Robust Stability and Stabilization of Time-Delay Systems via Integral Quadratic Constraint Approach

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Abstract. In this chapter, we consider two problems associated with time-delay systems: robust stability analysis and robust stabilization. We first obtain two results for robust stability using the integral quadratic constraint approach and the linear matrix inequality technique. Both results give an estimate of the maximum time-delay which preserves robust stability. The first stability result is simpler to apply while the second one gives a less conservative robust stability condition. We then apply these stability results to solve the associated robust stabilization problem using static state feedback. Our results provide new design procedures involving linear matrix inequalities.

1 Introduction

Consider a time-delay system described by

$$\dot{x}(t) = A_0 x(t) + A_d x(t-\tau) + B_u u(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, τ is an unknown constant time delay, A_0, A_d and B_u are constant matrices.

The system above has been analyzed by many researchers. Two types of robust stability conditions have been reported in the literature: the so-called *delay independent* conditions and *delay-dependent* conditions. In comparison, the delay independent conditions are simpler to apply, but the delay-dependent conditions are less conservative in general. With the recent advances in convex optimization (see, e.g., [2]), the focus of the current research is towards finding less conservative delay-dependent conditions by allowing more complex convex optimization. See [13] for a review of robust stability results.

One of the goals in this chapter is to provide new conditions under which the robust stability of the autonomous system of (1) is guaranteed. Our work is based on two ingredients: 1) a sufficient condition for robust stability expressed in the frequency domain; and 2) the integral quadratic constraint (IQC) approach to robustness analysis. Two stability results are presented. Both results are expressed in terms of linear matrix inequalities (LMIs), and they give an estimate of the maximum time-delay which preserves robust stability. The first stability result is simpler to apply while the second one is less conservative. We point out that the stability results in this chapter generalize those in [13].

After derived the two stability results, we then apply these results to solve the associated robust stabilization problem for the system (1) using state feedback control. We also provide explicit formula for controllers. Finally, we show several examples which demonstrate the applications as well as the advantages of the results obtained in this chapter.

2 Preliminaries

Several preliminary results are required for robust stability analysis of the autonomous system of (1). Throughout this chapter, we denote $A = A_0 + A_d$.

Lemma 1. *The autonomous system of (1) is asymptotically stable if A is asymptotically stable and that*

$$\mathcal{A}(j\omega,\tau) := j\omega I - A - \tau\rho_1(j\omega\tau)A_dA_0 - \tau\rho_2(j\omega\tau)A_dA_d \tag{2}$$

is nonsingular for all $\omega \in \mathbb{R}$, where

$$\rho_1(jv) = -e^{-jv/2}\frac{\sin(v/2)}{(v/2)}, \quad \rho_2(jv) = \rho_1(jv)e^{-jv}. \tag{3}$$

Proof. It is well-known that the autonomous system of (1) is asymptotically stable if and only if

$$\hat{A}(j\omega,\tau) = j\omega I - A_0 - A_d e^{-j\omega\tau}$$

is nonsingular for all $\omega \in \mathbb{R}$.

Suppose $\mathcal{A}(j\omega,\tau)$ is nonsingular, we need to show that $\hat{A}(j\omega,\tau)$ is nonsingular. Let x be such that $\hat{A}(j\omega,\tau)x = 0$. We need to show that $x = 0$. To see this, we note

$$\begin{aligned}
0 &= (j\omega I - A_0 - A_d e^{-j\omega\tau})x \\
&= (j\omega I - A - A_d(e^{-j\omega\tau} - 1))x \\
&= (j\omega I - A - \tau\rho_1(j\omega\tau)A_d j\omega)x \\
&= (j\omega I - A - \tau\rho_1(j\omega\tau)A_d(A_0 + A_d e^{-j\omega\tau}))x \\
&= \mathcal{A}(j\omega,\tau)x
\end{aligned} \tag{4}$$

So x must be zero due to the nonsingularity of $\mathcal{A}(j\omega,\tau)$. □

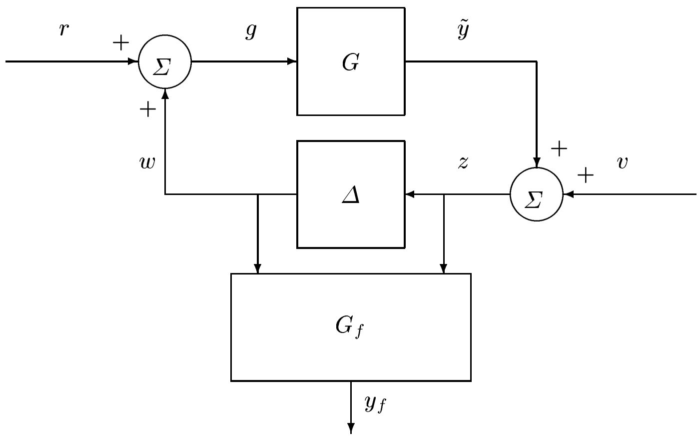

Fig. 1. Interconnected Feedback System

Consider the interconnected system in Figure 1 which is also described by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bg(t) \\ \tilde{y}(t) &= Cx(t) + Dg(t) \\ z(t) &= \tilde{y}(t) + v(t) \\ g(t) &= r(t) + w(t) \\ w(t) &= \Delta(z(t)) \end{aligned} \tag{5}$$

where $\Delta(\cdot) \in \underline{\Delta}$ which is a set of linear or nonlinear dynamic operators to be specified later. Denote

$$G(s) = C(sI - A)^{-1}B + D \tag{6}$$

and assume A to be asymptotically stable in the preliminaries and stability analysis sections.

The feedback block $\Delta(\cdot)$ is assumed to satisfy an IQC which is constructed via a *filter* given as follows:

$$\begin{aligned} \dot{x}_f &= A_f x_f + B_f u_f, \quad x_f(0) = 0 \\ y_f &= C_f x_f + D_f u_f \\ u_f &= \begin{bmatrix} z \\ w \end{bmatrix} \end{aligned} \tag{7}$$

where A_f is asymptotically stable. Denote the transfer function of the filter by

$$G_f(s) = C_f(sI - A_f)^{-1}B_f + D_f \tag{8}$$

The IQC used in this chapter is then described by the following inequality:

$$\int_0^T y_f^{'}\tilde{\Phi}y_f \geq 0, \quad \text{as } T \to \infty, \quad \forall\ \Delta \in \underline{\Delta}, z \in \mathcal{L}_2[0,\infty), \tag{9}$$

where $\tilde{\Phi}$ is a constant symmetric matrix.

Remark 1. The definition above does not require $w \in \mathcal{L}_2[0,\infty)$. But if this is the case, then the IQC (7)-(9) becomes, following the Parseval Theorem,

$$\int_{-\infty}^{+\infty} [z^*(j\omega) \quad w^*(j\omega)]\Phi(j\omega)\begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix} d\omega \geq 0, \quad \forall\ \Delta \in \underline{\Delta} \tag{10}$$

where $z(j\omega), w(j\omega)$ are Fourier transforms of $z(t), w(t)$, respectively, and

$$\Phi(s) = G_f^*(s)\tilde{\Phi}G_f(s) \tag{11}$$

The following results serve the foundation of the IQC approach.

Theorem 2. (The IQC Theorem) *[19, 16, 15] Given a connected set of operators $\underline{\Delta}$, containing the zero operator, for the feedback block of the system (5), the system is absolutely stable if there exists some $\Phi(s)$ of the form (11) and a constant $\epsilon > 0$ such that both (9) and the following condition are satisfied:*

$$[G^*(j\omega) \quad I]\Phi(j\omega)\begin{bmatrix} G(j\omega) \\ I \end{bmatrix} + \epsilon I \leq 0, \quad \forall\ \omega \in (-\infty, \infty) \tag{12}$$

Further, for causal and asymptotically stable linear time-invariant (LTI) $\Delta(\cdot)$, (9) is equivalent to the following:

$$[I \quad \Delta^*(j\omega)]\Phi(j\omega)\begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0, \quad \forall\ \omega \in (-\infty, \infty), \ \Delta \in \underline{\Delta} \tag{13}$$

That is, the system (5) is absolutely stable if there exists $\Phi(s)$ of the form (11) such that (12) and (13) hold.

Lemma 3. (KYP Lemma) *[1, 17] Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$ and symmetric $\Omega \in \mathbb{R}^{(n+k)\times(n+k)}$, there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + \Omega < 0 \tag{14}$$

if and only if there exists some constant $\epsilon > 0$ such that

$$[B^T((j\omega I - A)^{-1})^* \ \ I]\Omega\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} + \epsilon I \leq 0, \quad \forall \ \ \omega \in (-\infty, \infty) \tag{15}$$

Further, if A is Hurwitz and the top-left $n\times n$ submatrix of Ω is positive semidefinite, then (14) implies $P > 0$.

We also recall the following two linear matrix inequality results:

Theorem 4. (Positive Real Parrot Theorem) *[2, 11, 10, 7] Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices U, V of column dimension m. There exists a matrix Θ of compatible dimensions such that*

$$\Psi + U^T\Theta^T V + V^T \Theta U < 0 \tag{16}$$

if and only if

$$U_\perp^T \Psi U_\perp < 0 \tag{17}$$

$$V_\perp^T \Psi V_\perp < 0 \tag{18}$$

where $U_\perp$ (resp. $V_\perp$) is any matrix whose columns form a basis of the null space of U (resp. V).

Lemma 5. *[6] Given matrices $A, B_1, B_2, C_1, C_2, X_d, Q = Q^T, W_1 = W_1^T, W_2 = W_2^T$ of appropriate dimensions, suppose $W_1 > 0$, $W_2 > 0$. Then there exists a matrix K of appropriate dimension such that*

$$\left[\begin{array}{cc|c} QA^T + AQ & QC_1^T & B_1 \\ C_1Q & -W_1 & 0 \\ \hline B_1^T & 0 & -W_2 \end{array}\right] + \left[\begin{array}{c} B_2 \\ X_d \\ \hline 0 \end{array}\right] K \left[\begin{array}{cc|c} I & 0 & 0 \end{array}\right] + \left[\begin{array}{c} I \\ 0 \\ \hline 0 \end{array}\right] K^T \left[\begin{array}{cc|c} B_2^T & X_d^T & 0 \end{array}\right] < 0 \tag{19}$$

holds if and only if the following LMI holds:

$$\left[\begin{array}{c|c} \mathcal{N} & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} QA^T + AQ & QC_1^T & B_1 \\ C_1Q & -W_1 & 0 \\ \hline B_1^T & 0 & -W_2 \end{array}\right] \left[\begin{array}{c|c} \mathcal{N} & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{20}$$

where $\mathcal{N}$ is any matrix whose columns form a basis of the null space of $[B_2^T \; X_d^T]$.

Denote K_1 the solution of the following formula:

$$\left[\begin{array}{c} K_1 \\ \hline * \end{array}\right] = -\left[\begin{array}{c|c} 0 & X_d^T \\ \hline X_d & -W_1 \end{array}\right]^+ \left[\begin{array}{c} B_2^T \\ \hline C_1 Q \end{array}\right], \tag{21}$$

where $^+$ denote the pseudoinverse. Further let K_2 be any solution of the LMI

$$\Psi(K_1) + B_2(I - X_d^+ X_d)K_2 + K_2^T(I - X_d^+X_d)B_2^T < 0, \tag{22}$$

where

$$\begin{aligned} \Psi(K_1) = {} & QA^T + AQ + K_1^T B_2^T + B_2 K_1 \\ & + \begin{bmatrix} C_1Q + X_dK_1 \\ B_1^T \end{bmatrix}^T \begin{bmatrix} W_1^{-1} & 0 \\ 0 & W_2^{-1} \end{bmatrix} \begin{bmatrix} C_1Q + X_dK_1 \\ B_1^T \end{bmatrix}. \end{aligned} \tag{23}$$

Suppose (20) holds. Then a desired K for (19) is given by $K = K_1$ if $(I - X_d^+X_d)B_2^T = 0$ or $K = K_1 + (I - X_d^+X_d)K_2$ otherwise.

3 Stability Analysis

Consider the time-delay system (1). For stability analysis, we assume that $u(t) \equiv 0$. Express

$$A_d = HE, \quad H \in \mathbb{R}^{n \times q}, \quad E \in \mathbb{R}^{q \times n} \tag{24}$$

where $q \leq n$, and H and E are of full rank. Define

$$B^T = \begin{bmatrix} H^T \\ H^T \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} EA_0 \\ EA_d \end{bmatrix}, \quad C_\tau = \tau C, \quad D = 0, \tag{25}$$

and $\underline{\Delta}$ being the set of LTI operators with Fourier transform given by

$$\Delta(j\omega) = \lambda \ \mathrm{diag}\{\rho_1(j\omega\tau)I_q, \rho_2(j\omega\tau)I_q\} \tag{26}$$

for some $\lambda \in [0,\ 1]$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are defined in (3).

Using Lemma 1, we know that the system (1) is robustly stable if the following system is robustly stable:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bg(t) \\ \tilde{y}(t) &= C_\tau x(t) + Dg(t) \\ z(t) &= \tilde{y}(t) + v(t) \\ g(t) &= r(t) + w(t) \\ w(t) &= \Delta(z(t)) \end{aligned} \tag{27}$$

Following the IQC Theorem, we assert that the system (1) is robustly stable if there exists some IQC, or equivalently, $\Phi(s)$ as in (11) such that (12) and (13) hold. Note that the notion of absolute stability coincides with the notion of robust stability for a linear uncertain block Δ. In the rest of this section, we study two IQCs which give two robust stability conditions.

The first IQC is a simple constant D-scaling used in the analysis of structured singular value. More precisely, we take

$$G_f(s) = \ \mathrm{diag}\{I_{2q}, I_{2q}\} \tag{28}$$

and

$$\tilde{\Phi} = \tau^{-1} \ \mathrm{diag}\{\Lambda_1, \Lambda_2, -\Lambda_1, -\Lambda_2\} \tag{29}$$

for some $q \times q$ symmetric and positive-definite Λ_i, $i = 1, 2$, which are to be chosen. Denote

$$\Lambda = \ \mathrm{diag}\{\Lambda_1, \Lambda_2\} \tag{30}$$

The resulting IQC has

$$\Phi(s) = \tau^{-1} \ \mathrm{diag}\{\Lambda, -\Lambda\} \tag{31}$$

It is straightforward to verify that (13) holds because $\rho_i(\cdot)$ are contractive. The sufficient condition (12) for robust stability becomes

$$\begin{bmatrix} B^T((j\omega I - A)^{-1})^* & I_{2q} \end{bmatrix} \begin{bmatrix} \tau C^T \Lambda C & 0 \\ 0 & -\tau^{-1}\Lambda \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I_{2q} \end{bmatrix} + \epsilon I \leq 0 \tag{32}$$

for all ω.

Using the KYP Lemma, the above is equivalent to the existence of $P = P^T > 0$ such that the following LMI holds:

$$\begin{bmatrix} A^TP + PA + \tau C_1^T \Lambda_1 C_1 + \tau C_2^T \Lambda_2 C_2 & PH & PH \\ H^TP & -\tau^{-1}\Lambda_1 & 0 \\ H^TP & 0 & -\tau^{-1}\Lambda_2 \end{bmatrix} < 0$$

Multiplying τ to the second and third row and column blocks, which does not alter the validity of the LMI, the above becomes

$$\Pi(\tau) = \begin{bmatrix} A^TP + PA + \tau C_1^T \Lambda_1 C_1 + \tau C_2^T \Lambda_2 C_2 & \tau PH & \tau PH \\ \tau H^TP & -\tau\Lambda_1 & 0 \\ \tau H^TP & 0 & -\tau\Lambda_2 \end{bmatrix} < 0 \tag{33}$$

Note that $\Pi(\tau)$ is affine in P, Λ_1 and Λ_2.

We summarize the analysis above as follows:

Theorem 6. *The autonomous time-delay system of (1) is robustly stable for all $0 < \tau \le \bar{\tau}$ if there exist $n \times n$ symmetric and positive definite matrices Λ_1, Λ_2 and P such that the LMI*

$$\Pi(\bar{\tau}) < 0 \tag{34}$$

holds, where $\Pi(\tau)$ is defined in (33).

Proof. Suppose (34) holds. It follows from the analysis above that the system (1) is robustly stable for $\bar{\tau}$. The conclusion that the above also implies the robust stability for all $0 \le \tau \le \bar{\tau}$ follows from the fact that the $\Pi(\tau)$ is convex in τ. More precisely, $\Pi(\tau) < 0$ when τ is sufficiently small, due to (34) and $\Lambda_i > 0$. The rest follows from the convexity of $\Pi(\tau)$. □

The second IQC we use to study the robust stability of the system (1) will be more involved but give a less conservative condition. Let

$$f(s) = c_f(sI - a_f)^{-1}b_f + d_f \tag{35}$$

be any asymptotically stable SISO filter with the following property:

$$|f(jv)| \ge \left| \frac{\sin(v)}{v} \right|, \quad \forall v \in \mathbb{R} \tag{36}$$

Denote the diagonal transfer matrix

$$F(s) = f(s)I_{2q} = C_f(sI - A_f)^{-1}B_f + D_f \tag{37}$$

We will discuss how to choose $f(s)$ later.

Now define

$$G_f(s) = \mathrm{diag}\{F(s\tau), I_{2q}\} \tag{38}$$

and $\tilde{\Phi}$ as in (29). This yields

$$y_f(s) = \begin{bmatrix} F(s\tau)z(s) \\ w(s) \end{bmatrix} \tag{39}$$

$$\Phi(s) = G_f^*(s)\tilde{\Phi}G_f(s) = \tau^{-1} \text{ diag}\{F^*(s\tau)\Lambda F(s\tau), -\Lambda\} \tag{40}$$

Subsequently, condition (13) is automatically satisfied because

$$\begin{aligned}
&\tau[I_{2q} \quad \Delta^*(j\omega)]\Phi(j\omega) \begin{bmatrix} I_{2q} \\ \Delta(j\omega) \end{bmatrix} \\
&= F^*(j\omega\tau)\Lambda F(j\omega\tau) - \lambda^2 \text{diag}\{\rho_1^*(j\omega\tau)\rho_1(j\omega\tau)\Lambda_1, \rho_2^*(j\omega\tau)\rho_2(j\omega\tau)\Lambda_2\} \\
&= \Lambda^{1/2}(F^*(j\omega\tau)F(j\omega\tau) - \lambda^2 \text{diag}\{\rho_1^*(j\omega\tau)\rho_1(j\omega\tau), \rho_2^*(j\omega\tau)\rho_2(j\omega\tau)\})\Lambda^{1/2} \\
&\geq 0
\end{aligned}$$

Therefore, a sufficient condition for robust stability of system (1) is the condition (12) which, in our case, becomes

$$G_\tau^*(j\omega)F^*(j\omega\tau)\tau^{-1}\Lambda F(j\omega\tau)G_\tau(j\omega) - \tau^{-1}\Lambda + \epsilon I_{2q} \leq 0, \quad \forall\, \omega \in (-\infty, \infty) \tag{41}$$

for some (sufficiently small) $\epsilon > 0$, where $G_\tau(s) = C_\tau(sI - A)^{-1}B$.

Our next step is to convert the frequency domain condition (41) into the state space. To this end, we denote by $\bar{C}_\tau(sI - \bar{A}_\tau)^{-1}\bar{B}_\tau$ a state-space realization of $F(s\tau)G_\tau(s)$. Then, it is straightforward to verify that

$$\bar{A}_\tau = \begin{bmatrix} \tau^{-1}A_f & B_f C \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \bar{C}_\tau = \begin{bmatrix} C_f & D_f C_\tau \end{bmatrix} \tag{42}$$

Condition (41) can be rewritten as

$$[\bar{B}^T((j\omega I - \bar{A}_\tau)^{-1})^* \; I] \text{ diag}\{\tau^{-1}\bar{C}_\tau^T \Lambda \bar{C}_\tau, -\tau^{-1}\Lambda\} \begin{bmatrix} (j\omega I - \bar{A}_\tau)^{-1}\bar{B} \\ I \end{bmatrix} + \epsilon I \leq 0$$

for all $\omega \in (-\infty, \infty)$.

Applying the KYP Lemma, the above is equivalent to the existence of some $\bar{P} = \bar{P}^T > 0$ such that the following linear matrix inequality holds:

$$\bar{\Pi}(\tau) = \begin{bmatrix} \bar{A}_\tau^T \bar{P} + \bar{P}\bar{A}_\tau + \tau^{-1}\bar{C}_\tau^T \Lambda \bar{C}_\tau & \bar{P}\bar{B} \\ \bar{B}^T\bar{P} & -\tau^{-1}\Lambda \end{bmatrix} < 0 \tag{43}$$

The above analysis is summarized as follows:

Theorem 7. *The autonomous time-delay system of (1) is robustly stable for all $\tau \leq \bar{\tau}$ if there exist an asymptotically stable filter $f(s)$, and symmetric and positive-definite matrices Λ_1, Λ_2 and $\bar{P}$ such that the following LMI holds:*

$$\bar{\Pi}(\bar{\tau}) < 0 \tag{44}$$

where $\bar{\Pi}(\cdot)$ is defined in (43).

Proof. The proof is very similar to that of Theorem 6, so the details are omitted. The only step worth of discussion is the fact that $\bar{\Pi}(\bar{\tau}) < 0$ implies $\bar{\Pi}(\tau) < 0$ for all $0 < \tau \leq \bar{\tau}$. This step is a bit tedious but not too difficult to verify too. □

4 Stabilization

Consider the time-delay system (1). Our objective is to design a static state feedback controller such that the closed-loop system of (1) is uniformly asymptotically stable. The results derived in this section are based on Theorems 6 and 7.

Let a desired controller be given in the following form:

$$u(t) = Kx(t) \tag{45}$$

where K is the gain matrix to be designed.

With the controller (45), the closed-loop system of (1) is as follows:

$$\dot{x}(t) = (A_0 + B_u K)x(t) + A_d x(t-\tau) \tag{46}$$

Applying Theorem 6 and Lemma 5 to the system (46), we obtain the following result:

Theorem 8. *There exists a state feedback controller (45) such that the closed-loop time-delay system of (1) with this controller is robustly stable for all* $0 < \tau \leq \bar{\tau}$ *if there exist* $n \times n$ *symmetric and positive definite matrices* Γ_1, Γ_2 *and* Q *such that the LMI*

$$\left[\begin{array}{c|c} \mathcal{N}_B & 0 \\ \hline 0 & I \end{array}\right]^T \Pi_c(\bar{\tau}) \left[\begin{array}{c|c} \mathcal{N}_B & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{47}$$

holds, where

$$\Pi_c(\tau) = \left[\begin{array}{cc|ccc} QA^T + AQ & QC_1^T & QC_2^T & H\Gamma_1 & H\Gamma_2 \\ C_1 Q & -\tau^{-1}\Gamma_1 & 0 & 0 & 0 \\ \hline C_2 Q & 0 & -\tau^{-1}\Gamma_2 & 0 & 0 \\ \Gamma_1 H^T & 0 & 0 & -\tau^{-1}\Gamma_1 & 0 \\ \Gamma_2 H^T & 0 & 0 & 0 & -\tau^{-1}\Gamma_2 \end{array}\right],$$

and $\mathcal{N}_B$ *is any matrix whose columns form a basis of the null space of* $[B_u^T \;\; B_u^T E^T]$.

Further, suppose (47) holds. Let K_1 *be the solution of the following formula:*

$$\left[\begin{array}{c} K_1 \\ \hline * \end{array}\right] = -\left[\begin{array}{c|c} 0 & B_u^T E^T \\ \hline EB_u & -\bar{\tau}^{-1}\Gamma_1 \end{array}\right]^+ \left[\begin{array}{c} B_u^T \\ \hline C_1 Q \end{array}\right], \tag{48}$$

and K_2 *be any solution of the LMI*

$$\Psi(K_1) + B_u(I - (EB_u)^+ EB_u)K_2 + K_2^T(I - (EB_u)^+ EB_u)B_u^T < 0, \tag{49}$$

where

$$\begin{aligned} \Psi(K_1) &= QA^T + AQ + K_1^T B_u^T + B_u K_1 \\ &+ \bar{\tau} \begin{bmatrix} C_1 Q + EB_u K_1 \\ C_2 Q \\ H^T \\ H^T \end{bmatrix}^T \begin{bmatrix} \Gamma_1^{-1} & 0 & 0 & 0 \\ 0 & \Gamma_2^{-1} & 0 & 0 \\ 0 & 0 & \Gamma_1 & 0 \\ 0 & 0 & 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} C_1 Q + EB_u K_1 \\ C_2 Q \\ H^T \\ H^T \end{bmatrix} \end{aligned} \tag{50}$$

Then, a desired controller gain matrix K is given by $K = K_1 Q^{-1}$ if

$$(I - (EB_u)^+ EB_u)B_u^T = 0$$

or $K = (K_1 + (I - (EB_u)^+ EB_u)K_2)Q^{-1}$ otherwise.

Proof. Applying Theorem 6 to the closed-loop system (46), we find that this system is robustly stable for all $0 < \tau \leq \bar{\tau}$ if there exist $n \times n$ symmetric and positive definite matrices Λ_1, Λ_2 and P such that the matrix inequality

$$\left[\begin{array}{c|cc} \begin{array}{c}(A + B_uK)^T P + P(A + B_uK) + \bar{\tau} C_2^T \Lambda_2 C_2 \\ +\bar{\tau}(C_1 + EB_uK)^T \Lambda_1 (C_1 + EB_uK)\end{array} & \bar{\tau}PH & \bar{\tau}PH \\ \hline \bar{\tau}H^T P & -\bar{\tau}\Lambda_1 & 0 \\ \bar{\tau}H^T P & 0 & -\bar{\tau}\Lambda_2 \end{array}\right] < 0 \qquad (51)$$

holds. Using Schur complements, (51) can be rewritten as

$$\left[\begin{array}{cc|ccc} A^T P + PA & \bar{\tau}C_1^T & \bar{\tau}C_2^T & \bar{\tau}PH & \bar{\tau}PH \\ \bar{\tau}C_1 & -\bar{\tau}\Lambda_1^{-1} & 0 & 0 & 0 \\ \hline \bar{\tau}C_2 & 0 & -\bar{\tau}\Lambda_2^{-1} & 0 & 0 \\ \bar{\tau}H^T P & 0 & 0 & -\bar{\tau}\Lambda_1 & 0 \\ \bar{\tau}H^T P & 0 & 0 & 0 & -\bar{\tau}\Lambda_2 \end{array}\right]$$
$$+ \left[\begin{array}{c} PB_u \\ \bar{\tau}EB_u \\ \hline 0 \\ 0 \\ 0 \end{array}\right] K[I \; 0 \mid 0 \; 0 \; 0] + \left[\begin{array}{c} I \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] K^T [B_u^T P \;\; \bar{\tau}B_u^T E^T \mid 0 \; 0 \; 0] < 0 \qquad (52)$$

Define $Q = P^{-1}$, $\Gamma_1 = \Lambda_1^{-1}$ and $\Gamma_2 = \Lambda_2^{-1}$, respectively. Multiplying

$$\mathrm{diag}\{Q, \bar{\tau}^{-1}I, \bar{\tau}^{-1}I, \bar{\tau}^{-1}\Gamma_1, \bar{\tau}^{-1}\Gamma_2\}$$

to both sides, the inequality above is equivalent to

$$\Pi_c(\bar{\tau}) + \left[\begin{array}{c} B_u \\ EB_u \\ \hline 0 \\ 0 \\ 0 \end{array}\right] (KQ)[I \; 0 \mid 0 \; 0 \; 0] + \left[\begin{array}{c} I \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] (KQ)^T [B_u^T \;\; B_u^T E^T \mid 0 \; 0 \; 0] < 0 \quad (53)$$

Then, the results in the theorem are obtained by applying Lemma 5. □

Corresponding to Theorem 7 for stability analysis, we can obtain the following less conservative result for robust stabilization:

Theorem 9. *There exists a state feedback controller (45) such that the closed-loop time-delay system of (1) with this controller is robustly stable for all* $\tau \leq \bar{\tau}$ *if there exist symmetric and positive definite matrices* $\bar{\Gamma}$, Q_f *and* Q *such that the LMIs*

$$\left[\begin{array}{c|c} \mathcal{N}_f & 0 \\ \hline 0 & I \end{array}\right]^T \bar{\Pi}_c(\bar{\tau}) \left[\begin{array}{c|c} \mathcal{N}_f & 0 \\ \hline 0 & I \end{array}\right] < 0 \tag{54}$$

$$\begin{bmatrix} A_f Q_f + Q_f A_f^T & Q_f C_f^T \\ C_f Q_f & -\bar{\Gamma} \end{bmatrix} < 0 \tag{55}$$

hold, where

$$\bar{\Pi}_c(\tau) = \left[\begin{array}{ccc|c} QA^T + AQ & QC^T B_f^T & QC^T D_f^T & B\bar{\Gamma} \\ B_f CQ & \tau^{-1}(A_f Q_f + Q_f A_f^T) & \tau^{-1} Q_f C_f^T & 0 \\ D_f CQ & \tau^{-1} C_f Q_f & -\tau^{-1}\bar{\Gamma} & 0 \\ \hline \bar{\Gamma} B^T & 0 & 0 & -\tau^{-1}\bar{\Gamma} \end{array}\right]. \tag{56}$$

and $\mathcal{N}_f$ *is any matrix whose columns form a basis of the null space of the matrix* $[B_u^T \;\; X_d^T]$ *with*

$$X_d = \begin{bmatrix} B_f \\ D_f \end{bmatrix} \begin{bmatrix} EB_u \\ 0 \end{bmatrix}. \tag{57}$$

Further, suppose (54)-(55) hold. Denote

$$X_c = \begin{bmatrix} B_f \\ D_f \end{bmatrix} C \tag{58}$$

and

$$W_f = \begin{bmatrix} Q_f A_f^T + Q_f A_f & Q_f C_f^T \\ C_f Q_f & -\bar{\Gamma} \end{bmatrix}, \tag{59}$$

Let K_1 *be the solution of the following formula:*

$$\left[\begin{array}{c} K_1 \\ \hline * \end{array}\right] = -\left[\begin{array}{c|c} 0 & X_d^T \\ \hline X_d & \bar{\tau}^{-1} W_f \end{array}\right]^+ \left[\begin{array}{c} B_u^T \\ \hline X_c Q \end{array}\right], \tag{60}$$

and let K_2 *be any solution of the LMI*

$$\Psi(K_1) + B_u(I - X_d^+ X_d)K_2 + K_2^T (I - X_d^+ X_d) B_u^T < 0, \tag{61}$$

where

$$\Psi(K_1) = QA^T + AQ + K_1^T B_u^T + B_u K_1 + \bar{\tau} \begin{bmatrix} X_c Q + X_d K_1 \\ B^T \end{bmatrix}^T \begin{bmatrix} W_f^{-1} & 0 \\ 0 & \bar{\Gamma} \end{bmatrix} \begin{bmatrix} X_c Q + X_d K_1 \\ B^T \end{bmatrix}. \tag{62}$$

Then, a desired controller gain matrix K *is given by* $K = K_1 Q^{-1}$ *if*

$$(I - X_d^+ X_d) B_u^T = 0$$

or $K = (K_1 + (I - X_d^+ X_d)K_2)Q^{-1}$ *otherwise.*

Proof. The proof of this theorem follows the same line as that for Theorem 8. Namely, we apply Theorem 7 and Lemma 5 to the closed-loop system (46). We first use Schur complements to rewrite the robust stability condition in Theorem 8, assuming $K = 0$. That is, $\bar{\Pi}(\bar{\tau}) < 0$ if and only if

$$\left[\begin{array}{cc|c} \bar{A}_\tau^T \bar{P} + \bar{P}\bar{A}_\tau & \bar{C}_\tau^T & \bar{P}\bar{B} \\ \bar{C}_\tau & -\bar{\tau}\Lambda^{-1} & 0 \\ \hline \bar{B}^T \bar{P} & 0 & -\bar{\tau}^{-1}\Lambda \end{array}\right] < 0$$

Furthermore, we take $\bar{P} = \mathrm{diag}\{P_f, P\}$ since there is no interaction between the closed-loop system and the filter. Let $\bar{Q} = \bar{P}^{-1} = \mathrm{diag}\{Q_f, Q\}$ and $\bar{\Gamma} = \Lambda^{-1}$. Multiplying $\mathrm{diag}\{\bar{Q}, \bar{\tau}^{-1} I, \bar{\Gamma}\}$ to the both sides, the above inequality is converted into

$$\left[\begin{array}{ccc|c} \tau^{-1}(A_f Q_f + Q_f A_f^T) & B_f C Q & \tau^{-1} Q_f C_f^T & 0 \\ Q C^T B_f^T & Q A^T + A Q & Q C^T D_f^T & B\bar{\Gamma} \\ \tau^{-1} C_f Q_f & D_f C Q & -\tau^{-1}\bar{\Gamma} & 0 \\ \hline 0 & \bar{\Gamma} B^T & 0 & -\tau^{-1}\bar{\Gamma} \end{array}\right] < 0.$$

Swapping the first two rows and columns, which does not affect the inequality, we further convert the above into

$$\bar{\Pi}_c(\bar{\tau}) < 0 \tag{63}$$

where $\bar{\Pi}_c(\bar{\tau})$ is defined in (56).

Now let static state feedback be used, i.e., A_0 becomes $A_0 + B_u K$. Subsequently, A becomes $A + B_u K$ and C becomes

$$C + \begin{bmatrix} E B_u K \\ 0 \end{bmatrix}.$$

Hence, the robust stability condition (63) becomes the following robust stabilization condition:

$$\bar{\Pi}_c(\bar{\tau}) + \left[\begin{array}{c} B_u \\ X_d \\ \hline 0 \end{array}\right] (KQ) \left[\begin{array}{c|c} I \; 0 & 0 \end{array}\right] + \left[\begin{array}{c} I \\ 0 \\ \hline 0 \end{array}\right] (KQ)^T \left[\begin{array}{c|c} B_u^T \; X_d^T & 0 \end{array}\right] < 0$$

Then, it is tedious but straightforward to prove the characterization of K by applying Theorem 7 and Lemma 5. □

Remark 2. If we take the filter $f(s) = 1$ and further constrain Λ to be $\Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2\}$, then it is obvious that Theorem 9 reduces to Theorem 8. To see this, we may select $B_f = 0$, $C_f = 0$ and $A_f = -I$, then clearly (54) reduces to (47) while (55) is trivially satisfied.

5 Examples

Before providing illustrative examples, we address the problem of finding a suitable filter $f(s)$. First, we note that $f(s)$ is a SISO transfer function, and that the constraint on $f(s)$ (36) is independent of the system (1). This means that once a "good" $f(s)$ is found, it can be applied to various time-delay systems of the form (1). The complexity of $f(s)$ is mainly determined by the degree of $f(s)$. A second order example is given below:

$$f(s) = \frac{2(s+0.9)}{(s+0.8)(s+2.216)} \tag{64}$$

with its Bode plot given in Figure 2. Also plotted in Figure 2 is $|\sin(\omega)/\omega|$ to justify (36).

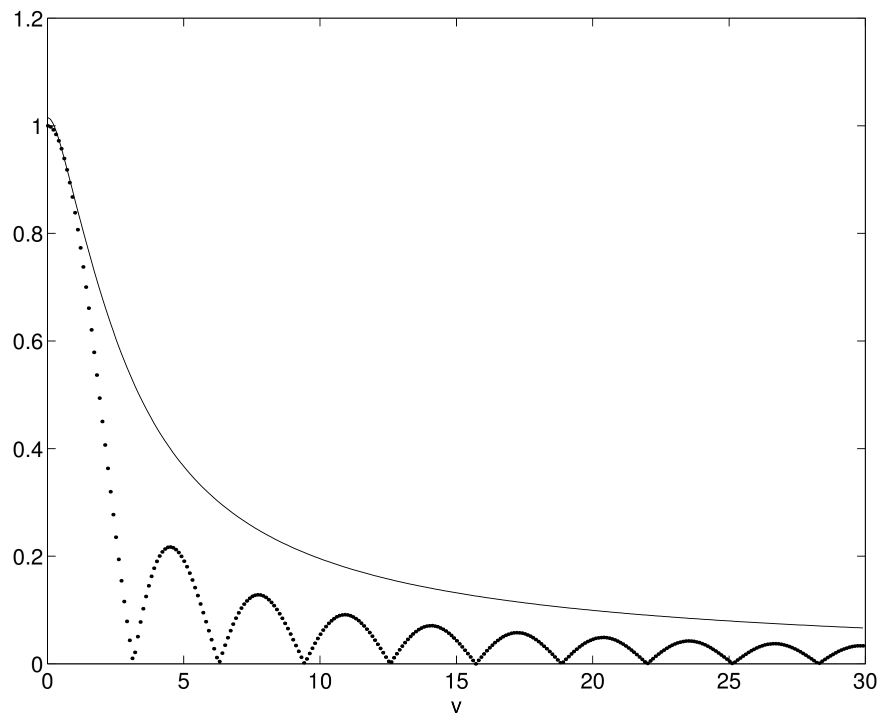

Fig. 2. Example of $f(s)$. Solid line: $|f(jv)|$; Dotted line: $|\sin(v)/v|$

Example 1: Consider the autonomous system of (1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0.25 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.85 \end{bmatrix} \tag{65}$$

Using Theorem 7 and the $f(s)$, the maximum τ is obtained to be $\tau_{\max} = 0.9848$.

Obviously, the conservatism of $\tau_{\max}$ depends on the filter $f(s)$. It is found in simulation that second order filters usually outperform first order ones. Also, higher order filters can be used to obtain slightly larger $\tau_{\max}$.

Using Theorem 6, the maximum τ is obtained to be $\tau_{\max} = 0.6417$.

As comparisons, we notice that the maximum τ using the results in [13, 12] is $\tau_{\max} = 0.58$ while the optimal τ for the system with the given parameters is $\tau_o = 1.54$[12].

Example 2: Consider the system (1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0.35 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.25 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}. \tag{66}$$

Since $A_0 + A_d$ is unstable, the system (1) with the above given parameters can't be stabilized independent of the time-delay using state feedback controller.

Using Theorem 9 and the $f(s)$, the maximum τ is obtained to be $\tau_{\max} = 0.9726$.

To avoid numerical difficulties, we synthesize the state feedback controller using $\tau_{\max} = 0.92$ instead. Following the explicit K formula in Theorem 9, we obtain that a desired controller gain matrix is given by

$$K = [-24.8250 \quad -62.2741]. \tag{67}$$

6 Conclusion

We have obtained two new robust stability conditions for time-delay systems by applying the IQC approach. These conditions are expressed in terms of LMIs and therefore easily solvable. Although a single delay is considered in this chapter, we stress that an extension to multiple delays can be simply derived. As applications of these new robust stability results, robust stabilization problems using static state feedback control have been tackled. Explicit controller formulas have also been provided.

We have not explained how to determine the maximal time delay $\bar{\tau}$. Generally, $\bar{\tau}$ can be obtained by a gradient method. First we set $\bar{\tau}$ to be sufficiently small, then gradually increase it until the corresponding robust stability or stabilization conditions are no longer feasible. A fine gradient can be adopted in the final critical region to obtain larger $\bar{\tau}$. Alternatively, we can use a bisection method. That is, we start with any lower bound and and a upper bound for $\bar{\tau}$. Then, choose the initial $\bar{\tau}$ to be the average of the bounds and test for the solvability of robust stability or stabilization conditions. The bounds will be improved according to the outcome of the test. This procedure is repeated until the gap between the bounds is sufficiently small.

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