1 Approximation of Complex μ

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1.1 Description of the problem

Given a matrix $M \in \mathbf{C}^{n\times n}$ and a set of positive integers $\mathcal{X} = (k_1, \cdots, k_m)$ with $k_1 + \cdots + k_m = n$, the so-called *complex structured singular value* $\mu_{\mathcal{X}}(M)$, (complex μ for short), is defined as follows:

$$\mu_{\mathbf{\Delta}}(M) = \inf \left\{ \rho : \rho > 0, \det(I_n - \rho^{-1}\Delta M) \neq 0,\ \forall\ \Delta \in B(\mathbf{\Delta}) \right\} \tag{1}$$

where

$$B(\mathbf{\Delta}) = \left\{ \Delta = \mathrm{diag}\{\Delta_1, \cdots, \Delta_m\} \mid \Delta_i \in \mathbf{C}^{k_i \times k_i}, ||\Delta_i|| \leq 1 \right\} \tag{2}$$

Let $\hat{\mu}$ be an approximation of μ. We call $\hat{\mu}$ an r-approximation, $r > 0$, if either

$$\mu \leq \hat{\mu} \leq (1+r)\mu \tag{3}$$

or

$$\frac{\mu}{1+r} \leq \hat{\mu} \leq \mu \tag{4}$$

Note that $\hat{\mu}$ is an upper bound in the former case and an lower bound in the latter.

We are interested in the computational complexity of the problem of approximating the complex μ. More specifically, we ask the following questions:

1. Does there exist some (arbitrarily small) constant $\epsilon > 0$ such that the problem of finding an ϵ-approximation for the complex μ is NP-hard?

2. For any (arbitrarily large) constant $R > 0$, is the problem of finding an R-approximation for the complex μ is NP-hard?

1.2 Motivations

The complex μ problem arises in robustness stability and robust performance problems where systems uncertainties and performance measures can be captured by the structure of $\mathcal{D}$. This problem was first formally proposed by Doyle in 1982 [1] where the so-called D-scaling method was introduced for computing an upper bound for μ. Many simulation results have been conducted by Doyle and his colleagues to demonstrate that the D-scaling method provides good approximaitons (with relative error no larger than 20%). However, there is no theoretical result supporting the simulation results. It is worth to know that the D-scaling method gives a polynomial algorithm.

In 1993, Megretski [2] showed that the D-scaling method gives a relative error r which grows at most linearly as the function of n, provided that $\mu \neq 0$. That is, the problem of finding a linearly growing r-approximation for the complex μ is a polynomial problem, provided $\mu \neq 0$.

The computational complexity of the μ problem became of interest since early 90's. A number of authors studied the so-called *real* μ problem where the complex blocks D_i are replaced with the so-called *repeated real blocks* $\delta_i I_{k_i}$, $\delta_i \in \mathcal{R}$, and the so-called *mixed* μ problem where there are real repeated blocks, complex blocks, and the so-called *repeated complex blocks* which are similar to the repeated real blocks except δ_i are complex variables. Coxson and DeMarco [3] (among many other people) showed that the real μ problem (and hence the mixed μ problem) is NP-hard. More specifically, the problem of determining if $\mu < 1$ is NP-complete. Subsequently, there exists no polynomial algorithm for computing μ, unless the commonly believed conjecture, $P \neq NP$, fails. The computational complexity of the complex μ is much harder to analyze than the real μ problem. Recently, Toker and Ozbay [4] used an elegant technique to show that the complex μ problem is still NP-hard.

Knowing that the problem of computing μ is NP-hard, the next logical question is how "hard" it is to approximate μ. To this end, a result in Coxson and DeMarco [3] shows that there exists some arbitrarily small $\epsilon > 0$ such that ϵ-approximation for the real μ is also NP-hard. Toker [5] offers a more negative answer for the real μ problem by proving that computing a $Cn^{1-\epsilon}$-approximation with some (very large) constant $C > 0$ and (very small) $\epsilon > 0$ is an NP-hard problem. Fu in a very recent paper [6] gives the following most negative result: The problem of $r(n)$-approximation for the real μ is NP-hard for any $r(n) > 0$. That is, the realtive approximation error can grow arbitrarily fast for any polynomial algorithm, unelss $P = NP$. The result in [6] is further extended by Fu and Dasgupta [7] to the case where the real blocks are bounded using an arbitrary p-norm rather than ∞-norm and imilar NP-hard results are obtained.

However, the computational complexity analysis for approximation of the complex μ appears to be much more involved. Hence, we invite the reader to study the proposed questions. We conjecture that the answers to both questions are affirmative. The first question appears to be easier to answer than the second one. So we rank the first question "moderate" and the second one "difficult".

References

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