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Robustness of the stability of feedback systems with respect to small time delays

Gjerrit Meinsma [a,*], Minyue Fu [b], Tetsuya Iwasaki [c]

[a] Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands
[b] Department of Electrical and Computer Engineering, University of Newcastle, Callaghan, NSW 2308, Australia
[c] Department of Computational Intelligence and System Science, Tokyo Institute of Technology, 4259 Nagatsuta, Midori-ku, Yokohama 226, Japan

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Abstract

It is shown that a feedback system is robustly stable with respect to small time delays if and only if it is stable for zero time delay and a structured singular value is less than one.

Keywords: Stability; Well-posedness; Robustness; Time delays; Structured singular value

1. Introduction

Is the feedback loop of Fig. 1 stable if the loop gain is a constant $P(s) \equiv 3$? This loop gain passes any standard stability test, but if P represents an I/O system with the measured output y being fed back, then the answer is no. The reason is that any active measuring brings in a small time delay in the loop and this may destroy stability no matter how small the time delay is. [1] More generally, if $P(s)$ is a constant gain $P(s) \equiv p$ and if there is a time delay of λ seconds in the loop somewhere, say right before y, then the response y to a unit step r satisfies $y(n\lambda) = p - p^2 + p^3 \cdots - (-p)^n$ and so, only if $|p| < 1$ does y converge to the expected $y = p/(1+p)r$. The amount of delay in the loop is irrelevant to stability, it only affects the speed with which the signals converge or diverge. (Oddly enough, for $|p| > 1$, the smaller the time delay the faster y blows up.)

In 1971, Willems ([10], pp. 90–97) showed that for SISO systems a closed loop with small time delays is in some sense "well-posed" and stable if it is stable in the usual sense and in addition $|P(\infty)| < 1$. Recently, the problem of robustness of stability with respect small time delays has been taken up again. Logemann et al. [4] have analyzed the problem for a class of regular infinite-dimensional MIMO systems P, and in [3] the results are taken further for ill-posed systems, and neutral systems are considered in [5]. Related to this is the work by Georgiou and Smith [2] on w-stability, which allows to deal with high-frequency perturbations, of which small time delays are an example.

One of the findings of [4] is that a stable feedback system remains stable when perturbed with small time delays if a certain structured singular value is less than one ([4], Theorem 6.5). It seems to be a difficult open problem whether or not this is in fact a necessary and sufficient condition (see [4], p. 589). We show that

* Corresponding author. E-mail: g.meinsma@math.utwente.nl.
[1] For general (behavioral) interconnected systems the presence of small time delays may not be justified.

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for the special case of *rational* loop gains and a more natural definition of stability, the structured singular value condition is indeed both necessary and sufficient. A consequence of this result is that the robust stability test is NP-hard, owing to a result by Toker [7] that computation of complex structured singular values is NP-hard.

2. Main result

We assume that the loop-gain P in Fig. 1 is a rational transfer function matrix, representing an LTI system, that u and y are subject to perturbations and that there is a small time delay in the loop in each of the components of the signals. Such a system may be represented by the block diagram in Fig. 2. Here v_1 and v_2 represent the perturbations of u and y, and the block $e^{-s\Lambda}$ represents the time delays. The matrix Λ is diagonal, $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, with real, nonnegative entries. In other words, the delays in the lower half of the loop in Fig. 2 are such that $\tilde{y}_k(t) = y_k(t - \lambda_k)$.

Definition 2.1 (*Robust stability with respect to small time delays*). The loop in Fig. 1 is *robustly stable with respect to small time delays* if for the loop in Fig. 2 there is an $\varepsilon > 0$ and an $N > 0$ such that the L_2-induced norm of the map from $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ is bounded by N for any diagonal time delay Λ with $\|\Lambda\| < \varepsilon$.

This definition combines input–output stability and well-posedness, where well-posedness expresses

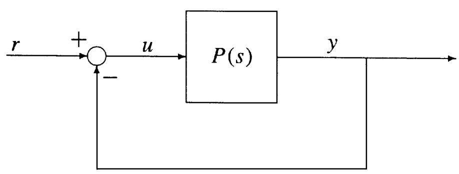

Fig. 1. The unity feedback loop.

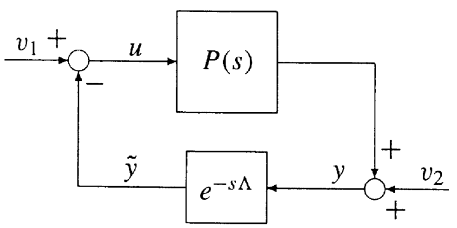

Fig. 2. The unity feedback with time delays and injected disturbances.

that small changes of the description result in small changes in the closed-loop signals over a given finite time. The reason to combine the two notions is mathematical convenience. The structured singular value $\mu(M)$ of $M \in \mathbb{C}^{n \times n}$ that we need is defined as

$$\mu(M) = \inf \left\{ \beta > 0\colon |I_n + M \operatorname{diag}(\delta_1, \dots, \delta_n)| \neq 0, \; \forall \delta_k \in \mathbb{C},\ |\delta_k| \leqslant \frac{1}{\beta} \right\}.$$

Theorem 2.2. *Let P be a rational transfer function matrix. The feedback loop in Fig.* 1 *is robustly stable with respect to small time delays, if and only if P is proper, $(I + P)^{-1}$ is stable and $\mu(P(\infty)) < 1$.*

Proof. Define $\mu_\infty = \mu(P(\infty))$ and let $\|\cdot\|$ denote the spectral norm (the largest singular value) for matrices.

(*If*): This is the straightforward but technical part of the proof. Suppose that P is proper, $(I + P)^{-1}$ is stable and $\mu_\infty < 1$. Let $P_{\mathrm{d}}^{-1} P_{\mathrm{n}} = P$ be a coprime factorization over the stable rational transfer function matrices. With this coprime factorization, the transfer function matrix from $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ can be expressed as

$$\begin{bmatrix} I - e^{-\cdot\Lambda}(P_{\mathrm{d}} + P_{\mathrm{n}} e^{-\cdot\Lambda})^{-1} P_{\mathrm{n}} & -e^{-\cdot\Lambda}(P_{\mathrm{d}} + P_{\mathrm{n}} e^{-\cdot\Lambda})^{-1} P_{\mathrm{d}} \\ (P_{\mathrm{d}} + P_{\mathrm{n}} e^{-\cdot\Lambda})^{-1} P_{\mathrm{n}} & (P_{\mathrm{d}} + P_{\mathrm{n}} e^{-\cdot\Lambda})^{-1} P_{\mathrm{d}} \end{bmatrix}. \tag{1}$$

The H_∞-norm of this transfer matrix equals the L_2-induced norm of the map from $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ (see e.g. [9] or [1], Section A.6.3).

It suffices to find an $N_1 > 0$ and $\varepsilon > 0$ such that

$$\sup_{\Lambda \text{ diagonal}, \Lambda \geqslant 0, \|\Lambda\| \leqslant \varepsilon} \|(P_{\mathrm{d}} + P_{\mathrm{n}} e^{-\cdot\Lambda})^{-1}\|_{H_\infty} < N_1 < \infty. \tag{2}$$

Indeed, this implies that the H_∞-norm of Eq. (1) is bounded by $1 + 2N_1(\|P_{\mathrm{n}}\|_{H_\infty} + \|P_{\mathrm{d}}\|_{H_\infty})$ for all such Λ's, which completes the proof of the if-part.

The complex structured singular value μ is continuous ([6], p. 73) hence, for any $c \in (0, 1 - \mu_\infty)$ we can find a large enough radius $R > 0$ such that

$$\sup_{s \in \Omega_R} \mu(P(s)) \leqslant \mu_\infty + c < 1$$

$$\text{where} \quad \Omega_R := \{ s \in \mathbb{C}\colon \operatorname{Re} s \geqslant 0,\ |s| \geqslant R \}. \tag{3}$$

The radius R can be chosen such that Ω_R does not contain a pole of P. On Ω_R we have that

$$\begin{aligned}
&\sup_{s\in\Omega_R} \|(P_\mathrm{d}(s)+P_\mathrm{n}(s)\mathrm{e}^{-s\Lambda})^{-1}\| \\
&\quad\leqslant \sup_{s\in\Omega_R} \|P_\mathrm{d}^{-1}(s)\|\,\|(I+P(s)\mathrm{e}^{-s\Lambda})^{-1}\| \\
&\quad\leqslant \sup_{s\in\Omega_R} \|P_\mathrm{d}^{-1}(s)\| \sup_{s\in\Omega_R} \|(I+P(s)\mathrm{e}^{-s\Lambda})^{-1}\| \\
&\quad\leqslant \sup_{s\in\Omega_R} \|P_\mathrm{d}^{-1}(s)\| \\
&\qquad\times \sup_{s\in\Omega_R,\, \Delta \text{ diagonal},\, \|\Delta\|\leqslant 1} \|(I+P(s)\Delta)^{-1}\| \\
&\quad\leqslant N_2 \quad \text{for some } N_2>0.
\end{aligned}$$

The last inequality holds for some $N_2>0$ for the following reason. Suppose such an N_2 does not exist, then, since $\|P_\mathrm{d}^{-1}(s)\|$ is bounded on Ω_R, there is a sequence $s_k\in\Omega_R$, Δ_k with $\|\Delta_k\|\leqslant 1$ such that $\lim_{k\to\infty}\|(I+P(s_k)\Delta_k)^{-1}\|=\infty$. Because $P(s_k)$ and Δ_k are bounded this would mean that $\lim_{k\to\infty}|I+P(s_k)\Delta_k|=0$. Let $P_{k_j}:=P(s_{k_j})$, Δ_{k_j} be a converging subsequence and note that $P(\Omega_R\cup\infty)$ is compact and, hence, that there is an $s^*\in\Omega_R\cup\infty$ such that $\lim_{j\to\infty}P_{k_j}=P(s^*)$. Then $|I+P(s^*)\Delta_{k_\infty}|=0$ contradicting Eq. (3), and therefore such N_2 exist.

Since $(I+P)^{-1}$ is stable we have that $P_\mathrm{d}+P_\mathrm{n}$ is bistable. Therefore, on the half disc with radius R we have that

$$\begin{aligned}
&\sup_{\operatorname{Re}s\geqslant 0,\,|s|\leqslant R} \|(P_\mathrm{d}(s)+P_\mathrm{n}(s)\mathrm{e}^{-s\Lambda})^{-1}\| \\
&\quad\leqslant \sup_{\operatorname{Re}s\geqslant 0,\,|s|\leqslant R} \|((P_\mathrm{d}(s)+P_\mathrm{n}(s)) \\
&\qquad +P_\mathrm{n}(s)(\mathrm{e}^{-s\Lambda}-I))^{-1}\| \\
&\quad\leqslant \|(P_\mathrm{d}+P_\mathrm{n})^{-1}\|_{H_\infty} \\
&\qquad\times \sup_{\operatorname{Re}s\geqslant 0,\,|s|\leqslant R} \|(I+(P_\mathrm{d}(s) \\
&\qquad +P_\mathrm{n}(s))^{-1}P_\mathrm{n}(s)(\mathrm{e}^{-s\Lambda}-I))^{-1}\| \\
&\quad\leqslant 2\|(P_\mathrm{d}+P_\mathrm{n})^{-1}\|_{H_\infty} \\
&\qquad \text{for some small enough } \varepsilon>0 \text{ and all } \|\Lambda\|\leqslant\varepsilon.
\end{aligned} \tag{4}$$

This shows that Eq. (2) holds for $N_1:=\max(N_2, 2\|(P_\mathrm{d}+P_\mathrm{n})^{-1}\|_{H_\infty})$ and with the ε determined in Eq. (4).

(*Only if*): Suppose the loop is robustly stable with respect to small time delays. This obviously implies that P is proper and that $(I+P)^{-1}$ is stable. Suppose, to obtain a contradiction, that $\mu_\infty\geqslant 1$. Therefore $|I+P(\infty)\Delta|=0$ for some diagonal $\Delta\in\mathbb{C}^{n\times n}$ with $\|\Delta\|=1/\mu_\infty\leqslant 1$. All entries δ_i of Δ can be chosen to have the same absolute value, $1/\mu_\infty$, ([6], Lemma 6.3), stated differently, without loss of generality, we may assume that Δ is of the form

$$\Delta=\mathrm{e}^{\log(1/\mu_\infty)I_n+\mathrm{j}Y} \quad \text{with } Y\in\mathbb{R}^{n\times n},\ Y \text{ diagonal.}$$

With Δ expressed like this we can find a sequence $\{s_k\}$ in the open right-half plane and a sequence of delays $\{\Lambda_k\}$ such that

$$\lim_{k\to\infty}|s_k|=\infty,$$

$$\lim_{k\to\infty}\|\Lambda_k\|=0,$$

$$\lim_{k\to\infty}\mathrm{e}^{-s_k\Lambda_k}=\Delta.$$

This is achieved for s_k and Λ_k defined as

$$s_k=k\left(-\log\left(\frac{1}{\mu_\infty}\right)+\frac{1}{k}+\mathrm{j}2k\pi\right),$$

$$\Lambda_k=\frac{1}{k}\left(I-\frac{1}{2k\pi}Y\right).$$

As a result we have that

$$\lim_{k\to\infty}|I+P(s_k)\mathrm{e}^{-s_k\Lambda_k}|=|I+P(\infty)\Delta|=0.$$

It is important to note that $\|\Lambda_k\|$ goes to zero as k goes to infinity. This shows that for every $\varepsilon>0$ and every $N>0$ we can find a time delay Λ_k smaller in norm than ε but such that the H_∞-norm of $(I+P(s)\mathrm{e}^{-s\Lambda})^{-1}$ exceeds N. Hence, the loop is not robustly stable with respect to small time delays, which is a contradiction. Therefore $\mu_\infty<1$. □

Corollary 2.3. *The problem of testing robust stability with respect to small time delays is NP-hard.*

Proof. This is a direct consequence of Theorem 2.2 and of a result by Toker ([7], Ch. 11, Remark 4) – see also [8] – which states that testing whether $\mu(M)$ is less than 1, is NP-hard, even if there are no repeated blocks. □

3. Multiple time delays

For more complicated loops with time delays occurring at various places in the loop, the way to test for robust stability is to “pull out” the time-delay blocks

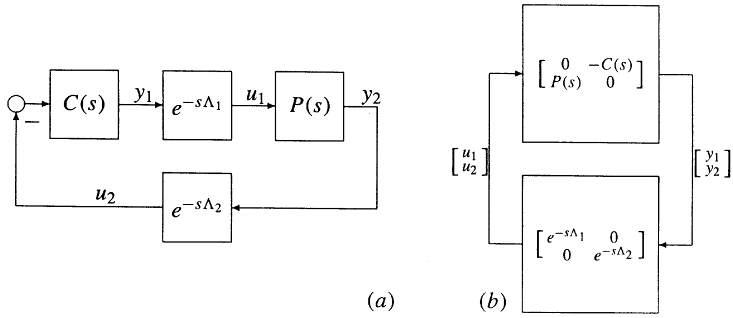

Fig. 3. Plant and controller with delays in between.

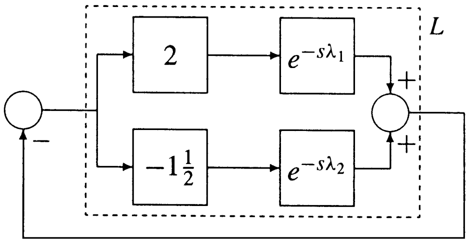

Fig. 4. Time delays in the loop gain.

as is done in the robust stability literature. For example, the delays $e^{-s\Lambda_1}$ and $e^{-s\Lambda_2}$ in the closed loop shown in Fig. 3a can be pulled out and stacked diagonally as shown in part b of Fig. 3. Rearranging the loop does not affect stability and, hence, for this loop to be robustly stable with respect to small time delays it is necessary and sufficient that it be stable for zero time delay, that P and C are proper and that

$$\mu\left(\begin{bmatrix} 0 & -C(\infty) \\ P(\infty) & 0 \end{bmatrix}\right) < 1. \tag{5}$$

In probably all practical applications where time delays may arise either due to the plant P or the controller C is strictly proper. In such cases inequality (5) is trivially satisfied (because then $\mu = 0$) and only the well-known stability condition remains.

The method of pulling out the uncertain delay blocks can also be used to show that the feedback loop of Fig. 4 is not robustly stable with respect to small time delays even though in open loop the gain L is practically $\frac{1}{2}$ for small time delays.

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