1, 2, 3, 6, 8\}$, and load buses are $\{2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14\}$. Note that the power transmission grid is not necessarily the same with the information communication network, so we do not

TABLE I
GENERATOR PARAMETERS (MU = MONEY UNIT)

Generator	Bus	α_i (MU/MW^2)	β_i (MU/MW)	$\bar{x}_i(MW)$
1	1	-25	12.5	80
2	2	-50	16.67	90
3	3	-57.14	14.29	70
4	6	-66.67	16.67	70
5	8	-31.25	12.5	80

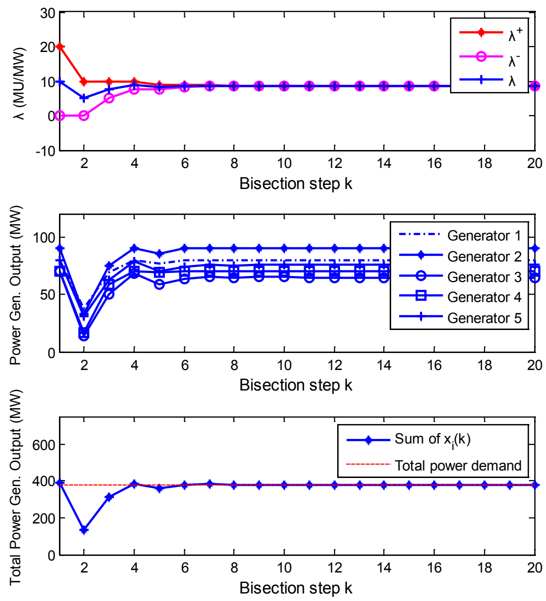

Fig. 1. Results of EDP by distributed bisection algorithm.

assign a node to bus 7. Define two corresponding node sets $V_m = \{1, 2, ..., 6, 8, 9, ..., 14\}$, $V_n = \{1, 2, 3, 6, 8\}$. For G_m and G_n, the edge sets E_m and E_n are properly chosen to set up two connected undirected graphs with self-loops.

The local power demands are: $P_1 = 0MW$, $P_2 = 9MW$, $P_3 = 56MW$, $P_4 = 55MW$, $P_5 = 27MW$, $P_6 = 46MW$, $P_8 = 0MW$, $P_9 = 8MW$, $P_{10} = 24MW$, $P_{11} = 53MW$, $P_{12} = 46MW$, $P_{13} = 16MW$ and $P_{14} = 40MW$. The total demand $P^\star = \sum_{i\in V_m} P_i = 380MW$, which is not known to the individual nodes.

We set $\lambda^+(0) = 20MU/MW$ and $\lambda^-(0) = 0MU/MW$. The result is shown in Fig. 1. The upper subplot of Fig. 1 shows the evolution of $\lambda(k)$, the middle subplot shows the evolution of each $x_i(k)$, and the lower subplot shows the evolution of $\sum_{i\in V_n} x_i$. We artificially set the bisection step to be 20, while the stopping criterion is already satisfied at $k = 13$. Taking the results at $k = 13$ to be the optimal solution, we have $x_1^* = 80.00MW$, $x_2^* = 90.00MW$, $x_3^* = 64.48MW$, $x_6^* = 70.00MW$, $x_8^* = 75.35MW$, and $\sum_{i\in V_n} x_i^* = 380.03MW$. The optimal incremental cost $\lambda^* = 8.5278MU/MW$, and the optimal solution x^* stays within the capacity constraints, where x_i^* of generator 1, 2 and 4 take their upper bounds of capacity constraints, respectively.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have proposed a distributed bisection method based on average consensus to solve the EDP with quadratic cost functions. The algorithm is fully distributed, with no need for a master node or leader. Also, each node only requires its local parameters, without any global information. Convergence of our algorithm is proved, and by simulations we show the performance of the algorithm. Future work includes the extension of our approach to the EDP with general cost functions (i.e., non-quadratic).

REFERENCES

[1] A. J. Wood and B. F. Wollenberg, *Power Generation, Operation, and Control*. John Wiley & Sons, 1996.
[2] C.-L. Chen and N. Chen, "Direct search method for solving economic dispatch problem considering transmission capacity constraints," *IEEE Trans. Power Syst.*, vol. 16, no. 4, pp. 764–769, 2001.
[3] J. Tippayachai, W. Ongsakul, and I. Ngamroo, "Parallel micro genetic algorithm for constrained economic dispatch," *IEEE Trans. Power Syst.*, vol. 17, no. 3, pp. 790–797, 2002.
[4] L. Bakule, "Decentralized control: an overview," *Annual Reviews in Control*, vol. 32, no. 1, pp. 87–98, 2008.
[5] A. Ipakchi and F. Albuyeh, "Grid of the future," *IEEE Power Energy Mag.*, vol. 7, no. 2, pp. 52–62, 2009.
[6] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.
[7] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, 2010.
[8] Z. Lin, *Distributed Control and Analysis of Coupled Cell Systems*. VDM Publishing, 2008.
[9] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems & Control Letters*, vol. 59, no. 3, pp. 209–217, 2010.
[10] M. Porfiri, D. G. Roberson, and D. J. Stilwell, "Tracking and formation control of multiple autonomous agents: a two-level consensus approach," *Automatica*, vol. 43, no. 8, pp. 1318–1328, 2007.
[11] Z. Zhang and M.-Y. Chow, "Incremental cost consensus algorithm in a smart grid environment," in *Proc. IEEE Power and Energy Society General Meeting*, San Diego, CA, Jul.24–29, 2011, pp. 1–6.
[12] Z. Zhang, X. Ying, and M.-Y. Chow, "Decentralizing the economic dispatch problem using a two-level incremental cost consensus algorithm in a smart grid environment," in *Proc. IEEE North American Power Symposium (NAPS)*, Boston, MA, Aug.4–6, 2011, pp. 1–7.
[13] A. D. Dominguez-Garcia, S. T. Cady, and C. N. Hadjicostis, "Decentralized optimal dispatch of distributed energy resources," in *Proc. IEEE Conference on Decision and Control*, Maui, Hawaii, Dec.10–13, 2012, pp. 3688–3693.
[14] S. Kar and G. Hug, "Distributed robust economic dispatch in power systems: A consensus + innovations approach," in *Proc. IEEE Power and Energy Society General Meeting*, San Diego, CA, Jul.22–26, 2012, pp. 1–8.
[15] S. Yang, S. Tan, and J. X. Xu, "Consensus based approach for economic dispatch problem in a smart grid," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4416–4426, 2013.
[16] A. Bemporad, M. Heemels, and M. Johansson, *Networked Control Systems*. Springer, 2010, vol. 406.
[17] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge Univ. Press, 2004.
[18] J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Distributed anonymous discrete function computation," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2276–2289, 2011.
[19] "Power system test case archive." [Online]. Available: http://www.ee.washington.edu/research/pstca/