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Unified Approach to Controller and MMSE Estimator Design with Intermittent Communications

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Abstract— **In this work we consider systems where the measurements and control signals are transmitted over networks that are affected by independent and identically distributed (i.i.d.) packet losses. In this setting, we study two separate problems. The first one is the design of an offline state estimator. The second one is the design of a linear quadratic regulator (LQR). Both designs take the statistics of the network and the system into account, and are optimal in the sense of minimizing given cost functions. It turns out that these two designs, of which each has an associated Riccati equation, are dual. We show that the convergence of the Riccati equation associated to this dual system formulation is a necessary and sufficient condition for stability of the estimator and controller. Finally, we compare the performance of the proposed offline estimator with those of other estimators available from the literature.**

I. INTRODUCTION

In this work we study the design of a controller and a state estimator when the network that connects the controller with the plant, or the sensor with the estimator, is affected by random packet dropouts. We also establish a form of duality that exists between these two designs. Although previous works have used the duality that exists between the Riccati equations that are used for the estimator and controller designs to establish stability when packet dropouts are involved, we will approach the problem slightly differently in this paper. As we explain later in more detail, the advantage of our approach is that our designs require solving a single Riccati equation, rather than a set of coupled ones.

We consider two separate problems. In the first one, we consider a system whose measurements arrive to the observer via a wireless network, as depicted in Fig. 1. In the second problem, we consider a system whose input is connected to the controller through a wireless network, as depicted in Fig. 2. Both networks are affected by random binary packet dropouts, which are independent and identically distributed (i.i.d.). We will look into the design of an optimal offline minimum mean square error (MMSE) state estimator, and an optimal linear quadratic regulator (LQR).

Most of the estimator designs for systems with intermittent communications, available in the literature today are based on two approaches: The first one is the time varying Kalman filter [1]–[3], which provides the optimal estimate, but requires heavy online computations. The second one is a pseudo-offline design method based on the jump linear system (JLS) framework [4]–[6]. In the JLS approach, the estimator is designed to take a finite number of recent packet transmission outcomes into account [6]. This is done by modeling the distribution of the packet arrivals as Markovian. Here a set of estimator gains are found by solving a set of coupled Riccati equations. These gains are then applied depending on the number of recent packet transmission outcomes. In [7], these estimators are called pseudo-steady-state estimators, since, instead of converging to a steady state error covariance, they converge to a set of error covariances, of which each one is paired with a pattern of transmission outcomes.

In the case of a controller design, there exists no online design that is dual to the online Kalman filter. The reason for this is that the controller, which utilizes information on the current state, will minimize a cost function that penalizes future states. However, since the system is causal, we do not know the future communication outcomes; and therefore, do not know which packets will be received. In fact, when computing the control input at time k, the controller does not even know the transmission outcome for the packet that it is going to transmit. It is in this case common to use the statistics of the network in the controller design [8]–[12] or to use a scenario based approach [13]. The JLS framework [5] can in this case be used to incorporate the probabilities for packet dropouts when computing an optimal control input. As for the estimator case, a set of optimal control gains is found by solving a set of coupled Riccati equations which are dual to the set of estimator Riccati equations. When using this framework, one assumes that the current network state is known at the controller when computing the control input. However, the controller can not know the current network state while computing the control input, since the current network state will first be known after transmitting the control input to the system. Therefore the JLS framework is not directly applicable for the system that we consider in the current work.

What we propose in this paper is to establish the duality between true offline controller and true offline estimator designs. This means that the design of the estimator and controller will be solely based on the statistics of the system and the network. The advantages of doing so is that we obtain a time-invariant solution to the estimator and the controller, which is easy to implement. Also, we only obtain a single Riccati equation instead of a set of coupled ones. In

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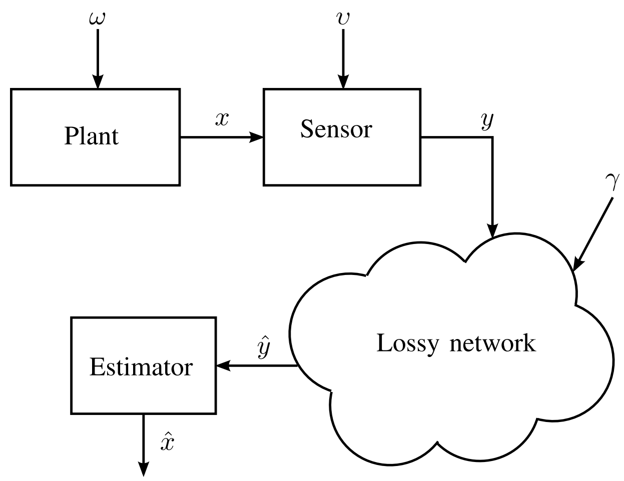

Fig. 1 – The system that is considered for the estimator design.

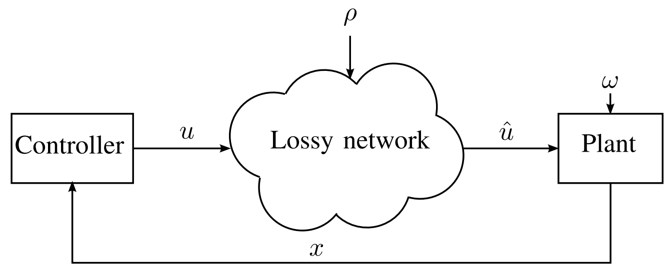

Fig. 2 – The system that is considered for the controller design.

the estimator case, this is done at the expense of certain performance detriment in comparison with a finite loss history or an online Kalman filter. In fact, by using i.i.d. packet dropouts in the JLS framework, one can obtain similar control and estimator laws, since in this case all the solutions to the coupled Riccati equations will be identical. However, in our current work, we present a simpler approach to the design of the offline estimator and controller, while we also, unlike the JLS framework, have duality in the closed loop response seen at the plant and estimator, respectively.

The rest of the paper is organized as follows: In Section II we introduce the estimator and controller designs. The main result is presented in Section III, where we first will show the stability results for the controller and then using the duality between the controller and observer show that stability of the controller is dual to stability of the observer. In Section IV we will, based on existing theory, discuss the convergence of the Riccati equation. Brief numerical examples and comparisons to other estimators are shown in Section V.

Notation: Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real and natural numbers, respectively. For a matrix A, denote that A is positive (semi)definite by $A > 0$ ($A \geq 0$). For two matrices of similar dimensions denote $A \leq B \Leftrightarrow A - B \leq 0$. The expectation of the random variable γ is denoted by $\mathcal{E}\{\gamma\}$.

II. PROBLEM DESCRIPTION

In this Section we will first introduce the optimal offline observer and then the optimal controller. Both minimize a given performance criterion, based on the statistics of the system and the network. For the observer we consider a system of the form

$$\begin{aligned} x_{k+1} &= Ax_k + \omega_k, \\ y_k &= \gamma_k \left(Cx_k + \upsilon_k\right), \end{aligned} \tag{1}$$

with $x_1 \sim \mathcal{N}(0, P)$, $\omega_k \sim \mathcal{N}(0, \Sigma_\omega)$, $\upsilon_k \sim \mathcal{N}(0, \Sigma_\upsilon)$ and the pair (A, C) being detectable. Also, γ_k is a stationary binary random process. Here $\gamma_k = 1$ means that the measurement y_k is successfully received by the estimator and $\mathcal{P}\{\gamma_k = 1\} = \mu$.

For the controller we consider another system which is of the form

$$x_{k+1} = Ax_k + \rho_k B u_k + \omega_k, \tag{2}$$

with $x_1 \sim \mathcal{N}(0, \Sigma_{x_1})$, $\omega_k \sim \mathcal{N}(0, \Sigma_\omega)$ and the pair (A, B) being stabilizable. Also ρ_k is a stationary binary random process, where $\rho_k = 1$ indicates that control signal u_k is successfully received by the plant. This occurs with probability $\mathcal{P}\{\rho_k = 1\} = \lambda$.

The random processes γ_k and ρ_k are stationary i.i.d. random processes, such that the probabilities for successful transmissions do not vary over time.

In the following we will derive the optimal offline observer followed by the optimal controller.

A. *Offline observer*

Let the matrix sequence $\mathcal{K}_N = (K_{1,N}, \cdots, K_{N,N})$. Define the following observer scheme

$$\tilde{x}_{k+1} = (A - \gamma_k K_{k,N} C)\,\tilde{x}_k + \gamma_k K_{k,N} y_k, \quad k = 1, \cdots, N,$$

with $\tilde{x}_1 \triangleq \mathcal{E}\{x_1\} = 0$.

We want the estimator to minimize the following cost function

$$E(\mathcal{K}_N) = \mathcal{E}\left\{\|x_N - \tilde{x}_N\|_2^2\right\}. \tag{3}$$

Hence, the sequence of estimator gains is given by

$$\mathcal{K}_N^\star = \underset{\mathcal{K}_N}{\arg\min}\; E(\mathcal{K}_N), \tag{4}$$

with $\mathcal{K}_N^\star = \left(K_{1,N}^\star, \cdots, K_{N,N}^\star\right)$.

The solution to the above problem is given in the following proposition.

Proposition 1. *For each $k \in \{1, \cdots, N\}$, the optimal observer gain is given by*

$$K_{k,N}^\star = AP_kC^T\left(\Sigma_\upsilon + CP_kC^T\right)^{-1}, \tag{5}$$

where

$$P_1 = P,$$
$$P_{k+1} = \Sigma_\omega + A\bigg(P_k - \mu P_k C^T \left(C P_k C^T + \Sigma_\upsilon\right)^{-1} C P_k\bigg)A^T, \tag{6}$$

and $\mu = \mathcal{E}\{\gamma_k\}$. Also, if the limit

$$\bar{P} = \lim_{k\to\infty} P_k \tag{7}$$

exists, then the asymptotic minimum cost $E^\star \triangleq \lim_{N\to\infty} E\left(\mathcal{K}_N^\star\right)$ is given by

$$E^\star = E\left(\bar{\mathcal{K}}_\infty\right) = \mathrm{trace}\left(\bar{P}\right), \tag{8}$$

where the infinite sequence $\bar{\mathcal{K}}_\infty \triangleq \left(\bar{K}, \bar{K}, \cdots\right)$ with

$$\bar{K} = \lim_{N\to\infty} K^\star_{N,N}. \tag{9}$$

In view of Proposition 1, whenever the limit in (7) exists, we define the offline estimator gain by (9). This choice then results in the following offline observer scheme

$$\hat{x}_{k+1} = \left(A - \gamma_k \bar{K} C\right)\hat{x}_k + \gamma_k \bar{K} y_k \tag{10}$$

where $\hat{x}_1 \triangleq \mathcal{E}\{x_1\} = 0$.

B. Linear Quadratic Regulator

Consider the system (2), which is depicted in Fig. 2. Let the matrix sequence $\mathcal{L}_N = \left(L_{1,N}, \cdots, L_{N,N}\right)$. Define the following control law

$$u_k = -L_{k,N} x_k, \quad k = 1, \cdots, N. \tag{11}$$

Consider the cost function

$$J\left(\mathcal{L}_N\right) = \frac{1}{N}\mathcal{E}\left\{\sum_{k=1}^{N+1}\|x_k\|_Q^2 + \sum_{k=1}^{N}\|\rho_k u_k\|_R^2\right\}, \tag{12}$$

where $Q > 0$ and $R \geq 0$ are design parameters, $\|x_k\|_Q^2 \triangleq x_k^T Q x_k$, and let

$$\mathcal{L}_N^\star = \arg\min_{\mathcal{L}_N} J\left(x_1, \mathcal{L}_N\right), \tag{13}$$

where $\mathcal{L}_N^\star = \left(L^\star_{1,N}, \cdots, L^\star_{N,N}\right)$.

The solution of the above minimization problem is given in the following proposition.

Proposition 2. *Let Q be such that the pair $\left(A, Q^{\frac{1}{2}}\right)$ is detectable and $R \geq 0$. Then for each $k \in \{1, \cdots, N\}$, the optimal control gain is given by*

$$L^\star_{k,N} = \left(R + B^T S_{N-k} B\right)^{-1} B^T S_{N-k} A, \tag{14}$$

with

$$S_0 = Q,$$
$$S_{k+1} = Q + A^T\bigg(S_k - \lambda S_k B\left(R + B^T S_k B\right)^{-1} B^T S_k\bigg)A \tag{15}$$

and $\lambda = \mathcal{E}\{\rho_k\}$. Also, if the limit

$$\bar{S} = \lim_{k\to\infty} S_k \tag{16}$$

exists, then the asymptotic minimum cost $J^\star_\infty \triangleq \lim_{N\to\infty} J\left(\mathcal{L}_N^\star\right)$ is given by

$$J^\star_\infty = J\left(\bar{\mathcal{L}}_\infty\right) = \mathrm{trace}\left(\Sigma_\omega \bar{S}\right),$$

where the infinite sequence $\bar{\mathcal{L}}_\infty \triangleq \left(\bar{L}, \bar{L}, \cdots\right)$ with

$$\bar{L} = \lim_{N\to\infty} L^\star_{1,N}. \tag{17}$$

In view of the above result, whenever the limit in (16) exists, we define the offline control gain by (17). This choice results in the following state feedback law

$$u_k = -\bar{L} x_k. \tag{18}$$

III. Stability Results

In this Section we will show that the convergence of the Riccati equation is a necessary and sufficient condition for stability of the observer and the controller. We will first state this for the controller and then show that the conditions for the estimator are dual to the control part. Here it is again important to note that both the controller and estimator depend on the channel statistics γ and μ, respectively. Conditions for when the Riccati equations converge are discussed in Section IV.

A. Controller

In this section we will present the conditions for mean square stability of the controlled system (2) and (18).

Let $\Sigma_k = \mathcal{E}\left\{x_k x_k^T\right\}$ denote the covariance of the state when using the model predictive control (MPC) scheme. Also, for a given matrix $\tilde{L}$, let

$$f_{\tilde{L}}(X) \triangleq (1-\lambda) A^T X A + \lambda H_{\tilde{L}}^T X H_{\tilde{L}} + Q + \lambda \tilde{L}^T R \tilde{L}, \tag{19}$$

$$g_{\tilde{L}}(X) \triangleq (1-\lambda) A X A^T + \lambda H_{\tilde{L}} X H_{\tilde{L}}^T + \Sigma_\omega. \tag{20}$$

with

$$H_{\tilde{L}} = A - B\tilde{L}.$$

The following lemma states a relation between the control Riccati equation and (19).

Lemma 1. *For every $k \in \mathbb{N}$,*

$$S_{k+1} = f_{L^\star_{N-k,N}}\left(S_k\right),$$

with $L^\star_{k,N}$ defined by (14). Also,

$$\Sigma_{k+1} = g_{\bar{L}}(\Sigma_k) \tag{21}$$

with $\bar{L}$ defined in (17).

Proof. The proof is shown in Appendix I. □

Denote, for $N \in \mathbb{N}$,

$$f_L^N(X) = f_L \circ f_L^{N-1}(X),$$

and $f_L^0(X) = X$.

For later reference, we introduce the following lemma. It states that, for a given L, the recursions induced by the map f_L are stable if and only if those induced by g_L are stable.

Lemma 2. *For any $X > 0$ and L, the following holds true*

$$\lim_{k\to\infty} \left\| f_L^k(X) \right\| < \infty \iff \lim_{k\to\infty} \left\| g_L^k(X) \right\| < \infty.$$

Proof. Using properties of the Kronecker product, we can state

$$\begin{aligned}\operatorname{vec}(f_{\bar{L}}(X)) &= F_L \times \operatorname{vec}(X) + \operatorname{vec}(K_L),\\ \operatorname{vec}(g_L(X)) &= G_L \times \operatorname{vec}(X) + \operatorname{vec}(M),\end{aligned}$$

where $K_L = Q + \lambda L^T R L$, $\operatorname{vec}(.)$ denotes the operation of converting a matrix into a vector by stacking its columns, and

$$\begin{aligned}F_L &= (1-\lambda)\left(A^T \otimes A^T\right) + \lambda\left(H_L^T \otimes H_L^T\right)\\ G_L &= (1-\lambda)(A \otimes A) + \lambda(H_L \otimes H_L).\end{aligned} \tag{22}$$

The result then follows since

$$F_L = G_L^T.$$

□

We are now ready to state the desired relation between stability of the closed loop system and the convergence of the Riccati equation. This result is presented in the following theorem:

Theorem 3. *The following holds true*

$$\lim_{k\to\infty} \|\Sigma_k\| < \infty \iff \lim_{k\to\infty} \|S_k\| < \infty. \tag{23}$$

Proof. Using the Kronecker product, we can write (21) as

$$\operatorname{vec}(\Sigma_{k+1}) = G_L \times \operatorname{vec}(\Sigma_k) + \operatorname{vec}(\Sigma_\omega),$$

where G_L is defined in (22).

Now $\lim_{k\to\infty} \Sigma_k < \infty$ holds if and only if $\lim_{k\to\infty} G_{\bar{L}}^k = 0$, which happens if and only if all eigenvalues of G are within the unit circle. This holds if and only if $\sqrt{1-\lambda}\sigma_A < 1$, where σ_A is the spectral radius of A. As stated in Lemma 2, $\lim_{k\to\infty} G_{\bar{L}}^k = 0$ if and only if $\lim_{k\to\infty} F_{\bar{L}}^k = 0$. The result then follows from [14, Theorem 5.2], where the authors state that the latter is true if $\lim_{k\to\infty} S_k$ is bounded. □

B. Observer

For the observer, we show that the convergence of (6) is necessary and sufficient for stability of the offline estimator.

Let $\Upsilon_k = \mathcal{E}\left\{e_k e_k^T\right\}$ denote the error covariance for the offline estimator, where

$$e_k \triangleq x_k - \hat{x}_k.$$

We define the map

$$\begin{aligned}h_K(P) \triangleq (1-\mu) A P A^T + \mu (A - KC) P (A - KC)^T\\ + \Sigma_\omega + \mu K \Sigma_\upsilon K^T.\end{aligned} \tag{24}$$

The relation to the estimation Riccati equation and the map (24) is stated in the following lemma.

Lemma 4. *The optimal error covariance for the estimator is given by*

$$P_{k+1} = h_{K_{k,N}^\star}(P_k), \tag{25}$$

where $K_{k,N}^\star$ is defined in (5). Also,

$$\Upsilon_{k+1} = h_{\bar{K}}(\Upsilon_k). \tag{26}$$

Proof. By substituting $A = A^T$, $C = B^T$, $\Sigma_\omega = Q$, $\Sigma_\upsilon = R$, $\mu = \lambda$, $K_{k,N} = L_{N-k,N}^T$, $P_k = S_k$, $\bar{K} = \bar{L}^T$ in (19), the map h turns into f (as defined in (19)) and the result then follows from Lemma 1. □

The following theorem then states the equivalence between the convergence of the Riccati equation and that of the offline estimator.

Theorem 5. *Consider Υ_k and P_k as defined in (25) and (26). We then have*

$$\lim_{k\to\infty} \|\Upsilon_k\| < \infty \iff \lim_{k\to\infty} \|P_k\| < \infty.$$

Proof. The argument is based on the fact that the Riccati equation (6) of the estimator is dual to the controller Riccati equation (15). Define for any finite positive definite X

$$\Sigma_{k+1}^\star = f_{\bar{L}}^k(X).$$

By doing the substitutions as in the proof of Lemma 4, it follows that

$$\lim_{k\to\infty} P_k < \infty \iff \lim_{k\to\infty} S_k < \infty \tag{27}$$

$$\lim_{k\to\infty} \Upsilon_k < \infty \iff \lim_{k\to\infty} \Sigma_k^\star < \infty, \tag{28}$$

Using (27) and (28), the result follows provided that

$$\lim_{k\to\infty} \|\Sigma_k^\star\| < \infty \iff \lim_{k\to\infty} \|S_k\| < \infty. \tag{29}$$

From Lemma 2 it follows that

$$\lim_{k\to\infty} \|\Sigma_k^\star\| < \infty \iff \lim_{k\to\infty} \|\Sigma_k\| < \infty. \tag{30}$$

Combining (29) and (30) we have that the result holds if

$$\lim_{k\to\infty} \|\Sigma_k\| < \infty \iff \lim_{k\to\infty} \|S_k\| < \infty,$$

and the latter follows from Theorem 3. □

Remark 1. As described in the proof of Lemma 4, the estimator can be designed by reformulating it as a dual controller design problem (or vice versa) using the substitutions

$$A^T \to A, C^T \to B, \Sigma_\omega \to Q, \Sigma_\mu \to R, \mu \to \lambda, P_1 \to S_0,$$

and designing the controller using (15) and (16). The offline estimator gain is then given by $\bar{K} = \bar{L}^T$ and the error covariance by $\bar{P} = \bar{S}$.

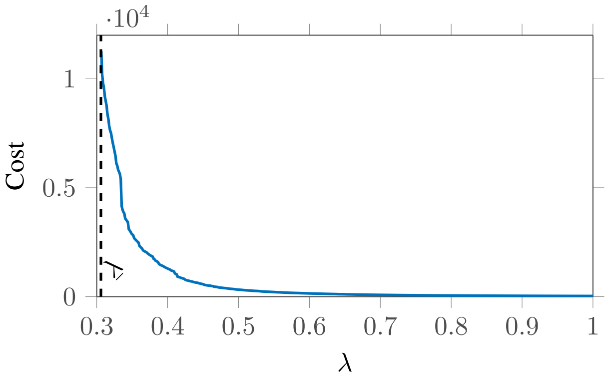

Fig. 3 – The cost $x_k^T Q x_k + u_k^T R u_k$ for the controller for different λ. The lower bound $\underline{\lambda}$, calculated using (31), is plotted with the dashed line.

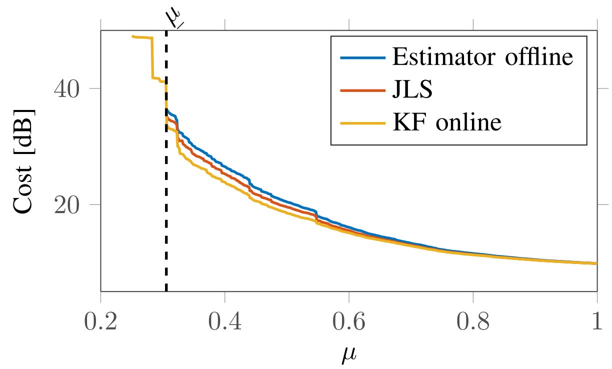

Fig. 4 – The cost $e_k^T e_k$ in dB for the different estimators for varying μ. The dashed horizontal line illustrates the lower bound $\underline{\mu}$, calculated using (31).

IV. Convergence of the Riccati equation

In this section we briefly discuss on conditions for when solutions to the Riccati equations (6) and (15) exist. We focus on the observer, but due to the duality that we established, the principles can easily be applied to the controller using the substitutions that are presented in Remark 1 of Section III-B.

It is worth noting that the form of the Riccati equations (6) and (15) are similar to the expression that the authors of [1] use to determine an upper bound for the performance of their online Kalman filter. The results on these bounds can therefore be applied directly on (6) and (15). The authors of [1] found that if the unstable poles can be canceled such that $A - \bar{K}C = 0$, the lower bound on μ for which a solution to (6) exists, is given by

$$\underline{\mu} = 1 - \frac{1}{\sigma_A^2}. \tag{31}$$

When multiple unstable poles are present and C is not invertible, the work [1] states conditions for the lower bound on μ, but did not state an analytical method to find $\underline{\mu}$. The more recent work [15] stated an analytical expression to find $\underline{\mu}$, which is given by

$$\underline{\mu} = \max_{\mathcal{I}_d} \left(1 - \frac{1}{\sqrt[r]{\left|\bar{A}_u\right|^2}} \right). \tag{32}$$

Here the pair $\left(\bar{A}_u, \bar{C}_u\right) = \left(A_u\left(\mathcal{I}_d\right), B_u\left(\mathcal{I}_d\right)\right)$ is the d'th partition of the unstable subspace of A and C and $r = \operatorname{rank}\left(\bar{C}_u\right)$. The variable $\mathcal{I}_d = (i_1, \cdots, i_\ell) \in (1, \cdots, n)$ selects which elements of A belong to this subsystem, while ℓ is the dimension of the subsystem. Also note that with only one unstable eigenvalue in A, (31) equals (32).

V. Numerical example

In this section we will illustrate the presented concepts in a numerical example where we separately design a controller and estimator. We will design the estimator by formulating the estimation problem into a dual control problem. We will also compare the performance of the offline estimator to the online Kalman filter and the JLS pseudo-offline design [6]. We consider the following system

$$A = \begin{bmatrix} 1.2 & 1 & 0 \\ 0 & 0.9 & 1 \\ 0 & 0 & 0.6 \end{bmatrix}$$

with the eigenvalues given on the diagonal,

$$B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$
$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

Remark that the pairs (A, B) and $\left(A^T, C^T\right)$ are both fully controllable. Further let $\Sigma_\omega = I_3$, where I_n is the $n \times n$ identity matrix, and $\Sigma_v = 1$. Packet dropouts are acknowledged, such that the estimator at time k knows both ρ_k and γ_k.

We first design the controller. Since is there only one unstable pole in A, we can use (31) to find $\underline{\lambda} = 0.3059$. This means that it is not possible to find a stabilizing controller if $\lambda < 0.3059$. The performance of the controller for varying λ is shown in Fig. 3. Note how the performance decreases as λ approaches $\underline{\lambda}$. The results are obtained by averaging over 1000 simulations each of 1000 time-steps.

An example of the controller design where the success probability is given by $\lambda = 0.5$, and the parameters $Q = I_3$ and $R = 0.1$, results in the control gain

$$\bar{L} = \begin{bmatrix} 0.3422 & 0.9728 & 1.3638 \end{bmatrix},$$

which gives closed loop eigenvalues located at $0.6677 \pm 0.045j$ and 0.0007.

We design the observer by reformulating it as a dual control problem, as explained in Remark 1. It is straightforward to see that in this case $\underline{\mu} = \underline{\lambda} = 0.3059$.

The performance of the offline observer for different values of μ is shown in Fig. 4. Here $\underline{\mu}$ is marked with the dashed line. The performance for the online Kalman filter [1] and for JLS with a loss horizon of length 2 [6] are plotted as well for comparison. Note that the JLS with horizon 1 would result in exactly the same performance as the offline estimator. It is worth noting, that while for small values of μ the Kalman filter and JLS estimator perform significantly better than the offline estimator, the performance difference decreases significantly as μ increases.

An example for the estimator design where the success probability is given by $\mu = 0.5$, results in the estimator gain

$$\bar{K} = \begin{bmatrix} 1.3468 & 0.1622 & 0.007 \end{bmatrix}^T.$$

This results in the closed loop eigenvalues $0.6693 \pm 0.0959j$ and 0.0075.

VI. Conclusions

We established a new link between the designs of an offline estimator and a LQR controller for systems with i.i.d. intermittent communications. We have designed an observer and a controller, both being offline in the sense of minimizing a cost function based solely on system and channel statistics, and therefore not requiring online computations. We have pointed out the duality of these two designs which permits the estimator design for a given system to be performed by formulating a dual system, and then design a controller for the latter. We have therefore extended the classical duality that exists between a LQR and a Kalman filter to the case of systems that are affected by packet losses. Future work involves investigating whether the separation principle holds for the studied controller and estimator. This is to be done for both cases, namely, when successful packet transmissions are acknowledged, and when they are not.

Appendix I
Proof of Lemma 1

Proof of Lemma 1. From (15), we have

$$S_{k+1} = Q + (1-\lambda) A^T S_k A + \lambda M, \tag{A.1}$$

with

$$\begin{aligned} M &= A^T S_k A - A^T S_k B \left(R + B^T S_k B\right)^{-1} B^T S_k A \\ &= A^T S_k A - A^T S_k B \left(R + B^T S_k B\right)^{-1} \\ &\quad \times \left(R + B^T S_k B\right) \left(R + B^T S_k B\right)^{-1} B^T S_k A \end{aligned}$$

Using (14) in the above equation, we get

$$M = A^T S_k A - L_{S_k}^T \left(R + B^T S_k B\right) L_{S_k}. \tag{A.2}$$

Also with $H_{L_{S_k}} \triangleq A - BL_{S_k}$ we have that

$$\begin{aligned} A^T S_k A =& H_{L_{S_k}}^T S_k H_{L_{S_k}} - L_{S_k}^T B^T S_k B L_{S_k} \\ &+ A^T S_k B L_{S_k} + L_{S_k}^T B^T S_k A \end{aligned} \tag{A.3}$$

and

$$A^T S_k B L_{S_k} = L_{S_k}^T \left(R + B^T S_k B\right) L_{S_k}. \tag{A.4}$$

Inserting (A.3) and (A.4) into (A.2) we obtain

$$M = H_{L_{S_k}}^T S_k H_{L_{S_k}} + L_{S_k}^T R L_{S_k}.$$

Replacing the above into (A.1), leads to

$$\begin{aligned} S_{k+1} &= (1-\lambda) A^T S_k A \\ &\quad + \lambda H_{L_{S_k}}^T S_k H_{L_{S_k}} + Q + \lambda L_{S_k}^T R L_{S_k} \\ &= f_{L_{S_k}}(S_k), \end{aligned}$$

and the result follows.

The covariance when using the controller (17) is given by

$$\begin{aligned} \Sigma_{k+1} =& (1-\lambda) A \Sigma_k A^T + \lambda H_{\bar{L}} \Sigma_k H_{\bar{L}}^T + \Sigma_\omega \\ =& g_{\bar{L}}(\Sigma_k), \end{aligned}$$

and the result follows from (20). □

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