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Infinite Horizon LQG Control with Fixed-Rate Quantization

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***Abstract*— In this paper, we consider infinite-horizon linear quadratic Gaussian (LQG) control systems with the constraint that the measurement signal is quantized by a fixed-rate quantizer before going into the controller. It has been shown recently that only weak separation principle holds for the LQG control system with communication channels. It has also been shown that the separation principle holds approximately for quantized LQG control in the finite-horizon setting under the assumption of high-resolution quantization. We propose an adaptive fixed-rate quantizer for feedback control design to achieve the mean-square stability and good LQG performance, where the long-term average cost is divided into two parts: the first part depends on the classical LQG cost, and the second part depends on the distortion of the quantizer.**

I. Introduction

There has been extensive research on quantized feedback control systems; see, e.g., [1], [2], [3] and a survey paper [4]. Most pertinent research to this paper is the problem of linear quadratic Gaussian control (LQG) with quantization data, the so-called quantized LQG (QLQG) problem [2], [5], [6], [7], [8]. The core question to ask is whether the classical separation principle for LQG still holds with the quantization constraint. It turns out that a *weak separation principle* holds which states that the QLQG problem can be separated into a full-state control design problem and the so-called *quantized state estimation* problem [6], but further separation between quantization and state estimation is in general not possible. However, for the finite-horizon QLQG problem, an linear predictive coding (LPC) scheme has been proposed for quantization in [6] to show that full separation of control, estimation and quantization can indeed be achieved approximately under a high resolution quantization assumption and a mild rank condition.

One of the core issues on QLQG design is how to quantize temporally correlated signals. There is abundant literature about quantization for autoregressive sources [9], [10], [11], [12]. In [13], Hui and Neuhoff analyze the asymptotic property of optimal fixed-rate uniform scalar quantization for a class of memoryless distributions. Explicit asymptotic formulas are presented for the distortion and optimal quantizer length approximation when the source is Gamma distribution. However the results can not be used directly to QLQG control since they are based on a key assumption that the system is stable and there is no feedback. In control problems, we also want to consider unstable systems and quantized feedback. A special difficulty with quantized feedback is that the signal to be quantized will become nonlinear due to quantization and feedback of past samples.

The method in [6] can not be directly extended to the infinite horizon case. The key difficulty is that a memoryless fixed-rate quantizer can not guarantee closed-loop stability, let alone the performance. This is caused by the saturation effect of the finite-support of the quantizer, as shown in [3]. To get around this difficulty, a simple LPC scheme with *adaptive* fixed-rate quantizer has been proposed for infinite-horizon quantized LQG control of *scalar* systems [20].

In this paper, we study the infinite-horizon QLQG problem for single-input, single-output systems with a *higher order*. For the high order case, the key difficulty is that the weighting matrix in the LPC-based quantizer is not full rank and the quantized state may be large although the distortion of the quantized output is small. We propose a new LPC-based quantizer modified from [20]. We show that the mean-square stability of the quantized feedback system is achieved, and the average distortion is in the order of $N^{-2}\log_2 N$, where $N = 2^R$, and R is the quantization bit rate (per sample).

The rest of the paper is organized as follows: Section II formally formulates the quantized LQG problem; Section III introduces a weak separation principle and studies the relationship between QLQG and quantized state estimation; Section IV is devoted to the quantized LQG problem for SISO systems; and Section V concludes the paper.

II. Problem Statement

Consider a discrete-time system described as follows:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $y_t \in \mathbb{R}^p$ is the measured output, $w_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^p$ are independent Gaussian random distributions with zero mean and covariances $W_t > 0$ and $V_t > 0$, respectively, and the initial state x_0 is also assumed to be an independent zero-mean Gaussian distribution with covariance Σ_0. In the sequel, we denote $z^t = \{z_0, z_1, \ldots, z_t\}$.

The cost function is defined as

$$J = \lim_{T\to\infty} \sup \frac{1}{T}\mathcal{E}\sum_{t=0}^{T-1}(x_t'Qx_t + 2u_t'Hx_t + u_t'Su_t) \tag{2}$$

where $\mathcal{E}$ is the expectation operator, $S > 0, Q \geq 0$ and $Q - HS^{-1}H' \geq 0$.

The problem is to design an observer based controller, and an R-bit uniform quantizer to minimize the cost J.

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It is well known that the optimal state feedback gain is given by

$$K = -(S + B^{'}PB)^{-1}(B^{'}PA + H), \tag{3}$$

where P is the solution of the Ricatti equation

$$P = Q + A^{'}PA - (B^{'}PA + H)'(S + B^{'}PB)^{-1}(B^{'}PA + H). \tag{4}$$

Let the optimal observer based controller is given by $u_t = K\hat{x}_t^q$, where K is the feedback gain matrix, and $\hat{x}_t^q$ is the quantized value of the estimated state from the following Kalman filter

$$\begin{aligned} \hat{x}_t &= \hat{x}_{t|t-1} + \Gamma(y_t - C\hat{x}_{t|t-1}) \\ \hat{x}_{t+1|t} &= A\hat{x}_t + Bu_t. \end{aligned} \tag{5}$$

where $\Gamma = EC'(CEC' + V)^{-1}$ and E is the solution of the following Ricatti equation

$$E = AEA' - AEC'(CEC^T + V)^{-1}CEA^T + W. \tag{6}$$

III. WEAK SEPARATION PRINCIPLE AND QUANTIZED STATE ESTIMATION

The first and most pertinent result is the so-called *weak separation principle*. This result was known by Fischer (1982) [8], although his interpretation that this result leads to separation of estimation and quantization is incorrect (see [6] for detailed comments). The weak separation principle (stated below) suggests that optimal quantized LQG control can be achieved by first constructing the optimal estimate $\hat{x}_t$, which is independent of the cost function, then quantizing it and the optimal control is given by $K\hat{x}_t^q$.

Lemma 1: Consider the system (1), the cost function (2), the quantized feedback controller $K\hat{x}_t^q$, with K given by (3) and $\hat{x}_t$ given by (5), and the R-bit fixed-rate quantization. Then, the quantized LQG controller is optimal if $\hat{x}_t^q$ is obtained by quantizer that minimizes the following distortion function

$$D = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathcal{E}[(\hat{x}_t - \hat{x}_t^q)^{'}\Omega(\hat{x}_t - \hat{x}_t^q)] \tag{7}$$

where $\Omega = K^{'}(S + B^{'}PB)K$. The corresponding cost function is given by

$$J = J_{LQG} + \min D = tr(PW) + tr(\Omega E) + \min D.$$

The implication of the weak separation principle is that the QLQG problem essentially becomes a quantized state estimation problem, as stated below [6]. This problem is depicted in Figure 1. The system we consider is given by

$$\begin{aligned} x_{t+1} &= Ax_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{8}$$

with $x_0, \{w_t\}, \{v_t\}$ being independent Gaussian random variables as before. Let $\hat{x}_t$ be the optimal (Kalman) estimate of x_t and consider $z_t = K_t\hat{x}_t$ for some given K_t. The task of quantized state estimation is to encode $\{y_t\}$ (or $\{z_t\}$ indirectly) with fixed bit rate R to minimize the following distortion function

$$D = \sum_{t=0}^{T-1} \mathcal{E}[(z_t - z_t^q)'\Omega_t(z_t - z_t^q)] \tag{9}$$

for some given Ω_t, where z_t^q is the quantized z_t.

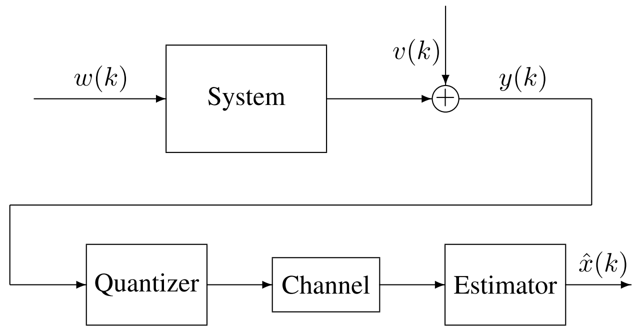

Fig. 1. Quantized State Estimation

The quantized state estimation problem above is similar to the traditional vector quantization problem in the sense that both problems consider quantizing a sequence of input signal $\{z_t\}$ to minimize some distortion function. However, in our problem the quantizer has the additional constraint of causality. That is, the encoding-decoding pair at time t is not allowed to "see" the "future" values of z_τ and a_τ, $\tau > t$.

The quantized state estimation problem can be viewed as a *generalized vector quantization problem*. Recall the standard fixed-rate vector quantization problem as follows: Given a vector of random variables $z \in \mathbb{R}^m$, $m \geq 1$, with probability density function f, a distortion measure $d(\cdot,\cdot)$, the standard vector quantization problem is to design an N-level quantizer to minimize $\mathcal{E}[d(z,\zeta)]$, where ζ is the quantized z. Quadratic distortion measures are most commonly used [14]. The quantizer has two parts: encoder and decoder. The encoder decomposes $\mathbb{R}^m$, the support of z, into N disjoint sets $I(k)$, $0 \leq k < N$ and maps z to k if $z \in I(k)$. The decoder maps each encoded value k to a quantized value $\zeta(k)$.

An optimal vector quantizer satisfies the well-known Llyod's conditions [16], [14] which state that

- For a given encoding partition $\{I(k)\}$ of $\mathbb{R}^m$, the optimal choice for decoding is given by
$$\zeta(k) = \mathcal{E}[z|z \in I(k)] \tag{10}$$
- For a given set of decoded values $\{\zeta(k)\}$, the optimal encoding partitions are such that $z \in I_k$ if and only if $k = \arg\min_i d(z, \zeta(i))$.

In many cases, the optimal quantizer can be obtained by iterating the two steps above [14], [17], [18], [19].

Returning to the quantized state estimation problem, we first note that the partition $\{I_t(k)\}$ of $\mathbb{R}^m$ at time t is conditioned on previous partitions. At $t = 0$, there are $L = 2^R$ partitions of $\mathbb{R}^m$. For each of these partitions, there will be L partitions of $\mathbb{R}^m$ at $t = 1$ and so on.

To make the dependencies of the partitions explicit, we denote the partitions $I_t(k)$ and the decoded values $\zeta_t(k)$ by $I_t(k_0, k_1, \cdots, k_t)$ and $\zeta_t(k_0, k_1, \cdots, k_t)$, respectively, where k_i is the corresponding partition index at time i. In general, there are L^{t+1} partitions at any t. All together, there will be $L_{\text{total}} = \sum_{t=1}^{T} L^t$ partitions to optimize. A generalized version of Llyod's conditions still holds:

- Given a sequence of encoding partitions $\{I_t(k_0, k_1, \ldots, k_t)\}$, $0 \le k_i < L, 0 \le i \le t, 0 \le t < T$, the optimal choice for decoding is given by

$$\zeta_t(k_0, k_1, \cdots, k_t) = \mathcal{E}[z_t | z_t \in I_t(k_0, k_1, \ldots, k_t)]$$

- Given a sequence of decoded values $\{\zeta_t(k_0, k_1, \cdots, k_t)\}, 0 \le k_i < L, 0 \le i \le t, 0 \le t < T$, the optimal encoding partitions are such that $z_t \in I_t(k_0, k_1, \ldots, k_t)$ if and only if

$$k_t = \arg\min_i d(z_t, \zeta_t(k_0, k_1, \cdots, k_{t-1}, i)).$$

Like the standard vector quantization case, the generalized Llyod's conditions can be used to iteratively optimize the quantizer. However, this is manageable only for very small R and T, but certainly not practical otherwise.

To get around the computational complexity as mentioned above, an LPC-based quantization scheme is proposed in [6]:

$$\begin{aligned} \hat{x}_t^q &= (A + BK)\hat{x}_{t-1}^q + \varepsilon_t^q \\ \varepsilon_t &= \Gamma(y_t - c\hat{x}_{t|t-1}) + A(\hat{x}_{t-1} - \hat{x}_{t-1}^q) \end{aligned}$$

$$\varepsilon_t^q = Q(\varepsilon_t) \tag{11}$$

with $\hat{x}_{-1}^q = 0$, where $Q(\cdot)$ is a memoryless quantizer. Under high-resolution quantization and a mild rank condition, it is shown that the complete separation principle holds for finite-horizon LQG control system, which means ε_t^q can be quantized by a memoryless quantizer, and the controller and estimator are the same as in Lemma 1. However, the quantization scheme (11) with memoryless quantizer can not guarantee stability if A is unstable, let alone the LQG performance. Hence we have to choose another type of quantizer, such that the quantized feedback system is stable and maintains a good LQG performance.

IV. Quantized LQG Control for SISO Systems

We now consider the aforementioned LPC based approach to QLQG for SISO systems (1) with an infinite horizon. That is, the control input u_t and the output y_t are all scalar signals. Our focus is on both stability and performance of the closed-loop system. More specifically, it was shown in [6] that for a finite horizon, the quantization distortion D decreases as the bit rate R increases in the way that D is in the order of $R2^{-2R}$. But this is for a fixed horizon T. For an infinite horizon ($T \to \infty$), we need to find a quantization scheme such that the quantization distortion remains bounded as $T \to \infty$. It is shown in [3] that memoryless fixed-rate quantizers cannot guarantee stability. To achieve our objective, an adaptive quantizer is needed.

A. Fixed-rate uniform scalar quantization

Let us consider a fixed-rate uniform scalar quantizer, which is the simplest and most common form of quantizer, and of which the asymptotic behavior has been understood recently for a class of source densities with infinite support, including Gaussian distributions[13]. In [13], explicit asymptotic formulas are presented for the distortion and optimal quantizer length approximation for Gamma distributions.

Next we introduce some basic concepts on uniform quantization. Consider an $N = 2^R$ level symmetric uniform scalar quantizer with step size Δ. Let $(-L, L]$ be the support of this quantizer, where $L = N\Delta/2$ is called the quantization length. Define $y_i = -N\Delta/2 + (i - 1/2)\Delta$ and $S_i = (y_i - \Delta/2, y_i + \Delta/2]$ for $i = 1, \ldots, N$. The quantizer is defined as

$$Q_\Delta(x) = \begin{cases} y_0, & \text{if } x \le -L, \\ y_i, & \text{if } x \in S_i, \quad i = 1, 2, \ldots, N, \\ y_N + \Delta, & \text{if } x > L. \end{cases}$$

Then the MSE granular and overload distortions are defined as follows:

$$\begin{aligned} D^{gran} &= \sum_{i=1}^{N} \int_{S_i} (x - y_i)^2 p(x) dx \\ D^{over} &= 2\int_L^\infty (x - y_N)^2 p(x) dx, \end{aligned}$$

where $p(x)$ is the source density function, and the total quantisation distortion is given by

$$D^{total} = \mathcal{E}[(x - Q_\Delta(x))^2] = D^{gran} + D^{over}$$

The following three results that are established in [13] will be used in this paper. In the rest of the paper, $X \approx Y$ means that $X/Y \to 1$ as N (or R) $\to \infty$.

Lemma 2: For a Gaussian distribution with zero mean and variance σ, the optimal quantization length for the uniform fixed-rate quantizer is given by $L \approx 2\sigma\sqrt{\ln N}$. Moreover, the distortions satisfy

$$\lim_{N\to\infty} \frac{D^{over}}{D^{gran}} = 0 \tag{12}$$

$$D^{total} \approx D^{gran} \approx \frac{\Delta_N^2}{12} = \frac{L^2}{3N^2}. \tag{13}$$

Lemma 3: For any distribution whatsoever, we have

$$\lim_{N\to\infty} \frac{D^{gran}}{\Delta_N^2/12} = 1.$$

Lemma 4: For a Gaussian random variable $x \sim \mathcal{N}(0, \sigma^2)$, define $W_{\sigma^2}(y)$ as

$$W_{\sigma^2}(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_y^\infty (x - y)^2 e^{-\frac{x^2}{2\sigma^2}} dx.$$

Then

$$W_{\sigma^2}(y) = \frac{2\sigma^5}{\sqrt{2\pi}} y^{-3} e^{-\frac{y^2}{2\sigma^2}} (1 + o(y)), \tag{14}$$

$$\lim_{N\to\infty} \frac{D_L^{over}}{2W_{\sigma^2}(L)} = 1, \tag{15}$$

where $o(y) \to 0$ as $y \to \infty$.

B. LPC-based approach to QLQG

Define $\eta_t = Ax_t - A\hat{x}_t$. Combining the system (1), the controller $u_t = K\hat{x}_t^q$, the state estimator (5) and the quantizer (11) together, we obtain the following equations

$$\begin{aligned}\eta_{t+1} &= (A - A\Gamma C)\eta_t + \Gamma C w_t + \Gamma v_{t+1} + w_t \quad (16)\\ \hat{x}_{t+1} &= (A+BK)\hat{x}_t - BK(\varepsilon_t - \varepsilon_t^q)\\ &\quad + \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} \quad (17)\\ \varepsilon_{t+1} &= \Gamma C\eta_t + \Gamma C w_t + \Gamma v_{t+1} + A(\varepsilon_t - \varepsilon_t^q). \quad (18)\end{aligned}$$

It follows from (16) that η_t is Gaussian for any t if the initial state x_0, w_t and v_t are all Gaussian. Note that $A - A\Gamma C$ is stable. Then $\Sigma_{\eta_t} = \mathcal{E}(\eta_t\eta_t^{'})$ is well defined for all $t \geq 0$ and we have

$$\begin{aligned}\Sigma_\eta &= (A - A\Gamma C)\Sigma_\eta(A - A\Gamma C)^{'}\\ &\quad + (\Gamma C + I)W(C^{'}\Gamma' + I) + \Gamma V\Gamma^{'},\end{aligned}$$

where $\Sigma_\eta = \lim_{t\to\infty}\Sigma_{\eta_t}$. Denote $z_{t+1} = C\eta_t + Cw_t + v_{t+1}$, then z_{t+1} is Gaussian with zero mean and variance σ_{t+1}^2, where

$$\sigma_{t+1}^2 = C(\Sigma_{\eta_t} + W)C' + V. \quad (19)$$

Denote σ_z^2 as

$$\sigma_z^2 := \lim_{t\to\infty}\sigma_t^2 = C(\Sigma_\eta + W)C' + V. \quad (20)$$

From now on, we consider the following system

$$\begin{aligned}\varepsilon_{t+1} &= \Gamma z_{t+1} + A(\varepsilon_t - \varepsilon_t^q) \quad (21)\\ \xi_{t+1} &= \sqrt{S + B'PB}K\varepsilon_{t+1}. \quad (22)\end{aligned}$$

The quantized LQG control problem becomes to design a quantizer for ε_t given by (21)-(22) such that the distortion

$$D = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathcal{E}[(\varepsilon_t - \varepsilon_t^q)^{'}\Omega(\varepsilon_t - \varepsilon_t^q)] \quad (23)$$

is minimized, where $\Omega = K'(S + B'PB)K$. This is equivalent to designing a quantizer for ξ_{t+1} such that the distortion

$$D = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mathcal{E}[(\xi_t - \xi_t^q)'(\xi_t - \xi_t^q)] \quad (24)$$

is minimized.

When A is stable, a memoryless quantizer can be designed to achieve the optimal distortion.

Theorem 1: Assume that A is stable in the system (21)-(22). Denote

$$\theta = (K\Gamma)^2(S + B^{'}PB). \quad (25)$$

Let the step size Δ_t of the R-bit uniform fixed-rate quantizer ξ_t^q be given by $\Delta_t = 2L_t/N$, where $N = 2^R$ and $L_t = 2\sigma_t\sqrt{\theta\ln N}$. Take

$$\varepsilon_t^q = \theta^{-\frac{1}{2}}\Gamma\xi_t^q. \quad (26)$$

The, for sufficiently large R, the above quantisation scheme achieves optimal performance given by

$$J = J_{LQG} + D^\star, \quad (27)$$

with $D^\star \approx \frac{4\theta\sigma_z^2\ln N}{3N^2}$, where σ_z^2 is defined by (20).

Because A is stable, Theorem 1 can be proved by induction following a similar idea in [6]. (Details are omitted here.)

The quantization scheme (11) with a memoryless quantizer can not guarantee stability of the whole feedback system when A is unstable [3]. For scalar systems where A is a scalar, an adaptive quantization scheme is proposed to guarantee mean square stability and maintain the same LQG performance (27) [20]. However there are typos in the main result of [20]. We provide the correct version in the following.

Theorem 2: For the scalar system (21) with $|A| > 1$, let the step size Δ_{t+1} of the R-bit uniform fixed-rate quantizer ε_t^q be given by $\Delta_{t+1} = 2L_{t+1}/N$, where L_{t+1} is chosen as follows

$$L_{t+1} = \begin{cases} L_{t+1,1} = (4\theta\sigma_{t+1}^2\ln N + A^2L_t^2)^{\frac{1}{2}} \\ \qquad \text{if } |\xi_t - \xi_t^q| > \frac{\Delta_t}{2} \\ L_{t+1,2} = (4\theta\sigma_{t+1}^2\ln N + N^{-2}L_t^2)^{\frac{1}{2}} \\ \qquad \text{if } |\xi_t - \xi_t^q| \leq \frac{\Delta_t}{2}\end{cases} \quad (28)$$

with $L_0 = 2\sigma_0\sqrt{\theta\ln N}$. Assume $N \gg |A|$. Then the distortion satisfies

$$D_{t+1} \approx \frac{4\theta\sigma_{t+1}^2\ln N}{3N^2} + \frac{A^2}{N^2}D_t. \quad (29)$$

As $t \to \infty$, we have

$$D_t \to \frac{1}{1 - A^2/N^2}\frac{4\theta\sigma_z^2\ln N}{3N^2}$$

and the performance J is given by (27) with

$$D^\star \approx \frac{4\theta\sigma_z^2\ln N}{3N^2}.$$

Theorem 3: Consider the scalar system (1), the cost function (2), the quantized feedback controller $K\hat{x}_t^q$ with K given by (3). Let $\hat{x}_t$ and ε_t^q be given by (5) and (11) respectively, where the quantizer ε_t^q is defined by (28). When $N = 2^R \gg |A|$, the whole cost function J is given by (27).

Remark 1: Detail proof of Theorem 2 and 3 can be found in [20]. The basic idea is that one can enlarge the step size of the quantizer once saturation happens. Although this may increase the distortion, the whole distortion is not changed in the sense that the probability of enlarging the step size is very small.

Remark 2: For a scalar system, K, Γ and A are all scalars. There is no difference between the quantization of ε_t and quantization of ξ_t. For a high-order system when A is a matrix, the quantization of ε_t and quantization of ξ_t are different because Ω is not full rank (it is actually rank 1 for SISO systems). A small distortion $\mathcal{E}[(\varepsilon_t - \varepsilon_t^q)'\Omega(\varepsilon_t - \varepsilon_t^q)]$ can not keep each element of $\varepsilon_t - \varepsilon_t^q$ small. Hence we have to deal with the null-space of $K\varepsilon_t$.

For the system (21)-(22), assume that $A \in \mathbb{R}^{n\times n}$ is unstable and $K\Gamma \neq 0$. Let $a = \bar{\lambda}(A'A)$ be the largest singular value and θ be given by (25). Let $L_0 = 2\sigma_0\sqrt{\theta\ln N}$.

For $t \geq 0$, define L_{t+1} as follows: If there exists j with $t-n \leq j \leq t$ such that $\left|\xi_j - \xi_j^q\right| > \frac{\Delta_j}{2}$, then

$$L_{t+1} := L_{t+1,1} = (4\theta\sigma_{t+1}^2 \ln N + a^n L_t^2)^{\frac{1}{2}}; \qquad (30)$$

else,

$$L_{t+1} := L_{t+1,2} = (4\theta\sigma_{t+1}^2 \ln N + N^{-2} L_t^2)^{\frac{1}{2}}. \qquad (31)$$

With the above definition, we have the following result.

Theorem 4: For the system (21)-(22), assume that A is unstable and $K\Gamma \neq 0$. Let the quantizer be designed as in (30)-(31). When $N \gg a^{n/2}$, the saturation probability satisfies

$$\Pr(|\xi_t - \xi_t^q| > \frac{\Delta_t}{2}) \leq N^{-2} \qquad (32)$$

for any $t \geq 0$.

Remark 3: For high order systems, recall that a small distortion

$$\mathcal{E}[(\xi_t - \xi_t^q)'(\xi_t - \xi_t^q)] = \mathcal{E}[(\varepsilon_t - \varepsilon_t^q)'\Omega(\varepsilon_t - \varepsilon_t^q)]$$

can not keep each element of $\varepsilon_t - \varepsilon_t^q$ small. But the quantizer designed in Theorem 4 makes sure that the distortion is bounded by $\beta a^n \frac{\ln N}{N^2}$, where β is a constant not depending on a, n and N. The remaining proof is similar to [20].

Theorem 5: For the system (21)-(22), assume that A is unstable and $K\Gamma \neq 0$. Let the quantizer be designed as (30)-(31). When $N \gg a^{n/2}$, we have

$$D_{t+1}^{gran} \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2} + \frac{a^n}{N^2} D_t^{gran}, \ \forall t \geq 0. \qquad (33)$$

and

$$\lim_{N\to\infty} \frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = 0, \ \forall t \geq 0 \qquad (34)$$

Proof: Assume that $\Pr(L_{t+1} = L_{t+1,1}) = \alpha$. It follows from Lemma 2 that the granular distortion at time $t+1$ is

$$\begin{aligned} D_{t+1}^{gran} &\approx \mathcal{E}(\frac{L_{t+1}^2}{3N^2}) = \Pr(L_{t+1} = L_{t+1,1})\frac{L_{t+1,1}^2}{3N^2} \\ &\quad + \Pr(L_{t+1} = L_{t+1,2})\frac{L_{t+1,2}^2}{3N^2} \\ &= \alpha\mathcal{E}(\frac{4\sigma_{t+1}^2 \ln N + a^n L_t^2}{3N^2}) \\ &\quad + (1-\alpha)\mathcal{E}(\frac{4\sigma_{t+1}^2 \ln N + N^{-2} L_t^2}{3N^2}) \\ &= \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + (\frac{1}{N^2} + \alpha\frac{N^2 a^n - 1}{N^2})\mathcal{E}(\frac{L_t^2}{3N^2}) \end{aligned}$$

Using Theorem 4, we know that

$$\alpha = \Pr(\left|\xi_{t+1} - \xi_{t+1}^q\right| > \frac{\Delta_{t+1,1}}{2}) \leq N^{-2}.$$

Therefore,

$$\begin{aligned} D_{t+1}^{gran} &\leq \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + (\frac{1}{N^2} + \frac{1}{N^2}\frac{N^2 a^N - 1}{N^2})\mathcal{E}(\frac{L_t^2}{3N^2}) \\ &= \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + \frac{(a^n+1)}{N^2}(1+o(N))D_t^{gran} \\ &\approx \frac{4\ln N}{3N^2}\sigma_{t+1}^2 + \frac{(a^n+1)}{N^2}D_t^{gran}. \end{aligned}$$

Under the high rate assumption, we have $\frac{(a^n+1)}{N^2} \ll 1$, hence D_t^{gran} is finite for all t, and

$$\lim_{t\to\infty} D_{t+1}^{gran} = \frac{4\ln N}{3N^2} \lim_{t\to\infty} \sigma_{t+1}^2 = \frac{4\ln N}{3N^2}\sigma_z^2.$$

This completes the proof of (33).

Using Lemma 4, the overload distortion can be computed as

$$\begin{aligned} D_{t+1}^{over} &= \Pr(L_{t+1} = L_{t+1,1}) \\ &\quad \cdot \int_{L_{t+1,1}}^{\infty} (x - L_{t+1,1} + \frac{\Delta_{t+1,1}}{2})^2 h_{t+1}(x)dx \\ &\quad + \Pr(L_{t+1} = L_{t+1,2}) \\ &\quad \cdot \int_{L_{t+1,2}}^{\infty} (x - L_{t+1,2} + \frac{\Delta_{t+1,1}}{2})^2 h_{t+1}(x)dx \\ &\leq N^{-2}N^{-2}(\sigma_{t+1}^2 + a^n\sigma_{\xi_t}^2) \\ &\quad + (1-N^{-2})N^{-2}(\sigma_{t+1}^2 + N^{-2}\sigma_{\xi_t}^2) \\ &= N^{-2}\sigma_{t+1}^2 + (a^n+1)N^{-4}\sigma_{\xi_t}^2(1+o(N)) \\ &\approx N^{-2}\sigma_{t+1}^2 + (a^n+1)N^{-4}(D_t^{gran} + D_t^{over}). \end{aligned}$$

Therefore, we have

$$\frac{D_{t+1}^{over}}{D_{t+1}^{gran}} \approx \frac{\sigma_{t+1}^2 + (a^n+1)N^{-2}(D_t^{gran} + D_t^{over})}{4\sigma_{t+1}^2 \ln N + (a^n+1)D_t^{gran}}. \qquad (35)$$

It follows from Lemma 3 that $\lim_{N\to\infty} \frac{D_0^{over}}{D_0^{gran}} = 0$. By induction of (35), we have

$$\lim_{N\to\infty} \frac{D_{t+1}^{over}}{D_{t+1}^{gran}} = 0$$

for any t. This completes the proof. ■

Theorem 6: For the system (21-22), assume that A is unstable and $K\Gamma \neq 0$. Let the quantizer be designed as (30)-(31). When $N \gg a^{n/2}$, the distortion satisfies

$$D_{t+1} \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2} + \frac{a^n}{N^2} D_t. \qquad (36)$$

The proof is obvious by using Theorem 5 and the fact that $D_{t+1} = D_{t+1}^{over} + D_{t+1}^{gran}$.

Theorem 7: Consider the system (1), the cost function (2), the quantized feedback controller $K\hat{x}_t^q$ with K given by (3). Let $\hat{x}_t$ and $\hat{x}_t^q$ be given by (5) and (11) respectively, where the quantizer ε_t^q is defined by (26), and ξ_t^q is defined in Theorems 4-6. When $N = 2^R \gg a^{n/2}$, the whole cost function J is given by (27).

Proof: When $N = 2^R \gg a^{n/2}$, it follows from (36) that

$$\lim_{t\to\infty} D_t \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2}.$$

Therefore the quantization distortion is given by

$$D_o = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} D_t \approx \frac{4\theta\sigma_{t+1}^2 \ln N}{3N^2}.$$

Using Lemma 1, we know that the cost function J is given by (27). ■

V. Conclusion

This paper has introduced the quadratic LQG control problem. Through a weak separation principle, this problem can be converted to a quantized state estimation problem. We have drawn the connection of the latter problem to vector quantization, as well as their differences. The bottom line is that the optimal solution to the quantized state estimation problem can not be easily separated into state estimation and quantization problems, thus there is Kalman filter-like recursive solution to the quantized LQG problem. We have studied the infinite-horizon quantized LQG control problem for a high-order single-input-single-output system by using a LPC-based approach to quantized state estimation. Under high resolution quantization assumption, an adaptive fixed-rate quantization scheme can indeed achieve stability for the closed-loop system, and its quadratic cost is simply characterized. We have shown that the average quantization distortion has the order of $R2^{-2R}$ under high resolution quantization, which is the same with that of LPC scheme with memoryless quantizer. We comment that although our result assumes high resolution quantization, in practice it is sufficient to have a very modest bit rate; see an example in [20].

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