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TEST OF CONVEX DIRECTIONS FOR ROBUST STABILITY *

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Abstract. We address the stability problem of a segment of polynomials. The polynomial which defines the direction of the segment is called a *convex direction* if the stability of the whole segment is implied by that of its extreme members, regardless where the segment lies. Such a property plays an important role in robust stability analysis, and a necessary and sufficient condition, called phase growth condition, has been given by Rantzer. In this paper, we provide alternative necessary and sufficient conditions which will allow us to determine in a finite number of rational operations whether a given polynomial is a convex direction.

1. INTRODUCTION

This paper is concerned with the robust stability of a segment of polynomials in the form

$$p_0(s) + \lambda p(s), \quad \lambda \in [0,\ 1] \tag{1}$$

where $p_0(s)$ and $p(s)$ are given polynomials, $p_0(s)$ is referred to as the nominal polynomial and $p(s)$ represents the direction of the segment. One of the important robust stability problems is under what conditions does the stability of the extreme members of the segment implies the stability of the whole segment. Such an "extreme point property" plays an important role in both robust stability analysis and robust synthesis for systems with real uncertain parameters. Examples of extreme point results range from the well celebrated Kharitonov theorem [1] to a recent synthesis result on interval plants [2]; see [3] for an excellent review.

A notion of convex direction is proposed by Rantzer [4] to study the extreme point property, and it is defined as follows: a polynomial $p(s)$ is called a convex direction if for any $p_0(s)$ with $\deg(p_0(s)) > \deg(p(s))$, the stability of both $p_0(s)$ and $p_0(s) + p(s)$ implies that of every $p_0(s) + \lambda p(s)$, $0 \le \lambda \le 1$.

Some sufficient conditions for convex directions are found in [5, 6, 7, 8] although the terminology is not used in these references. It is shown by Petersen [5] that a polynomial $p(s)$ is a convex direction if it is antistable (i.e., all the zeros of $p(s)$ are outside of the stability region). This result is generalized by Fu [6] and Rantzer [7] to allow the $p(j\omega)$ to have nonincreasing phase (as ω increases). Hollot and Yang [8] show that any first order polynomial is a convex direction, this result is then used by Barmish *et. al.* to design lead/lag compensators of interval plants [2]. A remarkable result is given by Rantzer [4] which shows that a polynomial is a convex direction if and only if its phase velocity is bounded by certain positive function. This condition is referred to as the "phase growth condition".

Although Rantzer's phase growth condition is necessary and sufficient, it is not clear how to test this condition efficiently. In particular, we are interested to know whether the condition can be tested in a finite number of elementary operations[1] and without sweeping the frequencies. This computational aspect is important to the application of the condition in robust stability analysis and synthesis. To simplify the computation, Barmish and Kang [9] propose the so-called Alternating Hurwitz Minor Condition (AHMC) which allows one to test convex directions by simply using the minors of a Hurwitz matrix. The disadvantage of this condition is that it is only sufficient. Indeed, we will show that this condition is necessary and sufficient for third or lower order polynomials, but not necessary for fourth or higher order ones.

In this paper, we propose alternative necessary and sufficient conditions for convex directions and use them to devise a computational procedure which determines if a polynomial is a convex direction in a finite number of rational operations. We first show that the phase growth condition can be represented in terms of certain nonnegativeness property of two polynomials. Each of these nonnegativeness conditions is then converted into checking the positivity of one polynomial at some zeros of another polynomial, a problem which can be solved by using the classical Sturm's theorem. Thus, the convex direction property can be determined in a finite number of elementary operations. We emphasize this computational procedure because the operations are done on the coefficients of the polynomial, thus good numerical accuracy can be guaranteed.

In addition to the above results, we also provide some useful necessary conditions for convex directions. We show that the zeros of even and odd part of a convex direction,

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[1] Elementary operations include arithmetic operations (addition, subtraction, multiplication and division), logical operations ("and", "or") and sign tests ("$a > b$ and $a = b$).

provided that they are coprime (an assumption without loss of generality), must be of odd multiplicity and interlacing in a certain way. Further, we analyze the conditions for low order convex directions.

The rest of the paper is organized as follows: Section 2 presents some preliminary results on convex directions and a short review on Cauchy indices. Section 3 provides necessary and sufficient conditions for convex directions which can be tested by using Cauchy indices. Also given in Section 3 are some useful necessary conditions for convex directions. A computational procedure is described in Section 4, along with an illustrating example. Conditions for low order convex directions are studied in Section 5.

2. PRELIMINARIES

This paper is concerned with the following robust stability problem: Given two real polynomials $p_0(s)$ and $p(s)$ with $\deg(p(s)) < \deg(p_0(s))$, determine whether the Hurwitz stability of $p_0(s)$ and $p_0(s)+p_1(s)$ implies that of all $p_0(s)+\lambda p(s)$, $0 \le \lambda \le 1$. In order to find out the conditions for such a convex property, Rantzer [4] introduces the concept of "convex directions" as follows:

Definition 1. [4] A real polynomial $p(s)$ is called a *convex direction* if for every real polynomial $p_0(s)$ with $\deg(p_0(s)) > \deg(p(s))$, the stability of $p_0(s)$ and $p_0(s)+p(s)$ implies that of $p_0(s)+\lambda p(s)$ for all $0 \le \lambda \le 1$.

The following necessary and sufficient condition for convex directions is given by Rantzer [4].

Lemma 1. *[4] A real polynomial $p(s)$ is a convex direction if and only if the following phase growth condition holds:*

$$\frac{d}{d\omega}\arg(p(j\omega)) \le \left|\frac{\sin(2\arg(p(j\omega)))}{2\omega}\right| \qquad (2)$$

for all frequencies $0 \le \omega < \infty$ at which the phase $p(j\omega) \ne 0$ (so that $\arg(p(j\omega))$ is well defined).

Write

$$p(s) = h(s)(f(-s^2)+sg(-s^2)) \qquad (3)$$

where $h(s)$ is either an even or odd polynomial, $f(-s^2)$ and $g(-s^2)$ are coprime, respectively. Without lose of generality, it is assumed throughout of the paper that $f(0) \ne 0$ (otherwise, an additional factor of s can be absorbed in $h(s)$). If either $f(-s^2)$ or $g(-s^2)$ is identically zero, the coprimeness of $f(-s^2)$ and $g(-s^2)$ should be understood in such a way that the other polynomial is a constant. We have the first result of the paper:

Theorem 2. *A given real polynomial $p(s)$ in (3) is a convex direction if and only if for every frequency $\omega > 0$, the value of either of the following two polynomials is nonnegative:*

$$\alpha(x) = f'(x)g(x) - f(x)g'(x) \qquad (4)$$

$$\beta(x) = x\alpha(x) - f(x)g(x) \qquad (5)$$

Equivalently, $p(s)$ in (3) is a convex direction if and only if both of the following conditions hold:

(i) $\alpha(x) \ge 0$ whenever $f(x)g(x) \ge 0$, $x > 0$; and

(ii) $\beta(x) \ge 0$ whenever $f(x)g(x) < 0$, $x > 0$.

See [12] for proof.

Defining complex polynomials

$$p_1(s) = f(-js) + jg(-js) \qquad (6)$$

$$p_2(s) = f(-js) + sg(-js) \qquad (7)$$

It can be verified that the phase velocities of $p_1(j\omega)$, $p_2(j\omega)$ and $p(j\omega)$ are given by

$$\frac{d}{d\omega}\arg(p_1(j\omega)) = -\frac{\alpha(\omega)}{|p_1(j\omega)|^2} \qquad (8)$$

$$\frac{d}{d\omega}\arg(p_2(j\omega)) = -\frac{\beta(\omega)}{|p_2(j\omega)|^2} \qquad (9)$$

$$\frac{d}{d\omega}\arg(p(j\omega)) = -\frac{\omega^2\alpha(\omega^2)+\beta(\omega^2)}{2|p(j\omega)|^2} \qquad (10)$$

This analysis yields a further alternative necessary and sufficient condition for convex directions:

Theorem 3. *A given real polynomial $p(s)$ in (3) is a convex direction if and only if the phase of $p_1(j\omega)$ is nonincreasing in the first and third quadrants and the phase of $p_2(j\omega)$ is nonincreasing in the second and forth quadrants as ω traverses from 0 to ∞.*

The main results to be presented in this paper involve establishing a relationship between the conditions in Theorem 2 and the Cauchy indices of some rational functions which can be tested in a finite number of rational operations.

Definition 2. [10] The Cauchy index of a real rational function $R(x)$ in a real interval (a,b) is denoted by $I_a^b R(x)$ and defined by the difference between the numbers of jumps of $R(x)$ from $-\infty$ to $+\infty$ and that of jumps from $+\infty$ to $-\infty$ as x traverses from a^+ to b^-, where a and b are real numbers or $\pm\infty$.

The Cauchy index can be computed by using the Sturm's theorem [10] which involves constructing a Sturm's chain (or called sequence). Let $f_1(x)$ and $f_2(x)$ be two real polynomials with $\deg(f_1(x)) > \deg(f_2(x))$. A Sturm's chain $\{f_1(x), f_2(x), \cdots, f_m(x)\}$ is constructed by polynomial division as follows:

$$f_i(x) = f_{i+1}(x)q_{i+1}(x) - f_{i+2}(x), \ i = 1, 2, \cdots, m-1 \quad (11)$$

where $q_{i+1}(x)$ and $-f_{i+2}(x)$ are the quotient and remainder of the division, respectively. Note that $\deg(f_{i+2}(x)) < \deg(f_{i+1}(x))$ and that the chain should be terminated when $f_{m+1}(x) = 0$. Then,

$$I_a^b \frac{f_2(x)}{f_1(x)} = V(a+\epsilon) - V(b-\epsilon) \quad (12)$$

where $V(x)$ is the number of sign variations of the Sturm's chain at x, and $\epsilon > 0$ is sufficiently small. Furthermore, $f_m(x)$ is a greatest common divisor of $f_1(x)$ and $f_2(x)$.

When $\deg(f_2(x)) \geq \deg(f_1(x))$, the following additional polynomial division is needed in the begining:

$$f_2(x) = f_1(x)q_0(x) + \hat{f}_2(x) \quad (13)$$

and the resulting $\hat{f}_2(x)$ should be used in place of $f_2(x)$. This is because (see Definition 2)

$$I_a^b \frac{f_2(x)}{f_1(x)} = I_a^b(q_0(x) + \frac{\hat{f}_2(x)}{f_1(x)}) = I_a^b \frac{\hat{f}_2(x)}{f_1(x)}$$

We emphasize that the construction can be done by using a Routh table; see [10], requiring only a finite number of rational operations on the coefficients of $f(x)$. The following results are well known.

Lemma 4. *[10] Given a real polynomial $f(x) \not\equiv 0$, then*

$$I_a^b \frac{f'(x)}{f(x)}, \quad a < b, \quad (14)$$

is equal to the number of distinct real roots of $f(x)$ in the interval (a, b).

Lemma 5. *[11] Given two real polynomials $f_1(x) \not\equiv 0$ and $f_2(x)$,*

$$I_a^b \frac{f_1'(x)f_2(x)}{f_1(x)}, \quad a < b, \quad (15)$$

is equal to the difference between the number of distinct real roots of $f_1(x)$ in (a, b) when $f_2(x)$ is positive and that when $f_2(x)$ is negative.

3. MAIN RESULTS

This section provides two alternative necessary and sufficient conditions for convex directions which can be determined in a finite number of rational operations. The alternative conditions involve testing the positivity or negativity of one polynomial at certain zeros of another polynomial, a problem solvable by using the Sturm's theorem. We also provide some simple necessary conditions for convex directions.

Theorem 6. *A given real polynomial $p(s)$ in (3) is a convex direction if and only if it belongs to either of the following two cases:*

(i) *both $f(x)$ and $g(x)$ are real constants;*

(ii) *All the following conditions hold:*

- (ii-1) *Either of the leading coefficients of $\alpha(x)$ and $\beta(x)$ is positive, where $\alpha(x)$ and $\beta(x)$ are defined in (4)-(5);*
- (ii-2) *$f(x)g(x) < 0$ for every zero $x = a > 0$ of $\alpha(x)$ with odd multiplicity;*
- (ii-3) *$f(x)g(x) > 0$ for every zero $x = a > 0$ of $\beta(x)$ with odd multiplicity.*

See [12] for proof.

Note that the conditions (ii-2) and (ii-3) above are the only ones which need further attention. The computational issue of these conditions will be discussed in the next section. Now we provide a modified version of Theorem 6.

Theorem 7. *A given real polynomial $p(s)$ in (3) is a convex direction if and only if it belongs to either of the following two cases:*

(i) *both $f(x)$ and $g(x)$ are real constants;*

(ii) *All the following conditions hold for all sufficiently small $\epsilon > 0$:*

- (ii-1) *Either of the leading coefficients of $\alpha(x)$ and $\beta(x)$ is positive, where $\alpha(x)$ and $\beta(x)$ are defined in (4)-(5);*
- (ii-2) *$f(x)g(x) < 0$ for every zero $x = a > 0$ of $\alpha(x) + \epsilon$;*
- (ii-3) *$f(x)g(x) > 0$ for every zero $x = a > 0$ of $\beta(x) + \epsilon$.*

Furthermore, if conditions (ii-1)-(ii-3) hold for $\epsilon = 0$, $p(s)$ is a convex direction.

See [12] for proof.

Remark 1. The tradeoff of Theorems 6 and 7 is clear: Theorem 6 needs to deal with multiplicities $\alpha(x)$ and $\beta(x)$'s zeros while Theorem 7 avoids doing so by introducing an ϵ. It should be mentioned that the introduction of ϵ is a standard technique for testing the nonnegativity of a polynomial, i.e., for converting the nonnegativity problem into a positivity one which is easier to test. In our case, we use it to deal with the nonnegativity of $\alpha(x)$ and $\beta(x)$.

The last result in this section deals with some necessary properties of convex directions. We show that the zeros of $f(x)$ and $g(x)$ must be of odd multiplicities and interlacing in certain way for $p(s)$ to be a convex direction. Furthermore, the orders of $f(x)$ and $g(x)$ are subject to certain constraints too. To this end, we first introduce the notation of *clockwise interlacing*.

Definition 3. Given a pair of coprime polynomials $(p_1(x), p_2(x))$, let $x_1 < \cdots < x_t$ and $y_1 < \cdots < y_v$ be the distinct real zeros of $p_1(x)$ and $p_2(x)$, respectively. These zeros are called clockwise interlacing if the following conditions hold:

(i) between every x_i and x_{i+1}, $1 \le i \le t-1$, there exists some y_j, $1 \le j \le v$, and that $p_2(y_j)p_1(x_i) > 0$ and $p_2(y_j)p_1(x_{i+1}) < 0$;

(ii) between every y_j and y_{j+1}, $1 \le j \le v-1$, there exists some x_i, $1 \le i \le t$, and that $p_1(x_i)p_2(y_j) < 0$ and $p_1(x_i)p_2(y_{j+1}) > 0$.

Remark 2. Conditions (i) and (ii) above basically mean that the plot of $p_1(x) + jp_2(x)$ in the complex plane intersect the real and imaginary axes alternatively in the clockwise direction as x traverses from 0 to ∞.

Theorem 8. *Suppose a given real polynomial $p(s)$ in (3) is a convex direction. Then the distinct zeros of $f(x)$ and $g(x)$ in $(0,\infty)$, if any, are of odd multiplicities and clockwise interlacing. Furthermore, $n \ge m$ if $f_0g_0 > 0$ and $n \le m+1$ if $f_0g_0 < 0$.*

See [12] for proof.

Remark 3. An alternative interpretation of Theorem 8 is that the phase of a convex direction $p(j\omega)$ as ω increases must intersect the real and imaginary axes alternatively, and the phase must be strictly decreasing at the crossing points. The condition of the relative degree of $f(x)$ and $g(x)$ is a restriction on the phase of $p(j\omega)$ at infinity.

4. COMPUTATIONAL PROCEDURE

This section discusses the computational aspect of the results in the previous section. As seen from Theorem 6, the key computational issue is the following problem: *Given two real polynomials $u(x)$ and $v(x)$ and an interval (a,b), determine if $v(x) > 0$ at every zero of $u(x)$ in (a,b) with odd multiplicity.* This problem can be solved by using Lemmas 4-5 in conjunction with the well-known Sturm's theorem [10]. The purpose of this section is to describe a computational procedure to this end.

Denote $u_1(x) = u(x)$ and by $u_i(x), i > 1$, a greatest common divisor of $u(x)$ and and its $(i-1)$th derivative $u^{(i-1)}(x)$. Then, $u_i(x)$ contains all zeros of $u(x)$ with multiplicities greater than or equal to i. A trivial property of $u_i(x), i > 1$ is that it is also a greatest common divisor of $u_{i-1}(x)$ and its derivative $u'_{i-1}(x)$.

We further denote by p_i, n_i and o_i the number of zeros of $u(x)$ in (a,b) with multiplicity i at which $v(x)$ is positive, negative, and zero, respectively. Then, we have the following lemma:

Lemma 9. *Given two polynomials $u(x)$ and $v(x)$, $v(x) > 0$ at every zero of $u(x)$ in (a,b) with odd multiplicity if and only if for every odd i,*

$$I_a^b \frac{u_i'(x)}{u_i(x)} - I_a^b \frac{u_i'(x)v(x)}{u_i(x)} = I_a^b \frac{u_{i+1}'(x)}{u_{i+1}(x)} - I_a^b \frac{u_{i+1}'(x)v(x)}{u_{i+1}(x)} \quad (16)$$

Proof. Using Lemmas 4-5, we know that the left hand side of (16) is equal to $\sum_{k \ge i} 2n_k + o_k$ and the right hand side is equal to $\sum_{k \ge i+1} 2n_k + o_k$. So their difference is $2n_i + o_i$ which is zero if and only if both n_i and o_i are zero. ▽▽▽

The result in Lemma 9 can be applied to Theorem 6 to test if a given polynomial is a convex direction. To this end, we define

$$I_{\alpha,i} = I_0^\infty \frac{\alpha_i'(x)}{\alpha_i(x)}, \quad J_{\alpha,i} = I_0^\infty \frac{\alpha_i'(x)f(x)g(x)}{\alpha_i(x)} \quad (17)$$

$$I_{\beta,i} = I_0^\infty \frac{\beta_i'(x)}{\beta_i(x)}, \quad J_{\beta,i} = I_0^\infty \frac{\beta_i'(x)f(x)g(x)}{\beta_i(x)} \quad (18)$$

Theorem 10. *A given real polynomial $p(s)$ in (3) is a convex direction if and only if it belongs to either of the following two cases:*

(i) *both $f(x)$ and $g(x)$ are real constants;*

(ii) *All the following conditions hold:*

(ii-1) *Either of the leading coefficients of $\alpha(x)$ and $\beta(x)$ is positive, where $\alpha(x)$ and $\beta(x)$ are defined in (4)-(5);*

(ii-2) *For all odd i:*

$$I_{\alpha,i} + J_{\alpha,i} = I_{\alpha,i+1} + J_{\alpha,i+1} \quad (19)$$

$$I_{\beta,i} - J_{\beta,i} = I_{\beta,i+1} - J_{\beta,i+1} \quad (20)$$

The proof is straightforward from Theorem 6 and Lemma 9, and is thus omitted.

Remark 4. As we mentioned in Section 2, the Cauchy indices in (17)-(18) can be computed by using the Sturm's theorem which involves constructing Sturm's chains. The last question we need to answer is how to compute $\alpha_i(x)$ and $\beta_i(x)$, $i > 1$, i.e., how to compute the greatest common divisor of two polynomials. Fortunately, the greatest common divisor falls out of the Sturm's chain automatically, as mentioned in Section 2. Hence, *the algorithm given in Theorem 10 involves only a finite number of elementary operations on the coefficients of $p(s)$ in order to determine whether $p(s)$ is a convex direction.*

Remark 5. In both (19)-(20), the left hand side is always greater than or equal to the right hand side, regardless whether $p(s)$ is a convex direction or not. Hence, if the left hand side turns out to be zero, no further verification is then necessary. This property can be used to terminate the test of $p(s)$. In particular, the sufficient condition in Theorem 7 (when $\epsilon = 0$) corresponds to this situation for $i = 1$.

Example To illustrate the computational procedure described above, we consider the following polynomial:

$$p(s) = s^6 + 3s^4 - 4s^3 + 3s^2 + s + 1 \tag{21}$$

The corresponding $f(x)$ and $g(x)$ are given by

$$f(x) = -x^3 + 3x^2 - 3x + 1 \tag{22}$$

$$g(x) = -4x + 1 \tag{23}$$

It is obvious that $f(x)$ and $g(x)$ are coprime because $f(1/4) \neq 0$.

The expression for $\alpha(x)$ and $\beta(x)$ are given by

$$\alpha_1(x) = \alpha(x) = 8x^3 - 15x^2 + 6x + 1$$

$$\beta_1(x) = \beta(x) = 4x^4 - 2x^3 - 9x^2 + 8x - 1$$

and their derivatives are given by

$$\alpha_1'(x) = 24x^2 - 30x + 6$$

$$\beta_1'(x) = 16x^3 - 6x^2 - 18x + 8$$

Using the Sturm's theorem [10], we obtain

$$I_{\alpha,1} = 1; \quad J_{\alpha,1} = 0; \quad I_{\beta,1} = 1; \quad J_{\beta,1} = 0$$

Since

$$I_{\alpha,1} + J_{\alpha,1} = 1 > 0; \quad I_{\beta,1} - J_{\beta,1} = 1 > 0$$

we must proceed. Again, by constructing a Sturm's chain starting from $\alpha_1(x)$ and $\alpha_1'(x)$, we obtain their common divisor

$$\alpha_2(x) = x^2 - 2x + 1; \quad \alpha_2'(x) = 2x - 1$$

Similarly, we get

$$\beta_2(x) = x^2 - 2x + 1; \quad \beta_2'(x) = 2x - 1$$

Applying the Sturm's theorem again, we obtain

$$I_{\alpha,2} = 1; \quad J_{\alpha,2} = 0; \quad I_{\beta,2} = 1; \quad J_{\beta,2} = 0$$

Since

$$I_{\alpha,1} + J_{\alpha,1} = I_{\alpha,2} + J_{\alpha,2}, \quad I_{\beta,1} - J_{\beta,1} = I_{\beta,2} - J_{\beta,2}$$

the conclusion is that $p(s)$ is a convex direction (see Theorem 10).

Incidentally, the AHMC does not apply in this example because two relevant hurwitz minors are given as follows (see [9] for notation):

$$\Delta_2 = 4 > 0; \quad \Delta_3 = -16 < 0.$$

5. LOW ORDER CONVEX DIRECTIONS

In this section, we point out that the AHMC in [9] is necessary and sufficient for third or lower order convex directions, but not necessary for higher order ones. Further, we demonstrate via an example that convex directions of fourth or lower order cannot be tested by the phase velocity of any of $p_1(j\omega), p_2(j\omega)$ or $p(j\omega)$ alone. It is known that all first order polynomials are convex directions [8]. So we proceed with second order polynomials.

Second Order Polynomials

Let $p(s) = s^2 + a_1 s + a_2$. Then,

$$f(x) = -x + a_2; \; g(x) = a_1$$

which are coprime if and only if $a_1 \neq 0$. If they are not coprime, $p(s)$ is obviously a convex direction because it is even. Otherwise,

$$\alpha(x) = -a_1; \quad \beta(x) = -a_1 a_2$$

which means that $p(s)$ is a convex direction if and only if either $a_1 \leq 0$ or $a_2 \leq 0$. Since this condition includes $a_1 = 0$, we conclude what is predicted by the AHMC [9]:

A second order polynomial $p(s) = s^2 + a_1 s + a_2$ is a convex direction if and only if either a_1 or a_2 is non-positive.

Third Order Polynomials

Let $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$. Then,

$$f(x) = -a_1 x + a_3; \quad g(x) = -x + a_2$$

Note that when $a_1 a_2 = a_3$, $f(x)$ and $g(x)$ are not coprime. In this case, $p(s)$ is a convex direction because both $f(x)$ and $g(x)$ becomes a constant after factoring out the common divisor. Now we suppose $a_1 a_2 \neq a_3$. Then,

$$\alpha(x) = -a_1 a_2 + a_3; \quad \beta(x) = -a_1 x^2 + 2a_3 x - a_3 a_2 \quad (24)$$

Obviously, $\alpha(x) \geq 0$ for all $x > 0$ if and only if $a_3 \geq a_1 a_2$. When this condition fails, we claim that $\beta(x) \geq 0$ for all $x > 0$ if only if $a_1 \leq 0$ and $a_3 \geq 0$. Indeed, if $a_3 < a_1 a_2$, $a_1 \leq 0$ and $a_3 \geq 0$, then $a_2 < 0$ and $\beta(x) \geq 0$ for all $x > 0$. Similarly, if $a_3 < a_1 a_2$ and $\beta(x) \geq 0$ for all $x > 0$, then $a_1 \leq 0$ and $a_3 \geq 0$ because

$$\beta(x) < -a_1 x^2 + (a_3 + a_1 a_2)x - a_2 a_3 = (-a_1 x + a_3)(x - a_2)$$

Note that the right hand side can be zero if $a_3 < 0$. So the claim is justified. Now according to Theorem 2 and noticing the case when $f(x)$ and $g(x)$ are not coprime, we again obtain what is predicted by the AHMC [9]:

A third order polynomial $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$ is a convex direction if and only if either $a_1 a_2 \leq a_3$ or both $a_1 \leq 0$ and $a_3 \geq 0$.

Fourth or Higher Order Polynomials

To demonstrate that fourth or higher order convex directions $p(s)$ may not satisfy the AHMC, or even the monotonic phase velocity of any of $p_1(j\omega)$, $p_2(j\omega)$ or $p(j\omega)$, we use the folowing simple example:

$$p(s) = s^4 + s^3 - s^2 + s + 1 \quad (25)$$

Simple computation shows that the corresponding $f(x)$ and $g(x)$ are coprime and

$$\alpha(x) = -x^2 + 2x + 2; \quad \beta(x) = 2x^2 + 2x - 1$$

For $0 < x \leq 1$, $\alpha(x) \geq 2x + 2 - 1 > 0$. Similarly, for $x > 1$, $\beta(x) > 2x^2 + x > 0$. Therefore, $p(s)$ is a convex direction, according to Theorem 2. However, $\alpha(\infty) < 0$ and $\beta(0) < 0$, implying that the AHMC will fail. Indeed, the two relevant Hurwitz minors to show it are given by $\Delta_1 = 1$ and $\Delta_4 = -3$ (see [9] for notation).

6. CONCLUSION

Several alternative necessary and sufficient conditions (Theorems 2-10) are given for a real polynomials to be a convex direction. They not only provide some interesting physical interpretations, but also lead to a computational procedure which can be performed in a finite number of rational operations. Also presented are some necessary conditions for convex direction. In particular, we have shown that if a polynomial is a convex direction, then the zeros of its real part and imaginary part, after factoring out their greatest common divisor, must be of odd multiplicities and clockwise interlacing. It is also interesting to see that the AHMC in [9] is necessary and sufficient for convex directions up to third order. For fourth or higher order convex directions, conditions are much more complex.

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