Proceedings of the 27th Conference
on Decision and Control
Austin, Texas • December 1988

ROBUST STABILITY FOR TIME–DELAY SYSTEMS: THE EDGE THEOREM AND GRAPHICAL TESTS

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Abstract

In this paper we consider the robust stability problem for a class of uncertain delay systems where the characteristic equations involve a polytope $\mathcal{P}$ of quasipolynomials (i.e., polynomials in one complex variable and exponential powers of the variable). Given a set D in the complex plane our goal is to find a constructive technique to verify whether all roots of every quasipolynomial in $\mathcal{P}$ belong to D (that is, to verify the D-stability of $\mathcal{P}$). Our first result is that, under a mild assumption on the set D, a polytope of quasipolynomials is D–stable if and only if the edges of the polytope are D–stable. Hence, the D-stability problem of a higher dimensional polytope of quasipolynomials is reduced to the D–stability problem of a finite number of pairwise convex combinations of vertic quasipolynomials of the polytope. This extends the "Edge Theorem" developed by Bartlett, Hollot and Lin [1] and Fu and Barmish [2] for the D-stability of a polytope of polynomials and is of particular interest since we show by counterexample that Kharitonov's Theorem does not hold for general delay systems. Our second result gives a constructive graphical test for checking the D-stability of a polytope of quasipolynomials which is especially simple when the set D is the open left half plane. The graphical test is based on the polar plots of some transfer functions associated with the vertic quasipolynomials of the polytope. In the special case when the vertic quasipolynomials are in a factored form, the graphical test is further simplified via a special mapping. An application example is used to demonstrate the power of the results.

1 Introduction

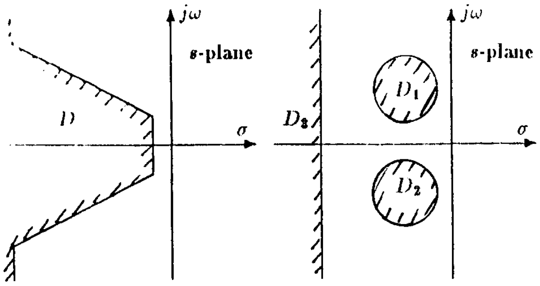
Research into robust stability of uncertain systems has become of great interest in the last few years. The general problem can be roughly formulated as follows: Given a family of linear systems $\mathcal{S}$ and a set D in the complex plane, provide computationally tractable techniques for determining the D-stability of $\mathcal{S}$, i.e., checking whether the eigenvalues of the systems in $\mathcal{S}$ stay within D. The first notable result regarding this problem was given by Kharitonov [3]. He demonstrated that if a family of polynomials $\mathcal{P}$ is a so–called "interval polynomial" with real coefficients and the set D is the open left half plane, then $\mathcal{P}$ is D-stable if and only if *four* special extreme polynomials are D-stable. If the coefficients are complex, then it is shown in Karitonov [4] that eight extreme polynomials are sufficient.

There are, however, two assumptions made by Kharitonov which limit the applicability of the results: 1) independent coefficient perturbations; this is often too restrictive in applications; 2) the set D must be the open left half plane, thus the result is not applicable to general D–stability problems such as stability of discrete-time systems where the stability set D is the open unit disk. Recently a considerable research effort has gone into attempts to remove these limitations [1,2], [5]-[12]; see Barmish and DeMarco [13] for a survey of related research and references priori to 1987. The most pertinent results to the problem we are addressing here are those by Bartlett, Hollot and Lin [1] and Fu and Barmish [2]. In [1] it was shown that given a simply connected set[1] D in the complex plane, a polytope of real polynomials $\mathcal{P}$ is D-stable if and only if the edges of $\mathcal{P}$ are D-stable. Therefore, the D-stability problem of a higher dimensional polytope of polynomials reduces to that of a finite number of pairwise convex combinations of vertic polynomials. This "Edge Theorem" is generalized in [2] to include disconnected D sets and complex polynomials. This is done by considering a set D with its complement D^c being continuously connected on the extended complex plane (the complex plane including the infinity point). It is demonstrated that the D–stability of the edges of a polytope of complex (including real) polynomials $\mathcal{P}$ is necessary and sufficient for the D–stability of $\mathcal{P}$. Thus the result of [1] becomes a special case of the result of [2] since any simply connected set D in the complex plane satisfies the requirement on D^c.

In this paper we consider the D–stability problem for a class of uncertain delay systems where the characteristic equations involve a polytope of quasipolynomials $\mathcal{P}$. Given a set D in the complex plane our goal is to find a constructive technique to verify whether all roots of every quasipolynomial in $\mathcal{P}$ belong to D (that is, to verify the D–stability of $\mathcal{P}$). Our first result is that, under a mild assumption on the set D, a polytope of quasipolynomials is D–stable if and only if the edges of the polytope are D–stable. Hence, the D-stability problem of a higher dimensional polytope of quasipolynomials is reduced to the D–stability problem of a finite number of pairwise convex combinations of vertic quasipolynomials of the polytope. This extends the "Edge Theorem" developed by Bartlett, Hollot and Lin [1] and Fu and Barmish [2] for the D-stability of a polytope of polynomials. One difficulty we have encountered in extending the results in [2] and[1] to delay systems is due to the fact a quasipolynomial usually has an infinite number of zeros. In other words, the set of zeros of a polytope of quasipolynomials is usually unbounded. As a consequence, for a given set D, assuming the simple connectedness of D or D^c being continuously connected on the extended complex plane may not necessarily lead to the "edge reduction." For this reason, a mild assumption is added on D. Roughly speaking, we require that D is such that for any point $x \in D^c$ there exists a continuous path in D^c connecting x to some point y with an arbitrarily large absolute value and the real part larger than some prescribed number. Examples of such D sets are shown in Figure 1.

The second result of this paper deals with the D–stability problem of a polytope of quasipolynomials when D is the open left half plane. Namely, we provide a constructive graphical test for checking the D-stability of a polytope of quasipolynomials. The graphical test is based on the polar plots of some transfer functions associated with the vertic quasipolynomials of the polytope. It is also easily extendable to the D-stability problem with a general open D sets (see Remark 4.2). We feel that the

[1] A set D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D [14]. For example, the open left half plane and the unit disk are simply connected, while an annulus defined by $\{s : 1 < |s| < 2\}$ is not.



$D = D_1 \cup D_2 \cup D_3$

Figure 1: Examples of admissible D sets

graphical test is especially attractive since no simple analytical test exists for checking the stability of a quasipolynomial (This contrasts with the polynomial case where we have, for example, the Routh–Hurwitz criteria).

When the generating quasipolynomials of the polytope are in a factored form (see Section 5 for definition), we further simplify the graphical test via a special mapping which transforms the stability problem of a quasipolynomial to that of a polynomial.

2 Problem Formulation and Notation

We consider a class of delay systems described by

$$F\dot{x}(t) = \sum_{i=0}^{\ell} A_i x(t - \tau_i) \tag{1}$$

where the trajectory vector $x(t) \in \mathbf{R}^n$, A_i and F are real (or complex) system matrices with F nonsingular, and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_\ell$ represent the delays. Then the characteristic equation of (1) is given using an n-th order quasipolynomial in the form of

$$p(s) \doteq \det(sF - \sum_{i=0}^{\ell} e^{-\tau_i s} A_i) = 0$$

where $p(s)$ can be written as

$$p(s) = a_{00}s^n + \sum_{i=1}^{n}\left(\sum_{k=0}^{N} a_{ik}e^{-h_k s}\right)s^{n-i} \tag{2}$$

where

$$a_{ik} = \alpha_{ik} + j\beta_{ik}; \quad \alpha_{ik}, \beta_{ik} \in \mathbf{R}$$

are constants, $a_{00} \neq 0$, and $0 = h_0 < h_1 < h_2 < \cdots < h_N$ correspond to τ_i.

Definition 2.1 Given a set D in the complex plane, the delay system (1) is called *D-stable* if the zeros of the characteristic quasipolynomial $p(s)$ in (2) stay in D. If so, $p(s)$ is called *D-stable*. In particular, $p(s)$ is called *stable* if $p(s)$ is D-stable for D being the open left half plane. (The latter case corresponds to exponential stability of solutions to (1) with integrable initial functions [16].) □

Suppose the coefficients of $p(s)$ in (2) involve uncertain parameters, then it is of interest to determine the D-stability of the system for all admissible parameter perturbations. Mathematically, we consider a family of n-th order (real or complex) quasipolynomials

$$\begin{aligned}\mathcal{P} \doteq \{p(s) &= a_{00}s^n + \sum_{i=1}^{n}\left(\sum_{k=0}^{N} a_{ik}e^{-h_i s}\right)s^{n-i}: \\ &a_{00} \neq 0; (a_{00}, a_{10}, a_{11}, \cdots, a_{1N}, \cdots, a_{nN}) \in \mathcal{T}\}\end{aligned} \tag{3}$$

for some $\mathcal{T} \in C^{nN+n+1}$ characterizing the parameter perturbations. Given a set D in the complex plane, we want to determine the D-stability of $\mathcal{P}$, i.e., whether $p(s)$ is D-stable for all $p(s) \in \mathcal{P}$.

In this paper, we consider a special family of quasipolynomials for which $\mathcal{P}$ is a polytope generated by the convex combinations of a number of n-th order quasipolynomials $p_1(s)$, $p_2(s)$, $\cdots$, $p_r(s)$ as in (2), i.e.,

$$\mathcal{P} \doteq \text{conv}\{p_1(s), p_2(s), \cdots, p_r(s)\} \tag{4}$$

and for which every member of $\mathcal{P}$ does not have vanishing leading coefficient. We will call the $p_i(s)$ in (4) generators of $\mathcal{P}$.

We denote by $E[X]$ the set of all edges of a polytope X. Recall that an edge of a polytope is its one-dimensional face [17], i.e., a closed segment $[x,y] \doteq \text{conv}\{x,y\}$ in X such that for any open segment $(x_0, y_0) \doteq \text{conv}\{x_0,, y_0\}\backslash\{x_0, y_0\}$ in X intersecting $[x,y]$, we have $[x_0, y_0] \subset [x,y]$. Note that any edge of the polytope (4) is of the form $\text{conv}\{p_i(s), p_j(s)\}$ but not all such closed segments are necessarily the edges (e.g., $\text{conv}\{p_3(s), p_2(s)\}$ is not an edge if $p_3(s) = (p_1(s)+p_2(s))/2, p_1(s) \neq p_2(s)$).

Remark 2.2 The requirement that the leading coefficient of every member of $\mathcal{P}$ does not vanish is equivalent to the assumption that the set of the leading coefficients of the generators $p_i(s)$ are on one side of some line through the origin in the complex plane. For the real case, this requires that the leading coefficients of $p_i(s)$ are of the same sign. □

Remark 2.3 Given a finite number of open loop transfer functions $G_i(s) = q_i(s)/p_i(s), i = 1, 2, \cdots, r$ where $q_i(s)$ and $p_i(s)$ are quasipolynomials, and an uncertainty model consisting of all convex combinations

$$G(s) = \sum_{i=1}^{r} \lambda_i G_i(s), \quad \lambda_i \geq 0, \quad \sum_{i=1}^{r} \lambda_i = 1,$$

the D-stability of the closed loop system with a unity feedback can be reduced to the D-stability of a polytope of quasipolynomials

$$\mathcal{P} = \text{conv}\{(G_1(s)+1)p(s), (G_2(s)+1)p(s), \cdots, (G_r(s)+1)p(s)\}$$

where $p(s)$ is a least common denominator of $G_i(s)$, $i = 1, 2, \cdots, r$. □

For an n-th order complex quasipolynomial $p(s)$ given by (2), we denote its coefficient vector by

$$\mathbf{p} = [\alpha_{00}\ \beta_{00}\ \alpha_{10}\ \beta_{10} \cdots \alpha_{1N}\ \beta_{1N} \cdots \alpha_{nN}\ \beta_{nN}]^T. \tag{5}$$

Then, it is straightforward to show that s is a zero of $p(s)$ if and only if

$$K(s)\mathbf{p} = 0$$

where

$$K(s) \doteq \begin{bmatrix} \mathrm{Re}\,(s^n) & \mathrm{Im}\,(s^n) \\ -\,\mathrm{Im}\,(s^n) & \mathrm{Re}\,(s^n) \\ \mathrm{Re}\,(e^{-h_0 s} s^{n-1}) & \mathrm{Im}\,(e^{-h_0 s} s^{n-1}) \\ -\,\mathrm{Im}\,(e^{-h_0 s} s^{n-1}) & \mathrm{Re}\,(e^{-h_0 s} s^{n-1}) \\ \cdots & \cdots \\ \mathrm{Re}\,(e^{-h_N s} s^{n-1}) & \mathrm{Im}\,(e^{-h_N s} s^{n-1}) \\ -\,\mathrm{Im}\,(e^{-h_N s} s^{n-1}) & \mathrm{Re}\,(e^{-h_N s} s^{n-1}) \\ \cdots & \cdots \\ \mathrm{Re}\,(e^{-h_N s}) & \mathrm{Im}\,(e^{-h_N s}) \\ -\,\mathrm{Im}\,(e^{-h_N s}) & \mathrm{Re}\,(e^{-h_N s}) \end{bmatrix}^T \qquad (6)$$

is a $2 \times (nN + n + 1)$ real matrix. For a family of n-th order quasipolynomials $\mathcal{P}$ given by (3) and ξ in the complex plane, we define

$$Q(\mathcal{P}, \xi) \doteq \{K(\xi)\mathbf{p} : p(s) \in \mathcal{P}\}. \qquad (7)$$

Note that for a polytope of quasipolynomials $\mathcal{P}$ and each fixed ξ, $Q(\mathcal{P}, \xi)$ is a polytope in the complex plane. With the definition above, a given polytope of quasipolynomials $\mathcal{P}$ as in (4) is D-stable if and only if $Q(\mathcal{P}, \xi)$ does not contain 0 for all $\xi \in D^c$.

3 D-Stability Criteria for a Polytope of Delay Systems

In this section we prove an Edge Theorem for delay systems and also give a counterexample showing that Kharitonov's Theorem does not extend to interval quasipolynomials.

Theorem 3.1 *Consider a polytope of n-th order (real or complex) quasipolynomials $\mathcal{P}$ as in (4) and a set D in the complex plane satisfying the following condition: There exists some real number α such that, for any point $x \in D^c$ (the complement of D) and any $M > 0$, we can find a continuous path in D^c connecting x and some point y with $|y| \geq M$ and $\mathrm{Re}\, y \geq \alpha$. Then, $\mathcal{P}$ is D-stable if and only if all the edges of $\mathcal{P}$ are D-stable.*

The following lemma is essential in the proof of Theorem 3.1.

Lemma 3.2 *Consider a polytope of quasipolynomials $\mathcal{P}$ as in (4) and $Q(\cdot,\cdot)$ defined in (7). Then, For any ξ in the complex plane,*

$$E[Q(\mathcal{P}, \xi)] \subset Q(E[\mathcal{P}], \xi) \qquad (8)$$

where $E[\mathcal{P}]$ (resp. $E[Q]$) denotes the set of the edges of $\mathcal{P}$ (resp. Q).

Proof: Denote $m \doteq \dim \mathcal{P} \doteq \dim \mathrm{aff}(\mathcal{P})$, where $\mathrm{aff}(\mathcal{P})$ is the affine extension of $\mathcal{P}$. Since the case $m \leq 1$ is trivial, we assume $m \geq 2$ in the proof. For any $a \in E[Q(\mathcal{P}, \xi)]$, we need to show that $a \in Q(E[\mathcal{P}], \xi)$. We proceed by defining

$$\mathcal{P}_a \doteq \{p(s) : p(s) \in \mathrm{aff}(\mathcal{P}), K(\xi)\mathbf{p} = a\}.$$

Note that $\mathcal{P}_a$ is an affine subspace in $\mathrm{aff}(\mathcal{P})$ with $\dim \mathcal{P}_a \geq m - 2$ and that, by assumption, $\mathcal{P}_a \cap \mathcal{P}$ contains at least one point, say $p_a(s)$. We claim that there must exist a 2-dimensional face F_2 of $\mathcal{P}$ such that $\mathcal{P}_a \cap F_2 \neq 0$. This is obvious when $m = 2$. When $m \geq 3$, $\mathcal{P}_a$ is at least one-dimensional and must intersect an $(m-1)$-dimensional face, say F_{m-1}, of $\mathcal{P}$. Indeed, either $p_a(s)$ lies on the relative boundary of $\mathcal{P}$ and we are done or it belongs to the relative interior of $\mathcal{P}$ and $\mathcal{P}_a$ contains a line through $p_a(s)$ which intersects F_{m-1}. If $m = 3$, this face is 2-dimensional and our claim holds. If $m > 3$, we replace $\mathcal{P}$ by F_{m-1} which is also a polytope [17] and repeat the argument above until we obtain a 2-dimensional face of $\mathcal{P}$ intersecting $\mathcal{P}_a$. The remaining part of the proof is divided into two cases:

Case 1: The linear mapping $p(s)|_{s=\xi}$: $\mathrm{aff}(F_2) \to C$ is one to one. Then there exists a unique point $p_a(s) \in F_2$ such that $p_a(\xi) = a$. Since the points from the relative interior of F_2 are mapped into interior points of $Q(\mathcal{P}, \xi)$, $p_a(s)$ is on the boundary of F_2 and therefore on an edge of F_2 (also an edge of $\mathcal{P}$). Thus, $a \in Q(E[\mathcal{P}], \xi)$.

Case 2: The kernel of the image of the linear mapping $p(s)|_{s=\xi}$: $\mathrm{aff}(F_2) \to C$ is at least one-dimensional. In this case, we have $\dim \mathcal{P}_a \cap F_2 \geq 1$. Therefore, $\mathcal{P}_a$ must intersect an edge of F (also an edge of $\mathcal{P}$). Again, $a \in Q(E[\mathcal{P}], \xi)$. □

Proof of Theorem 3.1: The necessity is obvious because $E[\mathcal{P}] \subset \mathcal{P}$. Now we proceed with the sufficiency by assuming, on the contrary, that there exists some $s_0 \in D^c$ such that $0 \in Q(\mathcal{P}, s_0)$. We need to show that there exists some $s_1 \in D^c$ such that $0 \in Q(E[\mathcal{P}], s_1)$. Indeed, because of the boundedness of $\mathcal{P}$, there exists some $M > 0$ such that $0 \notin Q(\mathcal{P}, s)$ for all s with $|s| \geq M$ and $\mathrm{Re}\, s \geq \alpha$. This follows from the fact that

$$\sup_{p(s) \in \mathcal{P}; \mathrm{Re}\, s \geq \alpha} \left| \frac{p(s)}{a_{00}s^n} - 1 \right| = \sup_{p(s) \in \mathcal{P}; \mathrm{Re}\, s \geq \alpha} \left| \sum_{i=1}^{n} \left(\sum_{k=0}^{N} a_{ik} e^{-h_k s} \right) s^{-i} \right| \to 0$$

as $|s| \to \infty$. Now let $\Gamma \subset D^c$ be any continuous path connecting s_0 and some point s_2 with $|s_2| \geq M$ and $\mathrm{Re}\, s_2 \geq \alpha$. For every $\xi \in \Gamma$, we define

$$d(\xi) \doteq \begin{cases} \min\{|q_\xi| : q_\xi \in E[Q(\mathcal{P}, \xi)]\} & \text{if } 0 \notin Q(\mathcal{P}, \xi) \\ -\min\{|q_\xi| : q_\xi \in E[Q(\mathcal{P}, \xi)]\} & \text{if } 0 \in Q(\mathcal{P}, \xi) \end{cases}$$

By the continuity of Γ, the minimum function, and the vertices with respect to ξ, we know that $d(\cdot)$ is continuous on Γ. Since $d(s_2) > 0$ and $d(s_0) \leq 0$, there must exist some $s_1 \in \Gamma$ such that $d(s_1) = 0$, i.e., $0 \in E[Q(\mathcal{P}, s_1)]$. Using Lemma 3.2, we conclude that $0 \in Q(E[\mathcal{P}], s_1)$. □

Remark 3.3 It can be seen that the "Edge Theorem" is extendable to a polyhedron of polynomials as well as a polyhedron of quasipolynomials using the same proof above. A polyhedron can be defined as the union of finitely many polytopes. This geometric object describes a more general class of linear perturbations or can be used to approximate nonlinear (including multilinear) perturbations. □

Remark 3.4 It is well known [3]–[4] that for the polynomial case where the polytope reduces to a hyper-rectangular region generated by varying the coefficients in some given intervals it is sufficient to check the Hurwitz stability of eight (or four for real coefficients) specially chosen vertic polynomials in order to determine the Hurwitz stability of the entire family of polynomials. In [18] a counterexample was given which demonstrates that checking the vertices of an interval polynomial is not sufficient for discrete time systems. Since delay systems fall somewhere between the continuous and discrete cases, it is of interest to know whether checking vertices may also be sufficient for continuous systems with time delays. Unfortunately, the following counterexample shows that this is not possible in general (This contradicts Theorems 3 and 5 of of Mori and Kokame [19], although the extension may stand in some special cases; see, for example, Theorem 2 of [19]). □

A counterexample showing that Kharitonov's theorem does not extend to delay systems Consider a closed loop system as in Figure 2. The characteristic equation of the system is given by

$$(s+e^{-s})^2+K=0.$$

The stability of the system is preserved for K close to zero since the equation

$$s+e^{-s}=0$$

has all its roots in the open left half plane; see, for example, Bellman and Cooke [16]. The Nyquist plot of $K/(s+e^{-s})^2$ corresponding to $K=K_1\doteq(3\pi/2+1)^2$ is presented in Figure 3, it crosses $(-1+j0)$ at $\omega=\omega_1\doteq 3\pi/2$. It can be easily verified that for

$$K=K_n\doteq\left(\frac{\pi}{2}+n\pi-(-1)^n\right)^2$$

and the corresponding frequencies

$$\omega=\omega_n\doteq\frac{\pi}{2}+n\pi,$$

$n=0,1,2,\cdots$, the Nyquist plot crosses $(-1+j0)$. Since the phase is oscillating around $-\pi$ for $\omega>\pi/2$ (see Figure 3), we have infinitely many switches from stability to instability and again stability when K grows to $+\infty$. Thus, for instance, the system is stable for $K\in[0,K_0)$ and $K\in(K_1,K_2)$ but unstable for $K\in(K_0,K_1)$. This shows that it is not possible to extend the Kharitonov's results to interval quasipolynomials.

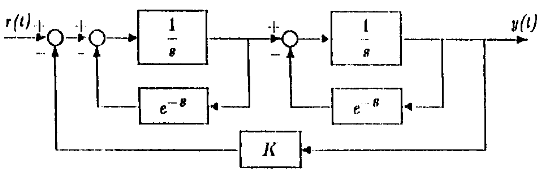


Figure 2: Block diagram of the counterexample

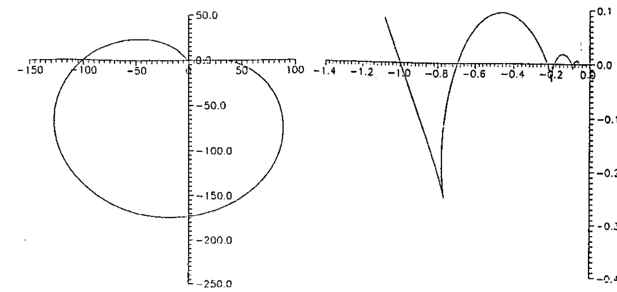
4 A Graphical Approach for Checking D-Stability of Delay Systems

In this section, we present an approach to checking the D-stability of delay systems which is based on polar plots. Although it can be argued that a closed form expression for testing the D-stability (or stability) of a quasipolynomial would be more desirable than a graphical test, this is not necessarily true. For example, for polynomials an alternative method (see Bialas [6] and Fu and Barmish [7]) would be to check the eigenvalues of $H_{k1}H_{k0}^{-1}$ (or, equivalently, $H_{k0}^{-1}H_{k1}$, or $H_{k1}^{-1}H_{k0}$, or $H_{k0}H_{k1}^{-1}$), where H_{k0} and H_{k1} are the Hurwitz matrices of $p_{k0}(s)$ and $p_{k1}(s)$, respectively. That is, stability of a polytope of polynomials $\mathcal{P}$ requires that, for every edge E_k, the eigenvalues of $H_{k1}H_{k0}^{-1}$ need to be either complex or positive. However, this involves calculating $H_{k1}H_{k0}^{-1}$ and its eigenvalues while for the graphical test to be proposed it is only necessary to evaluate the values of rational functions $p_{k1}(j\omega)/p_{k0}(j\omega)$. Furthermore, every control engineer is familiar with polar plots. With readily available graphics workstations and polar plot software, we feel that graphical tests are as good or better than analytic tests which often involve complicated numerical calculations for realistic systems.

Theorem 4.1 *Consider a polytope of n-th order (real or complex) quasipolynomials $\mathcal{P}$ as in* (4). *We use $E_1,E_2,\cdots,E_t$ to denote the edges of $\mathcal{P}$ and $p_{k0}(s)$ and $p_{k1}(s)$ to denote the vertic quasipolynomials of E_k. Then, $\mathcal{P}$ is stable if and only if the following two conditions hold for every E_k, $1\le k\le t$:*

i) The polar plot of $p_{k0}(j\omega)/(j\omega+1)^n$ does not encircle the origin;

ii) The polar plot of $p_{k1}(j\omega)/p_{k0}(j\omega)$ does not cross $(-\infty,0]$ (the nonpositive part of the real axis).



Global plot Zoomed around the origin

Figure 3: Nyquist plot of the counterexample

Proof: Note that if D is the open left half plane then the condition on D^c in Theorem 3.1 is easily satisfied (by choosing $\alpha=0$). According to Theorem 3.1, $\mathcal{P}$ is stable if and only if $E_1,E_2,\cdots E_t$ are stable. Hence, we need to show that conditions i) and ii) are necessary and sufficient for the stability of every E_k, $1\le k\le t$. For this purpose, we write E_k as

$$E_k=\{p_{k\lambda}(s)=(1-\lambda)p_{k0}(s)+\lambda p_{k1}(s):\lambda\in[0,1]\}$$

and need to show that condition i) is necessary and sufficient for the stability of $p_{k0}(s)$ and condition ii) is necessary and sufficient for the stability of $p_{k\lambda}(s)$ for all $\lambda\in(0,1]$ when condition i) holds. Indeed, using the argument principle, the number of zeros of $p_{k0}(s)$ in the closed right half plane is equal to the encirclements of $p_{k0}(j\omega)/(j\omega+1)^n$ around the origin since $(s+1)^n$ is stable; see, for example, El'sgol'ts and Norkin [15]. Therefore, $p_{k0}(s)$ is stable if and only if condition i) holds.

Assuming $p_{k0}(s)$ is stable, we now need to show that condition ii) is necessary and sufficient for the stability of all $p_{k\lambda}(s),\lambda\in(0,1]$. To see the necessity, note that, for $0<\lambda\le 1$, $p_{k\lambda}(s)$ being stable implies that

$$p_{k\lambda}(j\omega)=(1-\lambda)p_{k0}(j\omega)+\lambda p_{k1}(j\omega)\neq 0 \qquad (9)$$

for all $\omega\in\mathbf{R}$. Since $p_{k0}(j\omega)\neq 0$ and $\lambda\neq 0$, equation (9) is equivalent to

$$\frac{1-\lambda}{\lambda}+\frac{p_{k1}(j\omega)}{p_{k0}(j\omega)}\neq 0.$$

Noticing that $(1-\lambda)/\lambda$ takes values in $[0,\infty)$ when λ varies in $(0,1]$, we conclude that condition ii) is necessary for the stability of all $p_{k\lambda}(s),\lambda\in(0,1]$.

To show the sufficiency of condition ii) we proceed by contradiction. Suppose there exists some $\alpha\in(0,1]$ such that $p_{k\alpha}(s)$ is not stable, then we need to find some $\beta\in(0,1]$ and $\omega_0\in\mathbf{R}$ such that $p_{k\beta}(j\omega_0)=0$. For this purpose, we let $s_1(\lambda),s_2(\lambda),\cdots$

be the set of zeros of $p_{k\lambda}(s)$. Note that the $s_i(\lambda)$ continuously depend on λ (by Rouche's Theorem). Since $p_{k\alpha}(s)$ is unstable, there must exist some j with $s_j(\alpha)$ having non-negative real part. Since $s_j(0)$ has negative real part (because $p_{k0}(s)$ is stable), there must exist some $\beta \in (0, \alpha]$ and some $\omega_0 \in \mathbf{R}$ such that $\mathrm{Re}\ (s_j(\beta)) = 0$, i.e.,

$$p_{k\beta}(j\omega) = (1 - \beta)p_{k0}(j\omega) + \beta p_{k1}(j\omega) = 0. \qquad (10)$$

Hence, condition ii) is sufficient for the stability of $p_{k\lambda}(s)$ for all $\lambda \in (0, 1]$. □

Remark 4.2 It can be seen from the proof that the number of tests in i) for stability of vertices can be reduced to checking just one arbitrarily chosen vertex $p_{10}(s)$. Then we can check the stability of those edges which contain $p_{10}(s)$ using the tests of form ii). In the next step we test the stability of those edges which have a common vertex with one of the previous edges, etc. Since the set of edges of a polytope is connected, we can verify in this way the stability of all edges in a finite number of steps. □

Remark 4.3 It should be noted that the graphical test given in Theorem 4.1 can be generalized to sets other than the open left half plane by using the argument principle. For example, in the case when D is the open unit disk, then we only need to replace $j\omega : \omega \in (-\infty, \infty)$ by $\cos\theta + j\sin\theta : \theta \in [-\pi, \pi]$ and $p_{k0}(j\omega)/(j\omega+1)^n$ by $(p_{k0}(\cos\theta + j\sin\theta)/(\cos\theta + j\sin\theta)^n$. When the quasipolynomials are reduced to polynomials, this case corresponds to the stability of discrete-time systems. In general, if the set D is an open set and the boundary of D is a continuous path (or a finite collection of such paths in the case when D is disconnected), then the graphical test can be carried over by substituting $j\omega$ by a point on the boundary and $(s+1)^n$ by $(s+p)^n$ for some arbitrary $p \in D$. □

5 Polytope of Quasipolynomials with Generators in a Factored Form

In this section, we consider the stability problem of a special polytope of quasipolynomials when the generating quasipolynomials are in a factored form. Let $p(s)$ be an n-th order quasipolynomial given by

$$p(s) = (s + a_1 e^{-hs})(s + a_2 e^{-hs}) \cdots (s + a_n e^{-hs}) \qquad (11)$$

where a_i are complex (or real) numbers and $h > 0$ represents the delay. When $n = 2$, the quasipolynomial has the following expansion:

$$p(s) = s^2 + \alpha_1 s e^{-hs} + \alpha_2 e^{-2hs}$$

where $\alpha_1 = a_1 + a_2$ and $\alpha_2 = a_1 a_2$. A simple example of such a quasipolynomial corresponds to the characteristic equation of the closed loop system shown in Figure 4, where each integrator has delay time h. It can be verified that the characteristic equation of the system is given by

$$\Delta(s) = s^2 + k_3 s e^{-hs} + k_1 k_2 e^{-2hs} = 0.$$

Note that a given complex number s is a zero of $p(s)$ in (11) if and only if $h^n e^{nh} p(s) = 0$. On the other hand,

$$h^n e^{nh} p(s) = (hse^{hs} + ha_1)(hse^{hs} + ha_2) \cdots (hs^{hs} + ha_n). \quad (12)$$

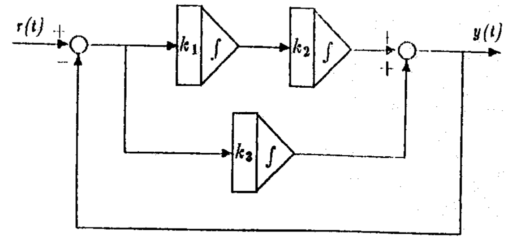


Figure 4: A system with characteristic quasipolynomial in the factored form

We define a mapping

$$\varsigma = hse^{hs}. \qquad (13)$$

Then, (12) becomes

$$\rho(\varsigma) = (\varsigma + ha_1)(\varsigma + ha_2) \cdots (\varsigma + ha_n). \qquad (14)$$

The analysis above indicates that the mapping (13) transforms a quasipolynomial in the factored form to a polynomial. It can be shown that every member of a polytope of quasipolynomials can be expressed in the factored form if its generators are in the factored form.

It can be shown that the mapping (13) maps the closed right half plane onto the complement of the bounded set D_- in the ς - *plane* with its boundary described by

$$\varsigma = -\nu\sin\nu + j\nu\cos\nu; \quad \nu \in [-\pi/2, \pi/2]; \qquad (15)$$

see Figure 5. Thus, the necessary and sufficient condition for stability of the quasipolynomial (11) is that the roots of (14) are in D_-. Hence, we obtain Theorem 5.1.

Theorem 5.1 *Consider a polytope of quasipolynomials $\mathcal{P}$ as in (4) with each generator $p_i(s)$ having the following factored form*

$$p_i(s) = (s + a_{i1}e^{-hs})(s + a_{i2}e^{-hs}) \cdots (s + a_{in}e^{-hs}). \quad (16)$$

Let $E_1, E_2, \cdots, E_t$ denote the edges of $\mathcal{P}$ and $p_{k0}(s)$ and $p_{k1}(s)$ denote the vertic quasipolynomials of E_k. Without loss of generality, we assume that all generating quasipolynomials are stable (i.e., $ha_{ij} \in D_-$ for all $p_i(s)$ and j). Then, $\mathcal{P}$ is stable (i.e., D-stable for D being the open left half plane) if and only if the

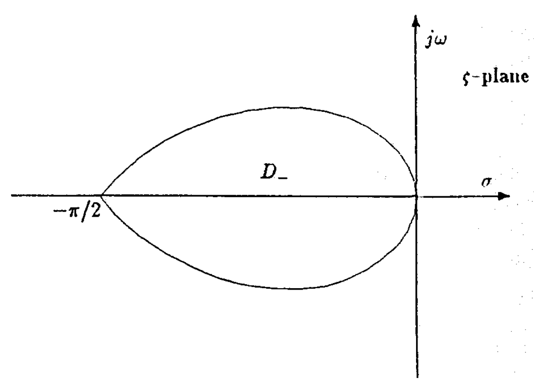


Figure 5: The set D_- in the ς-plane

following condition holds for every $E_k, 1 \leq k \leq t$:

$$\left.\frac{\rho_{k1}(\varsigma)}{\rho_{k0}(\varsigma)}\right|_{\varsigma = \nu\sin\nu + j\nu\cos\nu}$$

does not cross $(-\infty, 0]$ (the nonpositive part of the real axis) for all $\nu \in [-\pi/2, \pi/2]$.

Proof: This result is a direct consequence of Theorem 4.1 by noting that 1) Condition i) in Theorem 4.1 is guaranteed by the stability of all vertic quasipolynomials of E_k and, 2) The imaginary axis of the s-plane is mapped to the curve described by (15) in the ς-plane. □

Remark 5.2 Although few real systems can be modeled by the factored form in (11), this is not surprising since Theorem 5.1 is a very strong result which reduces the D-stability problem of a polytope of quasipolynomials to the D_ς-stability of a polytope of polynomials. Also, Remark 4.2 applies here too; i.e., the graphical test in Theorem 4.1 can be generalized to the D-stability problem of a polytope of quasipolynomials in the factored form. □

6 An Application Example

In this section, we provide an example of real life control system involving a time delay and uncertain parameters for which the problem of robust stability is crucial. This is a wind tunnel control problem addressed in [20,21] where the main objective of the feedback control is to provide a fast Mach number response so as to reduce the cost of liquid nitrogen losses during the transient regimes. Using the methods developed in this paper we shall examine the asymptotic stability and the exponential stability of a given decay rate of the closed loop system for the whole range of uncertain parameters. The latter case corresponds to a shifted left half plane. Moreover, the graphical tests we use provide additional information on the frequency domain nature of unmodelled uncertainties which may endanger the robust stability.

Let us start with the system equations [20]

$$\begin{aligned}\dot{x}_1(t) &= -ax_1(t) + akx_2(t-h)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= -\omega^2 x_2(t) - 2\varsigma\omega x_3(t) + \omega^2 u(t)\end{aligned} \tag{17}$$

where ς and ω are fixed at 0.8 and 6.0, respectively, the delay h varies slightly but can, for all practical purposes, be assumed equal to a nominal value $h = 0.33$. The parameters $a = 1/\tau$ and k, however, depend on the operating point, varying within a range approximated by

$$\tau \in [0.739, 2.58], \quad k \in [-0.0144, -0.0029]. \tag{18}$$

A feedback controller of the form

$$u(t) = -k_1x_1(t) - k_{20}x_2(t) - k_{21}\int_{-h}^{0} e^{a\theta}x_2(t+\theta)d\theta - k_3x_3(t)$$

was proposed in [20]. The main goal there was, for the nominal parameters $k = \bar{k} = -0.0117$ and $\tau = \bar{\tau} = 1/\bar{a} = 1.964$, that the characteristic polynomial of the feedback system be

$$(s+2.5)(s^2+4s+6.25).$$

It turned out that the control parameters were chosen to be

$$k_1 = -1305; \quad k_{20} = -22.75; \quad k_{21} = 9.0; \quad k_3 = -3.6.$$

Furthermore, to simplify the realization of the integral in the control law, the following 3-point Simpson approximation was used in [20]:

$$\begin{aligned}&k_{21}\int_{-h}^{0} e^{a\theta}x_2(t+\theta)d\theta\\ \approx\ &0.5(x_2(t) + 3.68x_2(t-0.165) + 0.85x_2(t-0.33)).\end{aligned}$$

Assuming this approximation and the the mismatch between the model parameters k and τ and their nominal values $\bar{k}$ and $\bar{\tau}$ which were used to design the controller, we arrive at the following characteristic equation for the closed loop system:

$$\begin{aligned}p(s,k,\tau) \doteq\ &\tau s^3 + (6\tau+1)s^2\\ &+ (13.75\tau + 6 + 1.82\tau e^{-0.165s} + 0.42\tau e^{-0.33s})s\\ &+ 13.75 + 1.82e^{-0.165s} + (0.42 - 1305k)e^{-0.33s} = 0\end{aligned}$$

and the associated polytope of quasipolynomials

$$P = \{p(s,k,\tau) : k \in [k_{\min}, k_{\max}], \tau \in [\tau_{\min}, \tau_{\max}]\}$$

where $[\tau_{\min}, \tau_{\max}]$ and $[k_{\min}, k_{\max}]$ are the assumed ranges for τ and k, respectively. Using the methods proposed in this paper, we checked and found that the closed loop system is asymptotically stable for all k and τ even exceeding the range given in (18). To save space, we do not present the corresponding numerical results. Then we asked the question whether for all k and τ within the assumed range of uncertainty the system eigenvalues remain in the half plane $D \doteq \{\mathrm{Re}\, s \leq -1\}$. Recall that the nominal system eigenvalues are -2.5 and $-2 \pm j2.5$ and they change only a little, after introducing the Simpson approximation, moving to -2.674 and $-1.97 \pm j1.503$, respectively. In order to check the D-stability, we proceeded by denoting

$$\begin{aligned}p_0(\sigma) &\doteq p(\sigma-1, \tau_{\max}, k_{\min}),\\ p_1(\sigma) &\doteq p(\sigma-1, \tau_{\max}, k_{\max}),\\ p_2(\sigma) &\doteq p(\sigma-1, \tau_{\min}, k_{\max}),\\ p_3(\sigma) &\doteq p(\sigma-1, \tau_{\min}, k_{\min}).\end{aligned}$$

We have translated the problem to a form tractable by Theorem 4.1, namely the stability of the polytope generated by the $p_i(\sigma), i = 0,1,2,3$. For several choices of ranges for k and τ we obtained a series of negative and positive results including a negative answer for the ranges given by (18). The latter means that the transient response of the closed loop system may decay slower that e^{-t} if the uncertain parameters differ from their nominal values. We present a set of computer generated figures for the case $[\tau_{\min}, \tau_{\max}] = [1.571, 2.357]$ and $[k_{\min}, k_{\max}] = [-0.0144, -0.0088]$ which corresponds to $\pm 20\%$ variation in τ and $\pm 25\%$ variation in k. This is a critical case as seen from the figures. Figure 6 shows the polar plot of $p_0(j\omega)/(j\omega+1)^3$ which proves the stability of $p_0(\sigma)$. The polar plots of $p_1(j\omega)/p_0(j\omega)$, $p_3(j\omega)/p_0(j\omega)$, $p_2(j\omega)/p_1(j\omega)$ and $p_3(j\omega)/p_2(j\omega)$ are given in Figures 7-10, respectively. It can be seen that these plots do not intersect $(-\infty, 0]$, thus by Theorem 4.1, the closed loop system is D-stable.

We want to point out that the presented figures not only prove the robust stability but also provide additional information on the nature of unmodelled uncertainties which could change stability into instability or vice versa. For instance, Figures 7-8 show that relatively small perturbations of $p(s,\tau,k)$ in high frequencies may lead to the loss of D-stability on the edges

$\text{conv}\{p_0(\sigma), p_1(\sigma)\}$ and $\text{conv}\{p_0(\sigma), p_3(\sigma)\}$, respectively. Similarly, Figures 9-10 show that somewhat larger perturbations for the mid-frequency range may destroy D-stability on the remaining two edges.

7 Conclusion

In this paper we have considered the D-stability problem for a class of uncertain delay systems where the characteristic equations involve a polytope $\mathcal{P}$ of quasipolynomials. Our first result shows that, under a mild assumption on the set D, a polytope of quasipolynomials is D-stable if and only if the edges of the polytope are D-stable. This extends the "Edge Theorem" developed by Bartlett, Hollot and Lin [1] and Fu and Barmish [2] for the D-stability of a polytope of polynomials. Our second result provides a polar-plot-based graphical test for checking the D-stability of a polytope of quasipolynomials. In a special case when the vertic quasipolynomials are in a factored form, the graphical test is further simplified via a special mapping. As shown in the example, the graphical tests we provided are quite useful in applications, allowing us to easily handle examples with many uncertain parameters.

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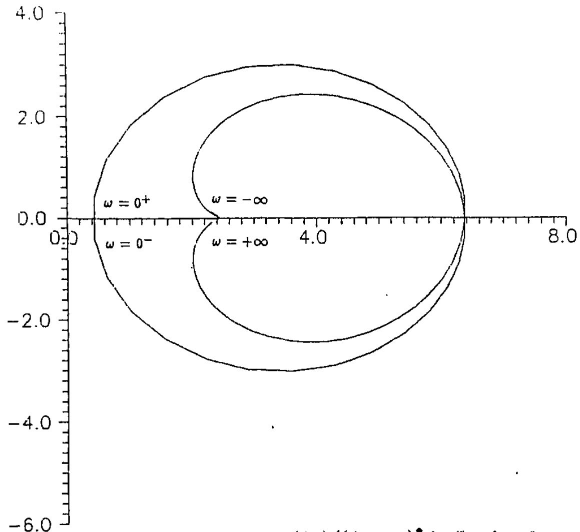

Figure 6: Polar plot of $p_0(j\omega)/(j\omega+1)^2$ in Section 6

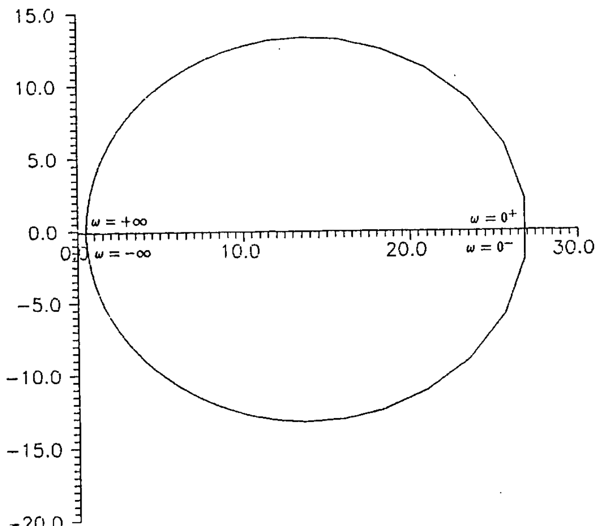

Figure 7: Polar plot of $p_1(j\omega)/p_0(j\omega)$ in Section 6

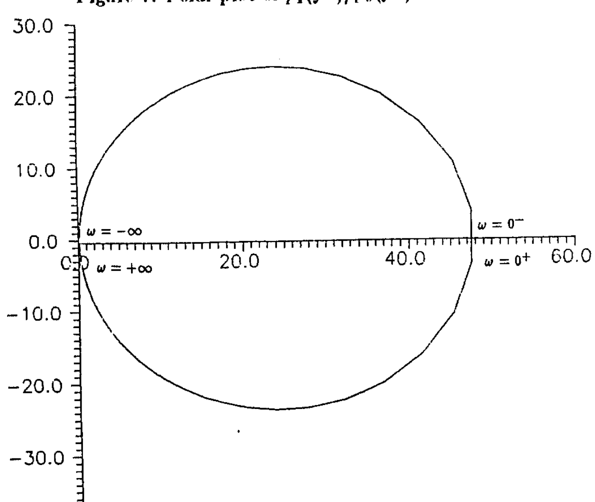

Figure 8: Polar plot of $p_3(j\omega)/p_0(j\omega)$ in Section 6

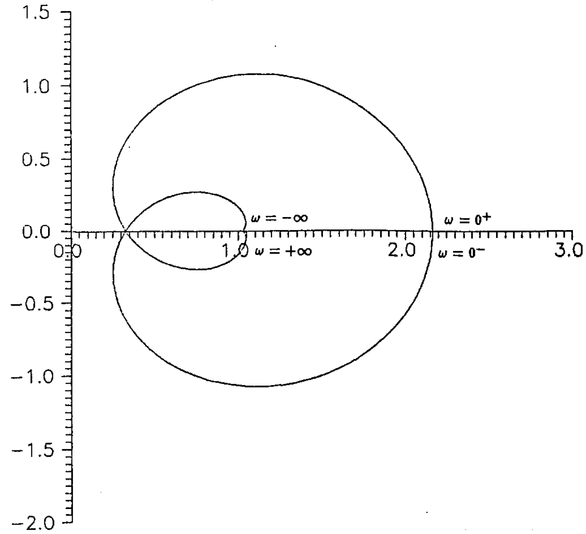

Figure 9: Polar plot of $p_2(j\omega)/p_3(j\omega)$ in Section 6

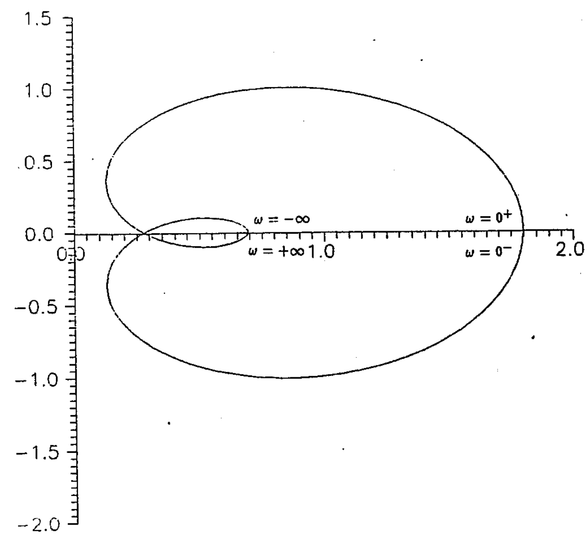

Figure 10: Polar plot of $p_1(j\omega)/p_3(j\omega)$ in Section 6