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Research paper

Quantitative bounds for general Razumikhin-type functional differential inequalities with applications

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ABSTRACT

This paper proposes the concept of Razumikhin-type functional differential inequalities and points out that certain quantitative properties can be established for the Razumikhin-type functional differential inequalities. By virtue of certain auxiliary functions, some fundamental results on the quantitative bounds for the Razumikhin-type functional differential inequalities are systemically established in the paper, and these bounds are applied to deduce the basic Razumikhin-type stability theorems, including those for Itô stochastic functional differential equations. Two examples are given to illustrate the application of the established quantitative properties and to verify the effectiveness of our approach.

1. Introduction

The well-known Razumikhin technique provides us an approach to overcome the difficulties brought about by time delays so as to establish stability theorems or criteria for functional differential equations. This technique was initially proposed by Razumikhin [1,2] to study the stability of deterministic functional differential equations. Later, this technique was further investigated by Hale [3], Hou and Qian [4,5], Hou and Gao [6], Teel [7], Sun et al. [23] and extended to some other models, such as stochastic neutral models [8], switching models [9], models driven by Levy noise [16] and impulsive models [22,31,34]. Especially in applications, several generalized problems have been considered, such as H_∞ control [25] and adaptive feedback control [32], stabilization [27,30,33]. The Razumikhin technique has become very popular in recent years since it is extensively applied in the fields of applied mathematics and control engineering. The corresponding results are generally referred to as stability theorems of the Razumikhin type.

For the functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t), \ \ t \geq t_0, \\ x_{t_0}(\theta) = \phi_0(\theta), \ \ \theta \in I_\tau = [-\tau, 0], \end{cases} \tag{1}$$

where $f(t, \phi) \in C(\mathbb{R}^+ \times C(I_\tau; \mathbb{R}^n); \mathbb{R}^n)$ is a completely continuous functional with $f(t, 0) = 0 \in \mathbb{R}^n$ for all $t \geq t_0$. Assume that for every initial condition $\phi_0 \in C(I_\tau; \mathbb{R}^n)$, there exists a unique global solution to the Eq. (1), which is denoted by $x(t) =$

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$x(t; t_0, \phi_0)$. So, under the assumption $f(t, 0) = 0$ for all $t \geq t_0$, the Eq. (1) has the solution $x(t) \equiv 0$ corresponding to the initial condition $\phi_0(\theta) = 0, \theta \in I_\tau$. This solution is called the trivial solution.

For the above Eq. (1), recall the Razumikhin theorem in [3] which states that if there exist a continuous Lyapunov function $V(t, x)$, three K_∞-functions u, v, w and a continuous nondecreasing function $q(s) > s$ for $s > 0$, such that the following two conditions are satisfied for all $t \in [t_0, \infty)$:

$$(1)\ u(\|x\|) \leq V(t,x) \leq v(\|x\|), \ \text{for all}\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n;$$

$$(2)\ \dot{V}(t,x(t)) \leq -w(\|x(t)\|), \ \text{if}\ V(t+\theta, x(t+\theta)) \leq q(V(t,x(t))), \ \forall\ \theta \in I_\tau,$$

then the trivial solution of the Eq. (1) is globally uniformly asymptotically stable.

By the above Razumikhin theorem, in order to guarantee asymptotic stability of the Eq. (1), we need to find a positive-definite function $V(t, x)$ whose time-derivative $\dot{V}(t, x(t))$ along the solution of the Eq. (1) is negative-definite under a Razumikhin-type condition. It is shown that the asymptotic behavior or stability properties of the solution of the Eq. (1) can be guaranteed by certain negative-definite conditions of the derivative of Lyapunov functions such as $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$, under some Razumikhin-type conditions such as $V(t+\theta, x(t+\theta)) \leq q(V(t,x(t)))$, for all $\theta \in I_\tau$. In other words, function $w(\cdot)$ is used to give qualitative properties of the solutions of functional differential equations; see [10,17,18]. Actually, the principle of the Razumikhin technique is to determine qualitative properties using the condition (2) above. The above condition (2) will be stated formally as a Razumikhin-type functional differential inequality in the sequel.

On the other hand, one may feel intuitively that different $w(\|x(t)\|)$ in the condition (2) above implies different decay rate for the solution of the Eq. (1). The larger $w(\|x(t)\|)$ is, the larger the decay rate for the solution of the Eq. (1) should be. That is, an inequality as $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ under some Razumikhin-type conditions may imply some quantitative information for the solution $x(t)$. This means that the so-called Razumikhin-type functional differential inequality stated in this paper can provide both quantitative description and qualitative description for the solutions of functional differential equations.

In general, what kinds of quantitative properties can be obtained depends on the form of the Razumikhin-type functional differential inequality. There usually exist three common forms of the negative-definite conditions of $\dot{V}(t,x(t))$ along the solution $x(t)$ of the Eq. (1) in the corresponding Razumikhin-type functional differential inequality. One form is $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$; see [3], another one is the form of $\dot{V}(t,x(t)) \leq -w(V(t,x(t)))$; see [19,20]. Actually, $w(\|x(t)\|)$ and $w(V)$ are equivalent in essence under some simple conditions, which will be proved in Theorem 2 of this paper. So we only refer to $w(V)$ in the following text. The last form involves two variables and is denoted by a function $\bar{w}(t,V)$, i.e., $\dot{V}(t,x(t)) \leq -\bar{w}(t, V(t,x(t)))$; see [4–6,10,13,17,18].

For the special case with $w(V) = \alpha V$, for $\alpha > 0$, the exponential stability of stochastic functional differential equations was easily obtained; see [19,20]. Later, [12,21,26] established the Razumikhin-type theorems to a time-varying case and extended the exponential stability to the general decay stability of stochastic functional differential equations. Ning et al. [13] proposed an improved Razumikhin-type stability theorems for input-to-state stability of nonlinear time-delay systems. Li et al. [24] focused on the finite-time stability of time-varying time-delay systems by the Razumikhin technique and weakened the negative-definite condition for $\dot{V}$.

The time-varying case with $\bar{w}(t,V) = \alpha(t)\beta(V)$ (where $\beta(V) \geq 0$ for all $V > 0$ and $\alpha(t) \geq 0$ for all $t > 0$) and its transformation were extended, and the decay estimate for applications of Razumikhin-type theorems and criteria for quantitative stability for a class of Razumikhin-type retarded functional differential equation were obtained; see [4–6]. Li and Song [14], Cheng et al. [28], Liu and Yang [29] relaxed the non-negative condition for $\alpha(t)$ to a more general case, and proposed new extensions of Razumikhin-type stability theorems and applied them to impulsive models. Teel [7] established the connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. Later, Ning et al. [13] proposed an improved Razumikhin-type stability theorems for input-to-state stability of nonlinear time-delay systems and also weakened the negative-definite condition of $\dot{V}$. Zhou and Egorov [11] generalized the results of [13] to a weaker one and obtained Razumikhin stability of time-varying time-delay systems.

In this aspect, we can see that the negative-definite condition of $\dot{V}$ in the existing literature admits a general case that time t and the Lyapunov function V are separated with each other. And the generalization of the negative-definite condition for $\dot{V}$ has been received much more attention; see [11,13,14,24,34]. However, little work has been done for the case that time t and the Lyapunov function V are inseparable in the negative-definite condition of $\dot{V}$. Recently, [10,17,18] considered the function $\bar{w}(t,V)$ with the case that time t and the Lyapunov function V are inseparable and obtained some novel asymptotical stability theorems for (hybrid) stochastic retarded systems by the Razumikhin technique. To the best of the authors' knowledge, the general result on quantitative properties of the Razumikhin-type functional differential inequalities has been barely systematically studied.

Motivated by the above analysis, we would like to investigate quantitative properties of the Razumikhin-type functional differential inequalities under a more relaxed condition. In this paper, general and uniform bounds for general Razumikhin-type functional differential inequalities are established. We will further show how to bound the solutions of the Razumikhin-type functional differential inequalities under the general case that time t and the Lyapunov function V are inseparable. Based on these bounds, we will present another approach to deduce some basic Razumikhin-type stability results, which are more direct than those reported so far in the related literature.

The structure of the paper is as follows: The next section introduces some preliminaries. A lemma for bounding Razumikhin-type functional differential inequalities is also presented. In Section 3, some quantitative bounds of the basic

Razumikhin-type functional differential inequalities are established. In Section 4, a Razumikhin-type stability theorem for deterministic functional differential equations is provided. In Section 5, Razumikhin-type stability theorems for stochastic functional differential equations are studied in detail. Finally, we illustrate our method using two examples in Section 6 and give a conclusion in Section 7.

2. Preliminaries

2.1. Notations

Throughout the paper, $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, P)$ is a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets. $\|\cdot\|$ is the vector norm. τ is a positive constant which stands for the upper bound for the bounded time-delay involved possibly in the inequalities or equations. Let $t_0 \in \mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}^- = (-\infty, 0]$, and T be a positive constant with $T > t_0 \geq 0$, or infinity. $I = [t_0 - \tau, T)$ is the existing interval for the solutions of inequalities or equations involved. $I_\tau = [-\tau, 0]$. $C = C(I_\tau; \mathbb{R}^n)$ denotes the family of continuous functions ϕ from I_τ to $\mathbb{R}^n$ with norm $|\phi| = \sup_{\theta\in I_\tau} \|\phi(\theta)\|$. As usual, for a given function $x(t) \in C(I; \mathbb{R}^n)$, the associated function $x_t \in C(I_\tau; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t+\theta), \theta \in I_\tau$. For a non-negative and non-decreasing function $u(\cdot)$, we define $u^{-1}(s) = \sup\{l \in \mathbb{R}^+ | u(l) = s\}, s \geq 0$. $\mathrm{sgn}(\cdot)$ refers to the sign function.

2.2. Razumikhin-type functional differential inequality

Here, we introduce the notion of Razumikhin-type functional differential inequality.

Definition 1. A functional differential inequality is said to be a Razumikhin-type functional differential inequality, if there exist a non-negative function $\mathcal{V}(t) \in C(I; \mathbb{R}^+)$, a non-positive function $F(t, \mathcal{V}) \in C(I \times \mathbb{R}^+; \mathbb{R}^-)$, and a continuous function $q(t) \geq 1$ with $t \geq t_0$ such that

$$\dot{\mathcal{V}}(t) \leq F(t, \mathcal{V}(t)), \text{ if } \mathcal{V}(t+\theta) \leq q(t)\mathcal{V}(t), \ \forall\ \theta \in I_\tau, \text{ and } t \geq t_0, \tag{2}$$

where $\mathcal{V}_{t_0} \in C(I_\tau; \mathbb{R}^+)$ is the initial condition for the functional differential inequality (2), then the functional differential inequality (2) is said to be a Razumikhin-type functional differential inequality. Moreover, $\mathcal{V}(t+\theta) \leq q(t)\mathcal{V}(t)$, for all $\theta \in I_\tau$, is said to be a Razumikhin condition for the Razumikhin-type functional differential inequality (2).

This means that if only a functional differential inequality is satisfied under certain Razumikhin-type condition, then the functional differential inequality is a Razumikhin-type functional differential inequality. By the notion of the Razumikhin-type functional differential inequality, we can see that the condition (2) in the Razumikhin-type stability theorem in [3] stated in the Introduction section is a kind of Razumikhin-type functional differential inequality. Of course, all the inequalities to guarantee the negative-definite conditions in the existing Razumikhin-type stability theorems can be unified under the form of the Razumikhin-type functional differential inequality (2).

Remark 1. For the Razumikhin-type functional differential inequality (2), once the Razumikhin-type condition $\mathcal{V}(t+\theta) \leq q(t)\mathcal{V}(t)$, for all $\theta \in I_\tau$ is satisfied at time $t^* \geq t_0$, i.e., $\mathcal{V}(t^*+\theta) \leq q(t^*)\mathcal{V}(t^*)$, for all $\theta \in I_\tau$, then $\dot{\mathcal{V}}(t^*) \leq F(t^*, \mathcal{V}(t^*))$.

Remark 2. An arbitrary function inequality, without any Razumikhin-type condition, implied by a Razumikhin-type functional differential inequality is considered to be an estimate for the solution of the inequality. The bounds in the estimations for the solutions may be either explicit or implicit. More specifically, the bounds for the solutions of the Razumikhin-type functional differential inequalities can be obtained without requirement on any specific functional differential equation. It implies that the quantitative properties of the solutions of the Razumikhin-type functional differential inequalities can be studied separately. Based on these bounds, we can obtain stable or asymptotic properties of the solutions of the Razumikhin-type functional differential inequalities, which are the essential foreshadowing to deduce the basic Razumikhin-type stability theorems. So we propose the notion of the Razumikhin-type functional differential inequality formally.

2.3. Assumptions

In this paper, the following assumptions for the function $F(t, \mathcal{V})$ in the Razumikhin-type functional differential inequality (2) will be applied respectively:

Assumption 1. $F(t, \mathcal{V})$ is a non-positive continuous function and is non-increasing in $\mathcal{V}$, and $F(t, \mathcal{V}) < 0$ for all $\mathcal{V} > 0$.

Assumption 2. $F(t, \mathcal{V}) \leq -\zeta(t)w(\mathcal{V})$, where $\zeta(t)$ is a non-negative continuous function, $w(\cdot)$ is a non-negative and non-decreasing continuous function.

For certain special cases, we will also use the following assumptions:

Assumption 3. $F(t, \mathcal{V}) \leq -\zeta w(\mathcal{V})$, where $\zeta > 0$ is a constant, $w(\cdot)$ is a non-negative and non-decreasing continuous function.

Assumption 4. $F(t, \mathcal{V}) \leq -\zeta(t)\mathcal{V}$, where $\zeta(t)$ is a non-negative continuous function.

Assumption 5. $F(t, \mathcal{V}) \le -\zeta\mathcal{V}$, where $\zeta > 0$ is a constant.

Remark 3. Assumption 1 admits the general case that t and $\mathcal{V}$ can not be separated from each other in the function $F(t, \mathcal{V})$, and also admits strong nonlinearity used to study the stability of the nonlinear functional differential equations. Assumptions 3–5 are special cases of Assumption 2, which can be applied to deduce varying degrees of bounds for Razumikhin-type functional differential inequalities.

Under Assumption 4, [11,13,24,34] extended the non-negative condition for $\zeta(t)$ to a more general case and obtained Razumikhin stability of time-delay systems. To the best of the authors' knowledge, little work has been done under Assumption 1. In this paper, the general and uniform results on quantitative properties of the Razumikhin-type functional differential inequalities are systematically studied.

2.4. Lemmas

We propose a direct approach to obtain bounds for the solutions of Razumikhin-type functional differential inequalities.

Lemma 1. *Under Assumption 1, we have the following bound for the solution $\mathcal{V}(t)$ of Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0, \ \forall\ t \in I,$$

where $\mathcal{V}_0 = \sup_{\theta\in I_\tau} \mathcal{V}_{t_0}(\theta) = \sup_{\theta\in I_\tau} \mathcal{V}(t_0+\theta) > 0$.

Proof. Let ε be an arbitrary positive constant, define an auxiliary function

$$\rho(t, t_0, \varepsilon) = \mathcal{V}(t) - \mathcal{V}_0 - \varepsilon, \ \forall\ t \in I.$$

Firstly, we have $\rho(t_0+\theta, t_0, \varepsilon) = \mathcal{V}(t_0+\theta) - \mathcal{V}_0 - \varepsilon \le 0 - \varepsilon < 0$, for all $\theta \in I_\tau$. By the continuity of $\mathcal{V}(t)$ and $\rho(t, t_0, \varepsilon)$ on I, we know that there exists a sufficiently small $t_1 > t_0$ such that $\rho(t, t_0, \varepsilon) < 0$ for all $t \in [t_0 - \tau, t_1)$.

We assert that $\rho(t, t_0, \varepsilon) < 0$ holds for all $t \in I$. Assume that the assertion were false. In this case, we define $t^* = \inf\{t \in I \mid \rho(t, t_0, \varepsilon) \ge 0\}$, then we have $t^* \ge t_1 > t_0$. By this definition, we have $\rho(t^*, t_0, \varepsilon) = 0$, i.e., $\mathcal{V}(t^*) = \mathcal{V}_0 + \varepsilon > 0$, and $\rho(t, t_0, \varepsilon) < 0$ for all $t \in [t_0 - \tau, t^*)$. Using this fact and noticing that $t_0 - \tau \le t^* + \theta \le t^*$ for all $\theta \in I_\tau$, we have $\dot{\rho}(t^*, t_0, \varepsilon) \ge 0$, and

$$\mathcal{V}(t^*+\theta) - \mathcal{V}_0 - \varepsilon \le \mathcal{V}(t^*) - \mathcal{V}_0 - \varepsilon, \ \forall\ \theta \in I_\tau.$$

It follows that

$$\mathcal{V}(t^*+\theta) \le \mathcal{V}(t^*) \le q(t^*)\mathcal{V}(t^*), \ \forall\ \theta \in I_\tau,$$

due to $q(t^*) \ge 1$. Namely, the Razumikhin condition is satisfied at t^*. By the Razumikhin-type functional differential inequality (2), we obtain $\dot{\mathcal{V}}(t^*) \le F(t^*, \mathcal{V}(t^*))$ and then $\dot{\rho}(t^*, t_0, \varepsilon) = \dot{\mathcal{V}}(t^*) \le F(t^*, \mathcal{V}(t^*)) < 0$ due to $\mathcal{V}(t^*) > 0$. This is a contradiction! It shows that $\rho(t, t_0, \varepsilon) < 0$ holds for all $t \in I$, namely $\mathcal{V}(t) \le \mathcal{V}_0 + \varepsilon$, for all $t \in I$.

Letting $\varepsilon \to 0^+$, we get $\mathcal{V}(t) \le \mathcal{V}_0$, for all $t \in I$. The proof is complete. □

Lemma 2 [15] [Barbalat Lemma]. *If a function $\kappa : \mathbb{R}^+ \to R$ is uniformly continuous, and the limit $\lim_{t\to\infty} \int_0^t \kappa(s)ds$ exists and is finite, then $\lim_{t\to\infty} \kappa(t) = 0$.*

3. Bounds for Razumikhin-type functional differential inequalities

Based on the above bound, we derive the quantitative bounds for Razumikhin-type functional differential inequalities, which are described by some integral inequalities, without any Razumikhin condition. Based on quantitative bounds, we can obtain stable or asymptotic properties of the solutions of Razumikhin-type functional differential inequalities. These bounds are the essential foreshadowing for us to deduce the basic Razumikhin-type stability theorems in the next section.

Theorem 1. *Under Assumption 1, and assuming that the function $q(t)$ in the Razumikhin condition satisfies $q(t) \ge \exp\{\int_{t-\tau}^t \hat{m}(s)ds\}$ for all $t \ge t_0$, where $\hat{m}(t) = -k(\mathcal{V}_0)F(t, \mathcal{V}_0)$ is a non-negative continuous function with $k(\mathcal{V}_0) = 1/\mathcal{V}_0$, we have a bound for the solution $\mathcal{V}(t)$ of the Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\int_{t_0}^t m(s)ds\right), \forall\ t \in I, \tag{3}$$

where $m(t) = -k(\mathcal{V}_0)F(t, \mathcal{V})$ is also a non-negative continuous function.

Proof. By Lemma 1, we first have $\mathcal{V}(t) \le \mathcal{V}_0$, for all $t \in I$. Let ε be an arbitrary positive constant, define auxiliary functions

$$\beta_1(t, t_0, \varepsilon) = \mathcal{V}_0 \exp\left(-\int_{t_0}^t m(s)ds\right) + \varepsilon,$$

$$\rho_1(t, t_0, \varepsilon) = \frac{\mathcal{V}(t)}{\beta_1(t, t_0, \varepsilon)} - 1, \ \ t \in I.$$

Firstly, since $m(t) \geq 0$, then for all $\theta \in I_\tau$,

$$\begin{aligned}\rho_1(t_0+\theta, t_0, \varepsilon) &= \frac{\mathcal{V}(t_0+\theta)}{\mathcal{V}_0 \exp\left(-\int_{t_0}^{t_0+\theta} m(s)\mathrm{d}s\right) + \varepsilon} - 1 \\ &\leq \frac{\sup_{\theta \in I_\tau} \mathcal{V}(t_0+\theta)}{\mathcal{V}_0 \exp\left(\int_{t_0+\theta}^{t_0} m(s)\mathrm{d}s\right) + \varepsilon} - 1 \\ &\leq \frac{\mathcal{V}_0}{\mathcal{V}_0 + \varepsilon} - 1 < 0.\end{aligned}$$

By the continuity of $\mathcal{V}(t)$ and $\rho_1(t, t_0, \varepsilon)$ on I, we know that there exists a sufficiently small $t_1 > t_0$ such that $\rho_1(t, t_0, \varepsilon) < 0$ for all $t \in [t_0 - \tau, t_1)$.

We assert that $\rho_1(t, t_0, \varepsilon) < 0$ holds for all $t \in I$. Assume that the assertion were false. In this case, we define $t^* = \inf\{t \in I \,|\, \rho_1(t, t_0, \varepsilon) \geq 0\}$, then we have $t^* \geq t_1 > t_0$. By this definition, we have $\rho_1(t^*, t_0, \varepsilon) = 0$, then we have $\mathcal{V}(t^*) = \beta_1(t^*, t_0, \varepsilon) > 0$, and $\rho_1(t, t_0, \varepsilon) < 0$ for all $t \in [t_0 - \tau, t^*)$. By this fact, noticing that $t_0 - \tau \leq t^* + \theta \leq t^*$ for all $\theta \in I_\tau$, we have $\dot{\rho}_1(t^*, t_0, \varepsilon) \geq 0$, and for all $\theta \in I_\tau$, $\rho_1(t^*+\theta, t_0, \varepsilon) \leq 0 = \rho_1(t^*, t_0, \varepsilon)$, then we have $\mathcal{V}(t^*+\theta)/\beta_1(t^*+\theta, t_0, \varepsilon) \leq \mathcal{V}(t^*)/\beta_1(t^*, t_0, \varepsilon)$, and then

$$\mathcal{V}(t^*+\theta) \leq \frac{\beta_1(t^*+\theta, t_0, \varepsilon)}{\beta_1(t^*, t_0, \varepsilon)} \mathcal{V}(t^*), \forall\ \theta \in I_\tau.$$

Denote

$$Q_1(t^*, \theta) = q_1(t^*, \theta) + 1 = \frac{\beta_1(t^*+\theta, t_0, \varepsilon)}{\beta_1(t^*, t_0, \varepsilon)}, \theta \in I_\tau.$$

By a simple computation, we obtain

$$\begin{aligned}q_1(t^*, \theta) &= \frac{\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*+\theta} m(s)\mathrm{d}s\right) + \varepsilon}{\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right) + \varepsilon} - 1 \\ &= \frac{\exp\left(\int_{t^*+\theta}^{t^*} m(s)\mathrm{d}s\right) - 1}{1 + \frac{\varepsilon}{\mathcal{V}_0} \exp\left(\int_{t_0}^{t^*} m(s)\mathrm{d}s\right)} \\ &\leq \exp\left(\int_{t^*+\theta}^{t^*} m(s)\mathrm{d}s\right) - 1 \\ &\leq \exp\left(\int_{t^*-\tau}^{t^*} m(s)\mathrm{d}s\right) - 1, \forall\ \theta \in I_\tau,\end{aligned}$$

thus we have $Q_1(t^*, \theta) \leq \exp\left(\int_{t^*-\tau}^{t^*} m(s)\mathrm{d}s\right)$ for all $\theta \in I_\tau$.

Based on Assumption 1, we can obtain $m(t) \leq \hat{m}(t)$ and then $\mathcal{V}(t^*+\theta) \leq \exp\left(\int_{t^*-\tau}^{t^*} m(s)\mathrm{d}s\right)\mathcal{V}(t^*) \leq \exp\left(\int_{t^*-\tau}^{t^*} \hat{m}(s)\mathrm{d}s\right)\mathcal{V}(t^*) \leq q(t^*)\mathcal{V}(t^*)$, for all $\theta \in I_\tau$. Namely, the Razumikhin condition is satisfied at t^*. By the Razumikhin-type functional differential inequality (2), we obtain $\dot{\mathcal{V}}(t^*) \leq F(t^*, \mathcal{V}(t^*))$ and then $\dot{\rho}_1(t^*, t_0, \varepsilon) = \bar{\rho}_1(t^*, t_0, \varepsilon)/\beta_1^2(t^*, t_0, \varepsilon)$, where

$$\begin{aligned}\bar{\rho}_1(t^*, t_0, \varepsilon) &= \dot{\mathcal{V}}(t^*)\beta_1(t^*, t_0, \varepsilon) - \mathcal{V}(t^*)\dot{\beta}_1(t^*, t_0, \varepsilon) \\ &\leq -\mathcal{V}_0 m(t^*)\beta_1(t^*, t_0, \varepsilon) - \mathcal{V}(t^*)\dot{\beta}_1(t^*, t_0, \varepsilon) \\ &= -\mathcal{V}_0 m(t^*)\beta_1(t^*, t_0, \varepsilon) + \mathcal{V}(t^*)\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right)m(t^*) \\ &= -\mathcal{V}_0 m(t^*)\left(\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right) + \varepsilon\right) + \mathcal{V}(t^*)\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right)m(t^*) \\ &\leq -\mathcal{V}_0 m(t^*)\left(\mathcal{V}_0 \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right) + \varepsilon\right) + \mathcal{V}_0^2 m(t^*) \exp\left(-\int_{t_0}^{t^*} m(s)\mathrm{d}s\right) \\ &= -\varepsilon\mathcal{V}_0 m(t^*) < 0,\end{aligned}$$

due to $0 < \mathcal{V}(t^*) \le \mathcal{V}_0$, and $m(t^*) > 0$. By this we have $\dot{\rho}_1(t^*, t_0, \varepsilon) < 0$. This is a contradiction! It shows that $\rho_1(t, t_0, \varepsilon) < 0$ for all $t \in I$, namely, $\mathcal{V}(t) \le \beta_1(t, t_0, \varepsilon) = \mathcal{V}_0 \exp\left(-\int_{t_0}^t m(s)\mathrm{d}s\right) + \varepsilon$ for all $t \in I$.

Letting $\varepsilon \to 0^+$, we get $\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\int_{t_0}^t m(s)\mathrm{d}s\right)$ for all $t \in I$. The proof is complete. □

Remark 4. Theorem 1 admits the general case that t and $\mathcal{V}$ can not be separated from each other in the function $F(t, \mathcal{V})$. From the result of Theorem 1, we notice that the bound for the solution $\mathcal{V}(t)$ of the Razumikhin-type functional differential inequality (2) has limitations, because of the presence of $\mathcal{V}(t)$ in $m(t)$, and this is due to that t and $\mathcal{V}$ can not be separated from each other in the function $F(t, \mathcal{V})$. Even so, we can still obtain the bound or asymptotic properties of the solution $\mathcal{V}(t)$. To the best of the authors' knowledge, the result of Theorem 1 has never been shown in the existing literature.

Based on further analysis for the properties of the exponential function in the above bound, we can obtain the final properties of the solution $\mathcal{V}(t)$. This bound is the essential foreshadowing for us to deduce the basic Razumikhin-type stability theorem in the next section.

Corollary 1. *Under Assumption 2, and assuming that the function $q(t)$ in the Razumikhin condition satisfies $q(t) \ge \exp\{\int_{t-\tau}^t \zeta(s)\mathrm{d}s\}$, for all $t \ge t_0$, we have the following bound for the solution $\mathcal{V}(t)$ of the Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\int_{t_0}^{t} m(s)\mathrm{d}s\right), \ \forall\, t \in I, \tag{4}$$

where $m(t) = k(\mathcal{V}_0)\zeta(t)w(\mathcal{V}(t))$ is a non-negative continuous function, and $k(\mathcal{V}_0) = \min\{\frac{1}{\mathcal{V}_0}, \frac{1}{w(\mathcal{V}_0)}\}$.

Remark 5. We note that Assumption 2 is a generalization of the assumptions used in [4–6,11], which can be applied to obtain asymptotic and stable properties of Razumikhin-type functional differential inequalities directly. Assumption 2 in this paper admits strong nonlinearity can be used to study the stability of the nonlinear functional differential equations, which will be shown below.

Based on Corollary 1 and special cases of Assumption 2, we have the following corollaries.

Corollary 2. *Under Assumption 3, and assuming that the function $q(t) = q \ge 1$ is a constant function in the Razumikhin condition, we have the following bound for the solution $\mathcal{V}(t)$ of the Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\int_{t_0}^{t} k(\mathcal{V}_0)w(\mathcal{V}(s))\mathrm{d}s\right), \ \forall\, t \in I,$$

where $k(\mathcal{V}_0) = \min\left\{\frac{\zeta}{\mathcal{V}_0}, \frac{\ln q}{\tau w(\mathcal{V}_0)}\right\}$.

Corollary 3. *Under Assumption 4, and assuming that the function $q(t)$ in the Razumikhin condition satisfies $q(t) \ge \exp\{\int_{t-\tau}^t \zeta(s)\mathrm{d}s\}$, for all $t \ge t_0$, then we have the following bound for the solution $\mathcal{V}(t)$ of the Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\int_{t_0}^{t} \zeta(s)\mathrm{d}s\right), \ \forall\, t \in I.$$

Corollary 4. *Under Assumption 5, and assuming that the function $q(t) = q \ge 1$ is a constant function in the Razumikhin condition, we have the following bound for the solution $\mathcal{V}(t)$ of the following Razumikhin-type functional differential inequality (2):*

$$\mathcal{V}(t) \le \mathcal{V}_0 \exp\left(-\lambda(t - t_0)\right), \ \forall\, t \in I,$$

where $\lambda = \min\{\zeta, \frac{\ln q}{\tau}\}$.

For the special cases of our results, Corollary 3 and Corollary 4 have been obtained in the existing literature; see [4–6,20]. In this section, we consider a more general Razumikhin-type functional differential inequality and obtain both quantitative and qualitative properties of the solutions of Razumikhin-type functional differential inequalities, which are applied to get the asymptotic stability or exponential stability of the corresponding equations directly.

4. Application to stability of deterministic functional differential equations

In this section, we will demonstrate the application of the above obtained bounds to a classical Razumikhin-type stability theorem for deterministic functional differential equations. This will be compared with the methods employed to establish Razumikhin-type stability theorems in the existing literature; see [3,19]. The method of using our bounds turns out to be more direct and distinctly different from those used in the existing literature.

Consider the deterministic functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t), \ t \ge t_0, \\ x_{t_0}(\theta) = \phi_0(\theta), \ \theta \in I_\tau, \end{cases} \tag{5}$$

where $f(t, \phi) \in C(\mathbb{R}^+ \times C(I_\tau; \mathbb{R}^n); \mathbb{R}^n)$ is a completely continuous functional with $f(t, 0) = 0 \in \mathbb{R}^n$ for all $t \ge t_0$. $\phi_0 \in C(I_\tau; \mathbb{R}^n)$ is the initial condition. The functional $f(t, \phi)$ is supposed to satisfy enough additional smoothness conditions to ensure a local continuous solution $x(t) = x(t; t_0, \phi_0)$ to the Eq. (5) for each initial condition $\phi_0 \in C(I_\tau; \mathbb{R}^n)$, such as the local Lipschitz condition. From now on, we assume that $I = [t_0 - \tau, +\infty)$.

Under the assumption $f(t, 0) = 0$ for all $t \geq t_0$, the Eq. (5) has the solution $x(t) \equiv 0$ corresponding to the initial condition $\phi_0(\theta) = 0, \theta \in I_\tau$. This solution is called the trivial solution.

Definition 2. The trivial solution of the Eq. (5) is said to be:

(1) stable if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t; t_0, \phi_0)\| < \varepsilon, \text{ for all } t \geq t_0,$$

provided $|\phi_0| < \delta$;

(2) uniformly stable if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x(t; t_0, \phi_0)\| < \varepsilon, \text{ for all } t \geq t_0,$$

provided $|\phi_0| < \delta$;

(3) uniformly asymptotically stable if it is uniformly stable, and

$$\lim_{t \to +\infty} x(t; t_0, \phi_0) = 0.$$

Lemma 3 (Uniform continuity)**.** *Assume that the functional f in the Eq. (5) satisfies the local Lipschitz condition. If the solution $x(t) = x(t, t_0, \phi_0)$ of the Eq. (5) is such that $x(t)$ is bounded for $t \in I$, then $x(t)$ is uniformly continuous.*

Proof. Assume that there exists a positive constant M such that $\|x(t)\| \leq M$, then $\|x(t)\| \leq [M] \vee 1 = \hat{M}$, where $[\ \cdot\ ]$ denotes the ceiling function. By the local Lipschitz condition and $f(t, 0) = 0$, for the positive integer $\hat{M} \geq 1$, there is a positive constant $L_{\hat{M}}$ such that for all $t \geq t_0$ and all $x \in \mathbb{R}^n$ with $\|x\| \leq \hat{M}$,

$$\|f(t, x)\| = \|f(t, x) - f(t, 0)\| \leq L_{\hat{M}}\|x\| \leq L_{\hat{M}}\hat{M},$$

That is, the functional f also satisfies the linear growth condition. Thus, for any positive increment h, based on the Eq. (5), we have

$$\|x(t + h) - x(t)\| = \|\int_t^{t+h} f(u, x_u)\mathrm{d}u\| \leq \int_t^{t+h} \|f(u, x_u)\|\mathrm{d}u \leq L_{\hat{M}}\hat{M}h,$$

this implies the uniform continuity of $x(t)$. The proof is complete. □

Based on Lemma 3, we need only to impose the local Lipschitz condition on the functional f to coordinate the application of the Barbalat Lemma.

Theorem 2. *Assume $u(\cdot), v(\cdot), w(\cdot), \hat{w}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions with $u(s), v(s), w(s), \hat{w}(s)$ being positive for $s > 0$ and $u(0) = v(0) = w(0) = \hat{w}(0) = 0$. Also assume $q > 1$. If there is a continuous Lyapunov function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that*

(1) *$u(\|x\|) \leq V(t, x) \leq v(\|x\|)$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;*
(2) *$\dot{V}(t, \phi(0)) \leq -\hat{w}(\|\phi(0)\|)$, or $\dot{V}(t, \phi(0)) \leq -w(V(t, \phi(0)))$, if $V(t + \theta, \phi(\theta)) \leq qV(t, \phi(0))$, $\forall \theta \in I_\tau$, then the solution $x(t) = x(t; t_0, \phi_0)$ exists globally and the trivial solution $x = 0$ of the Eq. (5) is uniformly asymptotically stable.*

Proof. Firstly, for the first case of the condition (2), we also have $\dot{V}(t, \phi(0)) \leq -w(V(t, \phi(0)))$, where $w(\cdot) = \hat{w}(v^{-1}(\cdot))$. So we need only to prove the conclusion for the second case, i.e., $\dot{V}(t, \phi(0)) \leq -w(V(t, \phi(0)))$.

Secondly, given the initial condition (t_0, ϕ_0) with $|\phi_0| > 0$. By Lemma 1, we have $V(t, x(t)) \leq V_0$ for all $t \in I$, where $V_0 = \sup_{\theta \in I_\tau} V(t_0 + \theta, \phi_0(\theta))$. Then we get the bound

$$\|x(t)\| \leq u^{-1}(V(t, x(t))) \leq u^{-1}(V_0) \leq \bar{X}_0 = u^{-1}(v(|\phi_0|)),\ \forall t \in I,$$

which means that it is impossible for the solution $x(t)$ to explode due to the boundedness of $x(t)$. In other words, the solution $x(t) = x(t; t_0, \phi_0)$ exists globally. At the same time, this bound also implies uniform stability.

Thirdly, by letting $F(t, \mathcal{V}) = -w(\mathcal{V})$ and applying Corollary 2, we have

$$V(t) \leq V_0 \exp\Big(- \int_{t_0}^{t} k(V_0)w(V(s, x(s))\mathrm{d}s\Big),\ \forall t \in I, \tag{6}$$

where $k(V_0) = \min\{\frac{1}{V_0}, \frac{\ln q}{\tau w(V_0)}\}$, and then $\|x(t)\| \leq u^{-1}(\beta^*(t, t_0))$, for all $t \in I$, where $\beta^*(t, t_0)) = V_0 \exp(-\int_{t_0}^{t} k(V_0)w(V(s, x(s))\mathrm{d}s)$.

If the integral $\int_{t_0}^{t} k(V_0)w(V(s, x(s)))\mathrm{d}s$ diverges, then by the above bound (6), we get $V(t, x(t)) \to 0$ as $t \to +\infty$ and then $x(t) \to 0$ as $t \to +\infty$. If the integral $\int_{t_0}^{t} k(V_0)w(V(s, x(s)))\mathrm{d}s$ converges, then by the condition (2) of the theorem, $\int_{t_0}^{t} k(V_0)w(u(\|x(s)\|))\mathrm{d}s \leq \int_{t_0}^{t} k(V_0)w(V(s, x(s)))\mathrm{d}s$ converges.

Based on the boundness of $x(t)$ and Lemma 3, we obtain the uniform continuity of $x(t)$. With this one can easily show that $\|x(t)\|$ and $w(u(\|x(t)\|))$ are also uniformly continuous. Therefore, by the Barbalat Lemma 2, the integrand $k(V_0)w(u(\|x(t)\|)) \to 0$ as $t \to +\infty$, by the assumptions on $w(\cdot)$, $u(\cdot)$ and $V(\cdot)$, this leads to $x(t) \to 0$ as $t \to +\infty$. The proof is complete. □

Remark 6. For only the uniform stability, the Razumikhin-type condition can be weakened to be $V(t+\theta, \phi(\theta)) \leq V(t, \phi(0)),\ \forall \theta \in I_\tau$, i.e., $p = 1$.

5. Application to stability of stochastic functional differential equations

Consider the following Itô stochastic functional differential equation

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dw(t), & t \geq t_0, \\ x_{t_0}(\theta) = \phi_0(\theta), & \theta \in I_\tau, \end{cases} \tag{7}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^+ \times C(I_\tau; \mathbb{R}^n) \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times C(I_\tau; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$ are assumed to be measurable functionals with $f(t, 0) = 0 \in \mathbb{R}^n, g(t, 0) = 0 \in \mathbb{R}^{n\times m}$ for all $t \geq t_0$. $w(t)$ is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, P)$. The initial condition for the Eq. (7) will given by (t_0, ϕ_0), where $\phi_0 = \{\phi_0(s) : s \in I_\tau\}$ is an $\mathscr{F}_{t_0}$-measurable $C(I_\tau; \mathbb{R}^n)$-valued random variable. The solution of the equation through (t_0, ϕ_0) is denoted by $x(t; t_0, \phi_0)$. The functionals f and g are supposed to be completely continuous and satisfy the local Lipschitz condition to ensure a local solution to the Eq. (7) for each initial condition $\phi_0 \in C(I_\tau; \mathbb{R}^n)$.

Under the assumptions $f(t, 0) = 0,\ g(t, 0) = 0$ for all $t \geq t_0$, the Eq. (7) has the solution $x(t) \equiv 0$ corresponding to the initial condition $\phi_0(\theta) = 0, \theta \in I_\tau$. This solution is called the trivial solution.

Let $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ be a positive function which is differentiable in t and twice continuously differentiable in $x \in \mathbb{R}^n$. Associated with the Eq. (7) define the differential operator $\mathscr{L}$ by

$$\begin{aligned} \mathscr{L}V(t, x_t) &= V_t(t, x) + V_x(t, x)f(t, x_t) + \frac{1}{2}\mathrm{Tr}\Big(g^T(t, x_t)V_{xx}(t, x)g(t, x_t)\Big), \\ V_t(t, x) &= \frac{\partial V(t, x)}{\partial t},\ V_x(t, x) = \Big(\frac{\partial V(t, x)}{\partial x_1}, \cdots, \frac{\partial V(t, x)}{\partial x_n}\Big), \\ V_{xx}(t, x) &= \Big(\frac{\partial^2 V(t, x)}{\partial x_i \partial x_j}\Big)_{n\times n}. \end{aligned}$$

Definition 3. The trivial solution of the Eq. (7) is said to be:

(1) stable in pth moment with $p \geq 2$ if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\mathbb{E}\|x(t; t_0, \phi_0)\|^p < \varepsilon, \text{ for all } t \geq t_0,$$

provided $\mathbb{E}|\phi_0|^p < \delta$;

(2) uniformly stable in pth moment with $p \geq 2$ if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\mathbb{E}\|x(t; t_0, \phi_0)\|^p < \varepsilon, \text{ for all } t \geq t_0,$$

provided $E|\phi_0|^p < \delta$;

(3) uniformly asymptotically stable in pth moment with $p \geq 2$, if it is uniformly stable in pth moment, and

$$\lim_{t\to+\infty} \mathbb{E}\|x(t; t_0, \phi_0)\|^p = 0.$$

Similar to Lemma 3, we have the following uniform continuity lemma for the solution of Eq. (7).

Lemma 4 [35] [Uniform continuity]**.** *Under the condition that the functionals f and g of the Eq. (7) satisfy the global Lipschitz condition, if the solution $x(t) = x(t, t_0, \phi_0)$ of the Eq. (7) is such that $\mathbb{E}\|x(t)\|^p$ is bounded for $t \in I$, then $\mathbb{E}\|x(t)\|^p$ is uniformly continuous, where $p \geq 2$.*

To apply Lemma 4 and Lemma 2, we impose the Lipschitz condition for the functionals f and g in the following Theorems 3 and 4.

Theorem 3. *Let $p \geq 2$. Suppose $u(s),\ v(s),\ w(s),\ \hat{w}(s) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous increasing functions, and they are positive for $s > 0$ with $u(0) = v(0) = w(0) = \hat{w}(0) = 0$, $v(\cdot)$ is concave, $u(\cdot)$, $w(\cdot)$ and $\hat{w}(\cdot)$ are convex. If there exists a continuous Lyapunov function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that*

(1) *$u(\|x\|^p) \leq V(t, x) \leq v(\|x\|^p)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$;*
(2) *$\mathbb{E}\mathscr{L}V(t, \phi) \leq -\mathbb{E}\hat{w}(\|\phi(0)\|^p)$, or $\mathbb{E}\mathscr{L}V(t, \phi) \leq -\mathbb{E}w(V(t, \phi(0)))$, if $\mathbb{E}V(t+\theta, \phi(\theta)) \leq q\mathbb{E}V(t, \phi(0))$, $\forall\ \theta \in I_\tau$, $q > 1$, then the solution $x(t) = x(t; t_0, \phi_0)$ exists globally and the trivial solution $x = 0$ of the Eq. (7) is uniformly asymptotically stable in pth moment.*

Proof. The proof is similar to that of Theorem 2. We prove the conclusion for the second case of the condition (2). Firstly, let $\phi_0 \in C(I_\tau;\mathbb{R}^n)$ be the initial condition with $|\phi_0| > 0$, define

$$\mathbb{E}V_0 = \mathbb{E}\sup_{\theta\in I_\tau} V(t_0+\theta,\phi_0(\theta)),$$
$$\rho(t,t_0,\varepsilon) = \mathbb{E}V(t,x(t)) - \mathbb{E}V_0 - \varepsilon, t\in I.$$

By Lemma 1, we can show that $\mathbb{E}V \le \mathbb{E}V_0$ for all $t\in I$. With this and the inequality $u(\mathbb{E}\|x(t)\|^p) \le \mathbb{E}V(t,x) \le v(\mathbb{E}|\phi_0|^p)$ or $\mathbb{E}\|x(t)\|^p \le \bar{X}_0 = u^{-1}(v(\mathbb{E}|\phi_0|^p))$ and then $\mathbb{E}|x_t|^p \le \bar{X}_0$, for $t\in I$. By Chebyshev's inequality, we can show that the solution $x(t)$ is not explosive, thus it exists globally. We also know that the trivial solution $x=0$ of the Eq. (7) is uniformly stable in pth moment.

Secondly, to complete the proof for the asymptotic stability, define

$$\beta_2(t,t_0,\varepsilon) = \mathbb{E}V_0 \exp\Big(-\int_{t_0}^{t} \bar{w}_1(\mathbb{E}V(s,x(s)))\mathrm{d}s\Big) + \varepsilon,$$
$$\rho_2(t,t_0,\varepsilon) = \frac{\mathbb{E}V(t,x(t))}{\beta_2(t,t_0,\varepsilon)} - 1,\ t\in I,$$

where $\bar{w}_1(\mathbb{E}V) = k(\mathbb{E}V_0)w(\mathbb{E}V),\ \ k(\mathbb{E}V_0) = \min\left\{\frac{1}{\mathbb{E}V_0}, \frac{\ln q}{\tau w(\mathbb{E}V_0)}\right\}$.

Due to $|\phi_0|^p > 0$, we have $\mathbb{E}V_0 > 0$ and $w(\mathbb{E}V_0) > 0$ by the assumption on $w(\cdot)$, then $k(\mathbb{E}V_0)$ and $\bar{w}_1(\mathbb{E}V)$ are well defined.

With these preliminaries, similarly to the proof of Theorem 1, we can show

$$\mathbb{E}V(t,x(t)) \le \mathbb{E}V_0 \exp\Big(-\int_{t_0}^{t} \bar{w}_1(\mathbb{E}V(s,x(s)))\mathrm{d}s\Big).$$

If the integral $\int_{t_0}^{t} \bar{w}_1(\mathbb{E}V(s,x(s)))\mathrm{d}s$ diverges, i.e., $\int_{t_0}^{t} \bar{w}_1(\mathbb{E}V(s,x(s)))\mathrm{d}s \to +\infty$ as $t\to+\infty$, then we get $\mathbb{E}V(t,x(t)) \to 0$ as $t\to+\infty$. According to the assumption $u(\|x(t)\|^p) \le V(t,x(t)) \le v(\|x(t)\|^p)$, obviously $\mathbb{E}\|x(t)\|^p \to 0$ as $t\to+\infty$; If the integral $\int_{t_0}^{t} \bar{w}_1(\mathbb{E}V(s,x(s)))\mathrm{d}s$ converges, then the integral $\int_{t_0}^{t} \bar{w}_1(u(\mathbb{E}\|x(s)\|^p))\mathrm{d}s$ converges too, due to $0 \le \bar{w}_1(u(\mathbb{E}\|x(t)\|^p)) \le \bar{w}_1(\mathbb{E}V(t,x(t))$. By Lemmas 3–4 and $\mathbb{E}\|x(t)\|^p \le \bar{X}_0$, then we have the uniform continuity of $\mathbb{E}\|x(t)\|^p$, so is $\bar{w}_1(u(\mathbb{E}\|x(t)\|^p))$. Based on Barbalat Lemma 2, we know that the integrand $\bar{w}_1(u(\mathbb{E}\|x(t)\|^p)) = k(V_0)w(u(\mathbb{E}\|x(t)\|^p)) \to 0$ as $t\to+\infty$, by the assumptions on $w(\cdot)$ and $u(\cdot)$, this leads to $\mathbb{E}\|x(t)\|^p \to 0$ as $t\to+\infty$. The proof is complete. □

For a special case, we have the following concrete result with decay rate for the solutions.

Theorem 4. *Assume $p\ge 2$. Let $\zeta(t)$ be a non-negative continuous function and $c_1(t), c_2(t)$ be positive continuous functions on I. Assume that there exists a Lyapunov function $V:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^+$ such that*

(1) *$c_1(t)\|x\|^p \le V(t,x) \le c_2(t)\|x\|^p$ for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$;*
(2) *$\mathbb{E}\mathscr{L}V(t,\phi) \le -\zeta(t)\mathbb{E}V(t,\phi(0))$, if $\mathbb{E}V(t+\theta,\phi(\theta)) \le q(t)\mathbb{E}V(t,\phi(0))$, $\forall\ \theta\in I_\tau$, $q(t) \ge \exp(\int_{t-\tau}^{t}\zeta(s)\mathrm{d}s)$, $\forall\ t\ge t_0$, then for all initial condition $\phi_0\in C(I_\tau;\mathbb{R}^n)$ we have*

$$\mathbb{E}\|x(t)\|^p \le K\mathbb{E}|\phi_0|^p \exp\Big(-\int_{t_0}^{t}\zeta(s)\mathrm{d}s\Big),\ \ \forall\ t\in I,$$

where $K = \sup_{t\in I}\left\{\frac{\sup_{\theta\in I_\tau} c_2(t_0+\theta)}{c_1(t)}\right\} > 0$.

Proof. Denote $\mathcal{V}(t) = \mathbb{E}V(t,x(t))$, then by the result of Corollary 3 and with the inequality $c_1(t)\|x\|^p \le V(t,x) \le c_2(t)\|x\|^p$, we directly obtain the result of the theorem. The proof is complete. □

Using Theorem 4, we also have the following result.

Corollary 5 [20]**.** *Let ζ, p, c_1, c_2 all be positive numbers and $p>1$. Assume that there exists a Lyapunov function $V:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^+$ such that*

(1) *$c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$ for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$;*
(2) *$\mathbb{E}\mathscr{L}V(t,\phi) \le -\zeta\mathbb{E}V(t,\phi(0))$, if $\mathbb{E}V(t+\theta,\phi(\theta)) \le q\mathbb{E}V(t,\phi(0))$, $\forall\ \theta\in I_\tau$, then for all initial condition $\phi_0\in C(I_\tau;\mathbb{R}^n)$ we have*

$$\mathbb{E}\|x(t)\|^p \le K\mathbb{E}|\phi_0|^p \exp(-\gamma(t-t_0)),\ \ \forall\ t\in I,$$

where $\gamma = \min\{\zeta, \frac{\ln q}{\tau}\}$ and $K = \frac{c_2}{c_1}$.

Proof. By the given condition, if $\mathbb{E}V(t+\theta,\phi(\theta)) \le q\mathbb{E}V(t,\phi(0))$ for all $\theta\in I_\tau$, we also have $\mathbb{E}\mathcal{L}V(t,\phi) \le -\gamma\mathbb{E}V(t,\phi(0))$. By Theorem 4, we directly obtain the result of the corollary. The proof is complete. □

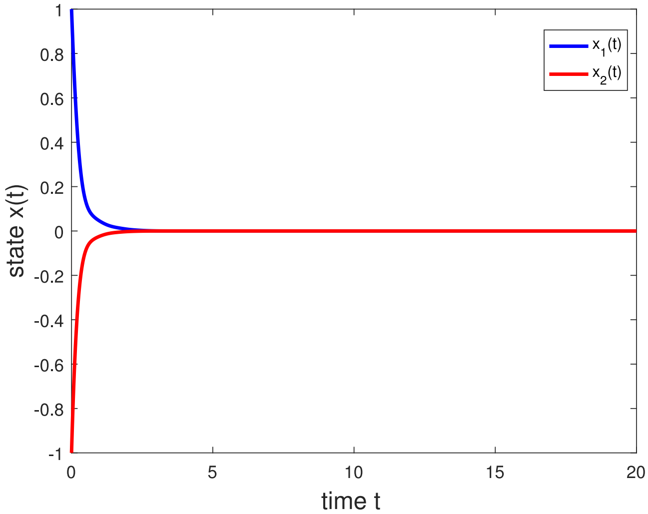

Fig. 1. Trajectory of x of the Eq. (8).

6. Illustrative examples

In this section, we consider two models described by a deterministic functional differential equation and a stochastic functional differential equation to verify the effectiveness of our established theorems, respectively.

Example 1. Consider a two-dimensional deterministic differential equation with a time-varying delay

$$\begin{cases} \dot{x}(t) & -3mm = A(t)x(t) + B(t)x(t-\tau(t)), \ t \geq t_0, \\ x_{t_0}(\theta) & -3mm = \phi_0(\theta), \ \theta \in I_\tau, \end{cases} \tag{8}$$

where $A \in C(\mathbb{R}^+; \mathbb{R}^{2\times2}), B \in C(\mathbb{R}^+; \mathbb{R}^{2\times2})$ are continuous matrix satisfying

$$A(t) = -\begin{bmatrix} 4 + 2\sin t + 2t & \frac{1}{t+1} \\ \frac{1}{t+1} & 5 + 2\sin t + 2t \end{bmatrix}, \ B(t) = \begin{bmatrix} \sin t + t & \frac{1}{2} \\ \frac{1}{2} & \sin t + t \end{bmatrix},$$

and $\tau(\cdot)$ is a continuous function with $0 \leq \tau(t) \leq \tau =$ constant.

Define Lyapunov function $V(x) = \left(x^{\mathrm{T}}x\right)^{\frac{1}{2}}$, denote $V(t) = V(x(t))$, we have

$$\begin{aligned} \dot{V}(t) &= \frac{1}{2}\left(x^{\mathrm{T}}(t)x(t)\right)^{-\frac{1}{2}} 2x^{\mathrm{T}}(t)\left[A(t)x(t) + B(t)x(t-\tau(t))\right] \\ &= \left(x^{\mathrm{T}}(t)x(t)\right)^{-\frac{1}{2}} x^{\mathrm{T}}(t)\left[A(t)x(t) + B(t)x(t-\tau(t))\right]. \end{aligned}$$

Based on the expressions of $A(t)$ and $B(t)$, we know that there exist two positive scalar continuous functions $a(t)$, $b(t)$ satisfying $a(t) = 3 + 2\sin t + 2t, b(t) = 1 + \sin t + t$ such that $A(t) \leq -a(t)I_{2\times2}$ and $B(t) \leq b(t)I_{2\times2}$, where $I_{2\times2}$ denotes a two-dimensional identity matrix. With this one has

$$\begin{aligned} \dot{V}(t) &= \left(x^{\mathrm{T}}(t)x(t)\right)^{-\frac{1}{2}} x^{\mathrm{T}}(t)\left[A(t)x(t) + B(t)x(t-\tau(t))\right] \\ &\leq -a(t)\|x(t)\| + |b(t)|\|x(t-\tau(t))\| \\ &= -a(t)V(t) + b(t)|V_t|. \end{aligned}$$

Let $q > 1$, and $V(t+\theta) \leq qV(t)$ for $\theta \in I_\tau$, then we have $|V_t| \leq qV(t)$. Under this assumption, we have $\dot{V}(t) \leq -(a(t) - qb(t))V(t)$. By Corollary 3, if $a(t) - qb(t) \geq 0$, then we have a bound for $V(t)$

$$V(t) = \|x(t)\| \leq V_0 \exp\left(-\int_{t_0}^{t}(a(s) - qb(s))\mathrm{d}s\right), \ t \geq t_0,$$

provided that $\exp(\int_{t-\tau}^{t}(a(s) - qb(s))\mathrm{d}s) \leq q$, where $V_0 = |\phi_0|$. For the parameters $a(t) = 3 + 2\sin t + 2t, b(t) = 1 + \sin t + t, t_0 = 0$ as well as $\tau(t) = \frac{2}{3}|\sin(t)|$.

Choose $q = 2$, we can verify that $\exp(\int_{t-\tau}^{t}(a(s) - qb(s))\mathrm{d}s) = e^{\tau} = 1.9477 < 2$ and then we have $\|x(t)\| \leq |\phi_0|e^{-t}, t \geq 0$.

We give a simulation in Fig. 1, with the initial condition $x(\theta) = [1 + \sin\theta \ -\cos\theta]^{\mathrm{T}}, \theta \in [-2/3, 0]$, and the step-size $h = 0.001$, which verifies our theory.

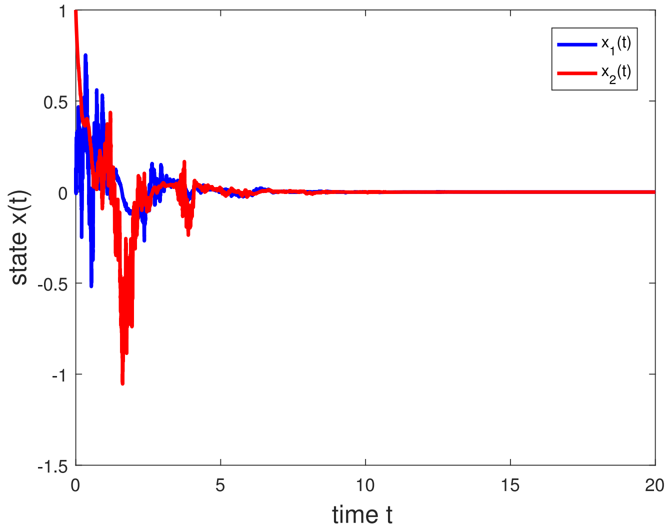

Fig. 2. Trajectory of x of the Eq. (9).

Example 2. Consider a two-dimensional nonlinear stochastic differential equation with a time-varying delay

$$\begin{cases} dx(t) = (A + B(x(t))x(t)dt + C(t)x(t-\tau(t))dw(t), \ t \geq t_0, \\ x_{t_0}(\theta) = \phi_0(\theta), \ \theta \in I_\tau, \end{cases} \tag{9}$$

where $A \in \mathbb{R}^{2\times 2}$, matrix functions $B \in C(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, $C \in C(\mathbb{R}^+; \mathbb{R}^{2\times 2})$ satisfy

$$A = \begin{bmatrix} -b & 0 \\ 0 & -b - \frac{1}{2} \end{bmatrix}, \ b > \frac{7}{2},$$

$$B(x) = \begin{bmatrix} -x^{\mathrm{T}}x & 1 \\ 4 & -x^{\mathrm{T}}x \end{bmatrix}, \ C(t) = \begin{bmatrix} \sqrt{2}\cos t & 0 \\ 0 & \sqrt{2}\sin t \end{bmatrix},$$

and $\tau(t)$ is a continuous function with $0 \leq \tau(t) \leq \tau =$ constant.

Define $V(x) = x^{\mathrm{T}}x$, and denote $V(t) = V(x(t))$, then we have

$$\begin{aligned}
&\mathscr{L}V(t, x_t) \\
=&2x^{\mathrm{T}}(t)(A + B(x(t))x(t) + x^{\mathrm{T}}(t-\tau(t))\begin{bmatrix} 2\cos^2 t & 0 \\ 0 & 2\sin^2 t \end{bmatrix}x(t-\tau(t)) \\
=&2x^{\mathrm{T}}(t)Ax(t) + 2x^{\mathrm{T}}(t)B(x(t))x(t) + x^{\mathrm{T}}(t-\tau(t))\begin{bmatrix} 2\cos^2 t & 0 \\ 0 & 2\sin^2 t \end{bmatrix}x(t-\tau(t)) \\
\leq&-2bx^{\mathrm{T}}(t)x(t) + \lambda_{\max}\big(B^{\mathrm{T}}(x(t)) + B(x(t))\big)x^{\mathrm{T}}(t)x(t) + 2x^{\mathrm{T}}(t-\tau(t))x(t-\tau(t)) \\
\leq&-2V^2(t) - 2(b - \frac{5}{2})V(t) + 2V(t-\tau(t)).
\end{aligned}$$

Take $q = b - \frac{5}{2}$, then we have $q > 1$, and then $\mathbb{E}\mathscr{L}V(t, x_t) \leq -2\mathbb{E}V^2(t) - 2(b - \frac{5}{2})\mathbb{E}V(t) + 2q\mathbb{E}V(t)$ whenever $\mathbb{E}V(t+\theta) \leq q\mathbb{E}V(t)$ for $\theta \in I_\tau$, namely

$$\big(\mathbb{E}V(t)\big)' \leq -2\Big((\mathbb{E}V(t))^2 + (b - q - \frac{5}{2})\mathbb{E}V(t)\Big) = -2(\mathbb{E}V(t))^2.$$

Denote $\mathcal{V}(t) = \mathbb{E}V(t)$, we have $w(\mathcal{V}) = 2\mathcal{V}^2$, which is increasing with $\mathcal{V} > 0$. By Theorem 3 with $p = 2$, the trivial solution $x = 0$ of the Eq. (9) is mean square asymptotically stable.

The simulation result is shown in Fig. 2 with $\tau = 2$, $b = 4$, the initial condition $x(\theta) = [\sin\theta \ \cos\theta]^{\mathrm{T}}$, $\theta \in [-\tau, 0]$, $t_0 = 0$, $x(0) = 1$, and the step-size $h = 0.001$, which verifies our theory.

What is interesting is that, the stability criterion here is time delay independent. This is a feature of the stochastic systems with delayed diffusive terms.

7. Conclusion

In this paper, we explicitly propose the notion of Razumikhin-type functional differential inequalities and establish the fundamental results on the quantitative bounds for the Razumikhin-type functional differential inequalities. By these quantitative bounds, Razumikhin-type stability results are deduced for both deterministic functional differential equations and Itô stochastic functional differential equations. It should be pointed out that the quantitative bounds established in this paper may not be the direct or the final bounds for the solutions of Razumikhin-type functional differential inequalities, thus maybe they can not provide concrete quantitative information for solutions, but with these bounds, we can deduce the Razumikhin-type stability theorems. This means that we have given a new approach to deduce the Razumikhin-type stability results and we show that this approach is more direct than the classical methods in the existing literature. By a careful observation, one may find that the Razumikhin condition $V(t+\theta,\phi(\theta))\le qV(t,\phi(0))$ can be replaced by $V(t+\theta,\phi(\theta))<qV(t,\phi(0))$. Of course, it does not make difference for our investigation, so it is unnecessary. Finally we also point out that, if the negative-definite conditions are given as $\dot{V}(t,x)\le-\omega(x)$, with the method of this paper, we can derive the Razumikhin version of the LaSalle's invariance principle.

Declaration of Competing Interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

CRediT authorship contribution statement

Xueyan Zhao: Conceptualization, Methodology, Software, Writing - original draft. **Minyue Fu:** Writing - review & editing. **Feiqi Deng:** Writing - original draft, Formal analysis, Validation, Writing - review & editing. **Qigui Yang:** Supervision.

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