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Comments on "Estimating the Robust Dead Time for Closed-Loop Stability"

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Abstract—**This short note illustrates how easy it is to find the upper bound on the robust dead time for closed-loop stability as compared to a recently proposed method for obtaining only an estimate of it.**

The above paper[1] does not have much practical use because the exact value of the robust dead time for closed-loop stability can be determined by using the methods recently developed in [1] and [2] and the references cited therein. The methods provided in [1] and [2] to find the exact value of the robust dead time are much easier to use than the one given in the above[1] for obtaining only an estimate, as illustrated herewith for the three examples used by El-Sakkary. The open-loop transfer function is of the form $e^{-sT}L(s)$ with T being the delay time (dead time).

Example 1: $L(s) = 1/s$. For $T = 0$, the only closed-loop pole $s = -1$ is the left half-plane. Using the method in [1], we find that the closed-loop poles cross the imaginary axis at $s = \pm j1$ when $T = (2n + 1/2)\pi$, $n = 0, 1, 2, \cdots$. Therefore, the upper bound on the robust dead time for closed-loop stability is $T = \pi/2$.

Example 2: $L(s) = 1/s(s + 1)$. The closed-loop poles for $T = 0$, $s = (-1 \pm j\sqrt{3})/2$ are in the left half-plane. Again, using the method in [1], we find the closed-loop poles on the imaginary axis at $s = \pm j0.78615$ when $T = 1.15061 + 2n\pi/0.78615$, $n = 0, 1, 2, \cdots$. Therefore, the upper bound on the robust dead time for closed-loop stability is $T = 1.15061$. (Notice the typographical error in the above[1] for this exact value.)

Example 3: $L(s) = 1/(s + 1)$. The closed-loop pole for $T = 0$ is $s = -2$, which is in the left half-plane. Using the method in [1], we determine that there are no closed-loop poles touching the imaginary axis and all poles will remain in the left half-plane for $0 \leq T < \infty$.

The above three examples have demonstrated how easy it is to find the exact value of the robust dead time for closed-loop stability as compared to the complicated approach by El-Sakkary to obtain only an estimate of it. It is noted that in the above,[1] Ω should be defined as the set of all positive frequencies w such that $M(w) \geq 0.5$ where $M(w)$ is the magnitude of $L(jw)/(1 + L(jw))$. The many typographical errors[1] also make the paper hard to follow.

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Author's Reply

AHMED K. EL-SAKKARY

The main result of the above paper[1] can be restated correctly as follows.

For $M(\omega) = |L(j\omega)/1 + L(j\omega)|$ and $\Omega = \{\omega \geq 0: M(\omega) \geq 0.5\}$, the robust dead time T for closed-loop stability can be found by determining the infinum of $(2/\omega)\sin^{-1}(1/2M(\omega))$ over all $\omega \in \Omega$.

The main emphasis in the above work[1] was to separate the parameter T from the rest of the system components which should be valuable for design purposes. The value of T can be estimated using the graph of a scalar function. The method is independent of the order of the system and can be a valuable tool for design of time delay system compensators. These facts have been overlooked by Zhou and Jong.

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Comments on "Stability of a Polytope of Matrices: Counterexamples"

Y. C. SOH AND Y. K. FOO

Abstract—**In this note, it is shown that the counterexample given in the above paper[1] does not invalidate the conjecture about the necessary and sufficient conditions for the stability of interval matrices. Some comments on the relationship between interval matrices and polytopes of polynomials are also given in this note.**

I. Introduction

Over the recent years, there has been a considerable amount of interest in the area of dynamical interval systems. In particular, there have been significant breakthroughs in the robust stability results for a family of

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[1] B. R. Barmish, M. Fu, and S. Saleh, *IEEE Trans. Automat. Contr.*, vol. 33, pp. 569-572, June 1988.

polynomials. For example, Kharitonov [1] has shown that a family of interval polynomials is stable if and only if four specially constructed extreme polynomials are stable. In fact, it has been shown that a family of interval polynomials will have all its zeros within a Kharitonov region if and only if all the extreme polynomials have their zeros within the region [2]–[4]. For the case where the family of polynomials can be described as a polytope of polynomials, we have the Edge Theorem to test the zero locations of the entire polytope of polynomials [5], [6]. Basically, the Edge Theorem states that the polytope of polynomials is D stable if and only if all the exposed edges of the polytope of polynomials are D stable where D is a region in the complex plane with certain properties [6].

While the robust stability results for a family of polynomials is encouraging, the robust stability problem for a family of matrices is far from completely solved. Bialas [7] made the first attempt to extend the robust stability result of interval polynomials to interval matrices. However, as the counterexamples in [8], [9] indicate, the robust stability of interval matrices cannot be inferred from all its vertex matrices. So the next logical step is to examine whether the stability of all the edge matrices is both necessary and sufficient to determine the stability of interval matrices. For the case of a more general polytope of matrices, it has been shown[1] by means of a counterexample that the stability of the edge matrices is not sufficient to determine the stability of the polytope of matrices. Barmish *et al.*[1] then proceed to show that the stability of the edge matrices is also not sufficient to determine the stability of interval matrices. However, as we shall show in the following section, the counterexample given in the above[1] does not invalidate the conjecture regarding the necessary and sufficient condition for the stability of interval matrices. The reason is that Barmish *et al.* have failed to check all the edge matrices.

II. Necessary and Sufficient Condition for Stability of Interval Matrices

Suppose that we have a family of n-dimensional interval matrices described by

$$A^- \le A \le A^+ \tag{2.1}$$

where $A^- \le A \le A^+$ implies that $a_{ij}^- \le a_{ij} \le a_{ij}^+, i, j = 1, 2, \cdots, n$. We denote by V_k, $k = 1, 2, \cdots, 2^{n^2}$ its vertex matrices where $[V_k]_{ij}$ is either a_{ij}^- or a_{ij}^+. Furthermore, we shall define all its edge matrices as

$$E_l \triangleq \alpha V_i + (1-\alpha)V_j, \qquad i, j = 1, 2, \cdots, 2^{n^2}. \tag{2.2}$$

In an attempt to show that the stability of all E_l is not sufficient to determine the stability of (2.1), Barmish *et al.* use the following interval matrix as the example:

$$M = \begin{bmatrix} m_{11} & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1.00 & 0.5 \\ 0.25 & -4.0 & -1.03 & 0 \\ 0 & 0.5 & 0 & m_{44} \end{bmatrix} \tag{2.3}$$

where

$$-1.5 \le m_{11} \le -0.5 \text{ and } -4.0 \le m_{44} \le -1.0.$$

For the above interval matrix, all the vertex matrices are

$$V_1 = M|_{m_{11}=-0.5,\, m_{44}=-1.0}$$
$$V_2 = M|_{m_{11}=-0.5,\, m_{44}=-4.0}$$
$$V_3 = M|_{m_{11}=-1.5,\, m_{44}=-1.0}$$
$$V_4 = M|_{m_{11}=-1.5,\, m_{44}=-4.0}$$

and all the edge matrices are

$$E_1 = \alpha V_1 + (1-\alpha)V_2 = M|_{m_{11}=-0.5,\, m_{44}=-4.0+3\alpha}, \qquad \alpha \in [0, 1]$$
$$E_2 = \alpha V_1 + (1-\alpha)V_3 = M|_{m_{11}=-1.5+\alpha,\, m_{44}=-1.0}, \qquad \alpha \in [0, 1]$$
$$E_3 = \alpha V_2 + (1-\alpha)V_4 = M|_{m_{11}=-1.5+\alpha,\, m_{44}=-4.0}, \qquad \alpha \in [0, 1]$$
$$E_4 = \alpha V_3 + (1-\alpha)V_4 = M|_{m_{11}=-1.5,\, m_{44}=-4.0+3\alpha}, \qquad \alpha \in [0, 1]$$
$$E_5 = \alpha V_1 + (1-\alpha)V_4 = M|_{m_{11}=-1.5+\alpha,\, m_{44}=-4.0+3\alpha}, \qquad \alpha \in [0, 1]$$
$$E_6 = \alpha V_2 + (1-\alpha)V_3 = M|_{m_{11}=-1.5+\alpha,\, m_{44}=-1.0-3\alpha}, \qquad \alpha \in [0, 1].$$

The matrices E_1, E_2, E_3, and E_4 correspond to case 1, case 2, case 4, and case 3, respectively, in the above paper.[1] However, in Barmish *et al.*, E_5 and E_6 were not considered. It has been shown there that E_i, $i = 1, 2, 3, 4$ are stable. If we let $\alpha = 0.5$ in E_5, then the eigenvalues of $E_5|_{\alpha=0.5}$ are -2.6215, -1.9411, $0.0013 - j\,1.3836$, and $0.0013 + j\,1.3836$. Hence, E_5 is not stable. Note also that $E_6 = E_5$ at $\alpha = 0.5$. Thus, E_6 is also not stable. Since some of the edge matrices of (2.3) are not stable, interval matrix (2.3) is not stable.

From the above result, we can conclude that the counterexample given in the above[1] does not invalidate the conjecture that the stability of all edge matrices of interval matrices is both necessary and sufficient for the stability of interval matrices. Further research is required to test the validity of the conjecture.

III. Interval Matrices and Polytopes of Polynomials

Since there are some strong results regarding the robust stability of a family of polynomials, it is natural to transform the problem of checking the robust stability of interval matrices to that of checking the stability of a polytope of polynomials. In this case, the polytope of polynomials is constructed from the convex hull of all the characteristic polynomials of the vertex matrices of the interval matrices. Although it has been shown[1] that the stability of the polytope of polynomials is not a necessary condition for the stability of the interval matrices, the sufficiency part of the result has been proved by Soh and Foo [10]. That is, the set of characteristic polynomials of interval matrices is a subset of the convex hull of the characteristic polynomials of the vertex matrices. While the result is not necessary, it is expected to be better than the currently available sufficient results on the stability of interval matrices [11]–[14].

Unfortunately, the result presented in [10] is only valid for interval matrices. It would be of more practical interest and importance if a corresponding result can be derived for a more general polytope of matrices. However, as the following counterexample indicates, such extension is not possible.

Example 3.1: Consider the polytope of matrices

$$S_A \triangleq \text{conv}\,\{A_1, A_2\} \tag{3.1}$$

where

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}.$$

The characteristic polynomials of A_1 and A_2 are

$$f_1(s) = s^2 - (a_1 + a_2)s + (a_1 a_2)$$

and

$$f_2(s) = s^2 - (b_1 + b_2)s + (b_1 b_2),$$

respectively. Clearly, $\text{conv}\{f_1(s), f_2(s)\}$ can be represented as a straight line in $\mathbb{R}^2$. However, for any $A \in S_A$, the characteristic polynomial is given by

$$f(s) = s^2 - (\alpha[a_1 + a_2] + (1-\alpha)[b_1 + b_2])s + (\alpha a_1 + (1-\alpha)b_1)(\alpha a_2 + (1-\alpha)b_2)$$

for $\alpha \in [0, 1]$. Thus, $f(s)$ can be represented as a curved line in $\mathbb{R}^2$. This implies that $f(s) \notin \text{conv}\{f_1(s), f_2(s)\}$. □

IV. Conclusions

The main purpose of this note is to show that the conjecture on the necessary and sufficient conditions for the stability of interval matrices is still a research topic to be investigated.

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Authors' Reply

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Conjecture 2 in our paper is motivated by the Edge Theorem in [1]. That is, viewing an interval matrix family as a rectangle $\mathbb{R}$ in $\boldsymbol{R}^{r^2}$, a natural conjecture is that stability of the exposed edges of $\mathbb{R}$ is both necessary and sufficient for the stability of $\mathbb{R}$. Our second counterexample shows that this conjecture is false. For the counterexample, the interval matrix family is a two-dimensional rectangle in $\boldsymbol{R}^{16}$, and it is quite easy to identify its exposed edges. They are E_1, E_2, E_3, and E_4, as given in Soh and Foo's comments. Note, however, that there is no need to check E_5 and E_6 (as Soh and Foo claim) since E_5 and E_6 are *not* exposed edges. Roughly speaking, they are "diagonals."

We accept blame for this misunderstanding because our wording in the Introduction was not sufficiently precise—hence, the confusion between "all pairwise combinations of extreme points" versus "exposed edges." We should have emphasized this difference for readers who are not totally familiar with the literature in this area.

However, note that even accepting Soh and Foo's interpretation, the conjecture remains false. That is, consider the interval matrix family $\boldsymbol{M}$ given in Conjecture 2 and form the new family

$$\boldsymbol{N} \triangleq \{M - 0.0026I : M \in \boldsymbol{M}\}$$

where $I \in \boldsymbol{R}^{4\times 4}$ is the identity matrix. It can be easily verified that all the edges (including the "diagonals") of $\boldsymbol{N}$ are stable, but the matrix corresponding to $q_1 = 0.436$ and $q_2 = 0.812$ is unstable.

We conclude our reply by noting that in Section III of Soh and Foo's comments, the authors "lay claim" to the following result: the set of characteristic polynomials associated with an interval matrix family is a subset of the convex hull of the characteristic polynomials of the vertex matrices. Note that this result is not new and is immediate from the so-called Mapping Theorem, e.g., see [2]. In fact, it has already appeared in the literature, e.g., see [3]. Furthermore, this result is also clearly noted in our paper[1] in the remark at the end of Section I and the discussion in the beginning of Section IV. In particular, we provide the formula

$$\text{conv } \boldsymbol{P_M} = \left\{ p_\lambda(s) = \sum_{i=1}^{m} \lambda_i p_i(s) : \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

where $\boldsymbol{P_M}$ is the set of characteristic polynomials associated with the interval matrix $\boldsymbol{M}$ and $p_i(s)$ are the characteristic polynomials of the vertex matrices.

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Correction to "An Approach for Pole Assignment in Singular Systems"

AMIT AILON

In a recent paper,[1] the Remark which appears in the second line after (2.16) was presented incorrectly.

Part (a) of that Remark should read as follows:

The condition $\boldsymbol{x}_0 \in R_0$ stated in Corollary 2.1 is essential. If $\boldsymbol{x}_0 \notin R_0$, (2.15) does not necessarily hold for $t \geq 0$.

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